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NUMERICAL STUDIES OF FLOW BETWEEN ROTATING COAXIAL DISKS

by

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ABSTRACT

A new algorithm, which is exceptionally fast for certain choices of numerical parameters, is described for the study of nonlinear, incompressible flow between two rotating disks. Typical examples for Reynolds number R in the range $10 \leq R \leq 2000$ are described and discussed. Comparisons are made with the limited available results generated by other methods.

1. INTRODUCTION

The study of fluid motion between rotating disks is of both practical and theoretical interest (see, e.g., references [1]-[7], [9]-[17], [18]-[25], and the references contained therein). It appears that the first mathematical paper on the subject was that of von Karmen [24], who dealt with the steady flow above an infinitely large rotating disk under the assumption that axial velocity was radius independent. This model was extended to steady flow between two coaxial rotating disks by Batchelor [2], and then to nonsteady models by Greenspan [9], Greenspan and Howard [10], and Pearson [15]. In studying the related mathematical and physical problems, various techniques have been applied, including asymptotic analysis ([2], [6], [20]-[22], [25]), linearization [10], [11], [16], numerical analysis [5], [12], [13], [15], [16], and experimentation [13], [21]. Unfortunately, the results of these analyses often are either unreasonable or contradictory. Thus, as the Reynolds number becomes infinite, Batchelor [2] and Stewartson [21] both find unique, steady, limiting flows, but which are qualitatively different, while Tam [22] claims that the problem admits an infinite number of flows. Pearson [15] and Lance and Rogers [12], using different numerical techniques, generate qualitatively different flows for certain classes of nonsteady problems. Mellor, Chapple and Stokes [13] claim to have produced several classes of solutions for a given problem by analytical-numerical means, but then can produce only one such class in laboratory experiments.

Our purpose in this paper is to study numerically only the steady motion of a viscous, incompressible fluid between two rotating, infinite coaxial disks. For simplicity, the first disk is positioned in (x,y,z) space in the plane $z = 0$ with its center at $(0,0,0)$ and is given an angular velocity Ω_1, while the second disk is positioned in the plane $z = 1$ with its center at $(0,0,1)$ and is given an angular velocity Ω_2. If the cylinderical coordinates of (x,y,z) are (r,θ,z), and if the fluid at (x,y,z) has velocity components (u,v,w), then the substitutions

$$u = -\frac{1}{2} r H'(z), \quad v = r G(z), \quad w = H(z) \tag{1.1}$$

enable one ([9], [15]) to transform the dimensionless, steady state Navier-Stokes equations to

$$H'' = M \quad , 0 \leq z \leq 1 \tag{1.2}$$

$$G'' + R(GH' - G'H) = 0 \quad , 0 \leq z \leq 1 \tag{1.3}$$

$$M'' - R(HM' + 4GG') = 0 \quad , 0 \leq z \leq 1, \tag{1.4}$$

where differentiation is with respect to z. For the coaxial flow under consideration, the boundary conditions for nonlinear system (1.2)-(1.4) are ([9], [15]) (1.5) $$G(0) = \Omega_1 , \quad G(1) = \Omega_2$$

(1.6) $$H(0) = 0 , \quad H(1) = 0$$

(1.7) $$H'(0) = 0 , \quad H'(1) = 0 .$$

The numerical method to be used is an extension of one developed for cavity flow problems [8], and which was convergent for all Reynolds numbers studied ($0 \leq R \leq 10^6$). Since the present work is largely experimental in nature, and since errors in computation often seem to be more the rule than the exception, the FORTRAN program used is being made accessible in a report [18], so that every aspect of the calculations can be reproduced by the reader.

2. THE NUMERICAL METHOD

In this section we give a precise description of the algorithm to be used for the numerical solution of (1.2)-(1.7).

Divide $0 \leq z \leq 1$ into n equal parts, each of length $h = \Delta z = \frac{1}{n}$. Let the points of subdivision be $0 = z_0 < z_1 < z_2 < \cdots < z_n = 1$. Thus, $z_j = jh = \frac{j}{n}$, $j = 0, 1, 2, \ldots, n$. Let S_h be the set of boundary grid points z_0 and z_n, while I_h is the set of interior grid points z_1, $z_2, \ldots, z_{n-1}$. If F is any function defined on $S_h + I_h$, then a convenient notation will be

$$F(z_i) = F_i \,.$$

We attempt to approximate H, G, and M by generating three sequences, $H^{(k)}$, $G^{(k)}$, and $M^{(k)}$, $k = 0, 1, 2, \ldots$, on $I_h + S_h$, each of which is convergent. This is done as follows. For all values of k, let

$$H_0^{(k)} = H_n^{(k)} = 0 \quad , \quad k = 0, 1, 2, \ldots \tag{2.1}$$

$$G_0^{(k)} = \Omega_1 , \quad G_0^{(k)} = \Omega_2 , \quad k = 0, 1, 2, \ldots \,. \tag{2.2}$$

Set

$$H_i^{(0)} = 0 \quad , \quad i = 1, 2, \ldots, n-1 \tag{2.3}$$

$$G_i^{(0)} = (1 - z)\Omega_1 + z\Omega_2 \quad , \quad i = 1, 2, \ldots, n-1 \tag{2.4}$$

(2.5) $$M_i^{(0)} = 0 \quad , \quad i = 0,1,2,\ldots,n \, .$$

By induction, $H^{(k+1)}$, $G^{(k+1)}$ and $M^{(k+1)}$ are generated from $H^{(k)}$, $G^{(k)}$ and $M^{(k)}$ as is shown next.

At the points z_1 and z_{n-1} write down the two equations

(2.6) $$4H_1 = H_2$$

(2.7) $$4H_{n-1} = H_{n-2} \, ,$$

which are difference approximations (see [4])of (1.7). At each of the remaining points of I_h, write down the difference analogue

(2.8) $$H_{i-1} - 2H_i + H_{i+1} = h^2 M_i^{(k)} \quad , \quad i = 2,3,\ldots,n-2$$

of (1.2). Insertion of (2.1) into (2.6)-(2.8) results in a diagonally dominant linear algebraic system. Solve this system by SOR (point successive over-relaxation) with over-relaxation factor r_H and convergence tolerance α_1, and denote the solution by $\bar{H}_i^{(k+1)}$, $i = 1,2,\ldots,n-1$. Define $H_i^{(k+1)}$, $i = 1,2,\ldots,n-1$, by the smoothing formula

(2.9) $$H_i^{(k+1)} = \rho \bar{H}_i^{(k+1)} + (1-\rho) H_i^{(k)}, \quad \begin{cases} i = 1,2,\ldots,n-1 \\ k = 0,1,2,\ldots \\ 0 \le \rho \le 1 \, . \end{cases}$$

At each point of I_h, write down next the following forward-backward difference analogues of (1.3):

$$(2.10)\qquad G_{i-1} + [-2 + RhH_i^{(k+1)}]G_i + [1 - Rh\,H_i^{(k+1)}]G_{i+1}$$
$$= -\frac{1}{2} RhG_i^{(k)} [H_{i+1}^{(k+1)} - H_{i-1}^{(k+1)}]; \text{ if } H_i^{(k+1)} < 0,$$

$$(2.11)\qquad [1 + RhH_i^{(k+1)}]G_{i-1} + [-2 - RhH_i^{(k+1)}]G_i + G_{i+1}$$
$$= -\frac{1}{2} RhG_i^{(k)} [H_{i+1}^{(k+1)} - H_{i-1}^{(k+1)}]; \text{ if } H_i^{(k+1)} \geq 0.$$

Solve the resulting diagonally dominant linear algebraic system by SOR with over-relaxation factor r_G and convergence tolerance α_2, call the solution $\bar{G}^{(k+1)}$, and on I_h define $G^{(k+1)}$ by

$$(2.12)\qquad G_i^{(k+1)} = \mu\,\bar{G}_i^{(k+1)} + (1-\mu)G_i^{(k)}, \quad \begin{cases} i = 1,2,\ldots,n-1 \\ k = 0,1,2,\ldots \\ 0 \leq \mu \leq 1. \end{cases}$$

Note that (2.10)-(2.11) avoid the possible eigenvalue problems inherent in viewing (1.3) as an equation in G by using the iterate $G^{(k+1)}$ to approximate G'' and G' and using the iterate $G^{(k)}$ to approximate G.

To construct $M^{(k+1)}$ on $I_h + S_h$, first set (see [4])

$$(2.13)\qquad M_0^{(k+1)} = \delta_1 [2H_1^{(k+1)}/h^2] + (1 - \delta_1) M_0^{(k)}, \quad \begin{cases} k = 0,1,2,\ldots \\ 0 \leq \delta_1 \leq 1 \end{cases}$$

$$(2.14)\qquad M_n^{(k+1)} = \delta_1 [2H_{n-1}^{(k+1)}/h^2] + (1 - \delta_1) M_n^{(k)}, \quad \begin{cases} k = 0,1,2,\ldots \\ 0 \leq \delta_1 \leq 1. \end{cases}$$

Using (2.13) and (2.14) as boundary values, write down at each point of I_h the following forward-backward difference analogues of (1.4):

$$(2.15)\quad M_{i-1} + [-2 + RhH_i^{(k+1)}]M_i + [1 - RhH_i^{(k+1)}]M_{i+1} = 2Rh\,G_i^{(k+1)}[G_{i+1}^{(k+1)} - G_{i-1}^{(k+1)}], \quad \text{if } H_i^{(k+1)} < 0,$$

$$(2.16)\quad [1 + RhH_i^{(k+1)}]M_{i-1} + [-2 - RhH_i^{(k+1)}]M_i + M_{i+1} = 2Rh\,G_i^{(k+1)}[G_{i+1}^{(k+1)} - G_{i-1}^{(k+1)}], \quad \text{if } H_i^{(k+1)} \geq 0.$$

Solve the resulting diagonally dominant, linear algebraic system by SOR with successive over-relaxation factor r_M and convergence tolerance α_3, call the solution $\bar{M}^{(k+1)}$, and define $M^{(k+1)}$ on I_h by

$$(2.17)\quad M_i^{(k+1)} = \delta_2 \bar{M}_i^{(k+1)} + (1 - \delta_2) M_i^{(k)}, \quad \begin{cases} i = 1,2,\ldots,n-1 \\ k = 0,1,2,\ldots \\ 0 \leq \delta_2 \leq 1 \end{cases}.$$

For given positive tolerances $\varepsilon_1, \varepsilon_2, \varepsilon_3$, the iteration proceeds for $k = 0,1,2,\ldots,$ until for some value $k = K$, one has

$$(2.18)\quad |H^{(K+1)} - H^{(K)}| < \varepsilon_1, \text{ uniformly in } I_h$$

$$(2.19)\quad |G^{(K+1)} - G^{(K)}| < \varepsilon_2, \text{ uniformly on } I_h$$

$$(2.20)\quad |M^{(K+1)} - M^{(K)}| < \varepsilon_3, \text{ uniformly on } I_h + S_h.$$

Finally one verifies whether or not $H^{(K+1)}$, $G^{(K+1)}$, $M^{(K+1)}$ are solutions of the difference equations being solved, and, if they are, then they are taken to be the respective approximations of H, G, M.

3. EXAMPLES

From the large number of examples run on the UNIVAC 1108, several which are typical, which are of physical interest, and which display readily the changes of flow patterns with increasing Reynolds number, will be presented in this section.

For $h = \frac{1}{50}$, $\Omega_1 = 1$, and $\Omega_2 = 0$, the results for H, H' and G, with H' determined by central differences, are shown graphically for Reynolds numbers 10, 100, 1000 in Figures 1, 2, 3, respectively. The other parameter choices were: (a) for $R = 10$, $\rho = 0.9$, $\mu = 0.9$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.0$, $\alpha_1 = \alpha_2 = \alpha_3 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.005$, (b) for $R = 100$, $\rho = 0.9$, $\mu = 0.9$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.5$, $\alpha_1 = \alpha_2 = \varepsilon_1 = \varepsilon_2 = 0.001$, $\alpha_3 = \varepsilon_3 = 0.05$, (c) for $R = 1000$, $\rho = 0.9$, $\mu = 0.05$, $\delta_1 = 0.9$, $\delta_2 = 0.2$, $r_H = 1.8$, $r_G = 1.1$, $r_M = 1.5$, $\alpha_1 = \varepsilon_1 = 0.005$, $\alpha_2 = \varepsilon_2 = 0.03$, $\alpha_3 = \varepsilon_3 = 0.3$. The number of outer iterations necessary for convergence for Reynolds numbers 10 and 100 were 72 and 182, respectively. To speed convergence for $R = 1000$, $H_i^{(0)}$ and $M_i^{(0)}$ were modified to agree with

$$(3.1) \qquad H^{(0)}(z) = \begin{cases} -0.6z, & 0 \le z \le \frac{1}{2} \\ (0.6)(z-1), & \frac{1}{2} \le z \le 1 \end{cases}$$

$$(3.2) \qquad M^{(0)}(z) = \begin{cases} -12 + 36z, & 0 \le z \le \frac{1}{2} \\ -6 - 24(z-1), & \frac{1}{2} \le z \le 1, \end{cases}$$

respectively, at the grid points, and convergence was attained in 19 outer iterations. Choices (3.1) and (3.2) were motivated by the results for $R = 100$ shown in Figure 2. The maximum running time of all cases discussed thus far was under thirty seconds and convergence resulted for a variety of other choices of parameters.

In $h = \frac{1}{50}$, $\Omega_1 = 1$, and $\Omega_2 = -1$, the results for H, H' and G are shown graphically for Reynolds numbers 10, 100, 1000 in Figures 4, 5, 6, respectively. The other parameter choices were: (a) for $R = 10$, $\rho = 0.9$, $\mu = 0.9$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.0$, $\alpha_1 = \alpha_2 = \alpha_3 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.005$, (b) for $R = 100$, $\rho = 0.9$, $\mu = 0.2$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.5$, $\alpha_1 = \alpha_2 = \varepsilon_1 = \varepsilon_2 = 0.001$, $\alpha_3 = \varepsilon_3 = 0.05$, (c) for $R = 1000$, $\rho = 0.1$, $\mu = 0.1$, $\delta_1 = 0.05$, $\delta_2 = 0.9$, $r_H = 1.8$, $r_G = 0.9$, $r_M = 1.1$, $\alpha_1 = \varepsilon_1 = 0.005$, $\alpha_2 = \varepsilon_2 = 0.03$, $\alpha_3 = \varepsilon_3 = 0.3$. The number of outer iterations necessary for convergence for Reynolds numbers 10 and 100 were 67 and 31. Convergence for Reynolds number 1000 was achieved in 22 outer iterations by modifying $H_i^{(0)}$ and $M_i^{(0)}$ to agree with

$$H^0(z) = \begin{cases} (-1.2)z & , \quad 0 \le z \le \frac{1}{4} \\ -0.3 + (1.2)(z - \frac{1}{4} & , \quad \frac{1}{4} \le z \le \frac{3}{4} \\ (-1.2)(z - 1) & , \quad \frac{3}{4} \le z \le 1 \end{cases} \tag{3.3}$$

$$(3.4) \qquad M^{(0)}(z) = \begin{cases} -10 + 52z & , \quad 0 \le z \le \frac{1}{4} \\ 3 - 12\,(z - \frac{1}{4}) & , \quad \frac{1}{4} \le z \le \frac{3}{4} \\ 10 + 52\,(z - 1) & , \quad \frac{3}{4} \le z \le 1 \end{cases}$$

respectively, at the grid points.

Finally, because of the broad interest in large Reynolds numbers and because the contradictary conclusions reached in certain cases of counter rotation, attention was turned to refined calculations for the case $R = 2000$, $\Omega_1 = 1$, and $\Omega_2 = -1$. In Figures 7 and 8 are shown the results for H, H', G, and M for the parameter choices $h = 1/400$, $\rho = 0.05$, $\mu = 0.05$, $\delta_1 = 0.1$, $\delta_2 = 0.925$, $r_H = 1.8$, $r_G = r_M = 0.8$, $\alpha_1 = 0.0003$, $\alpha_2 = 0.001$, $\alpha_3 = 0.002$, $\varepsilon_1 = \varepsilon_2 = 0.001$, $\varepsilon_3 = 0.01$ and for initial functions (3.3) and (3.4). Convergence was achieved in 128 outer iteration, which required seven minutes of running time. The increase in precision is readily apparent by comparing Figures 6 and 7. Moreover, since Figure 7 shows clearly that the fluid has separated into two distinct parts which rotate with relatively large, but with opposite, angular velocities, it is concluded the numerical solution supports Batchelor in the Batchelor-Stewartson controversy ([2], [20], [22]).

With regard to the computations byother methods, only the work of Pearson and that of Lance and Rogers appear to be numerically rigorous and moderately successful. For $R \leq 100$, their results and ours for the case of a single rotating disk are completely comparable. Thereafter, various results differ widely. For example, for $R = 1000$, $\Omega_1 = 1$ and $\Omega_2 = -1$, Pearson produced two, distinct solutions but failed to produce the symmetric one. By relaxing the convergence tolerances, we too were able to produce more than one solution. However, sharpening these tolerances always resulted in one and only one solution. It is also rather interesting to observe that if, for Pearson's results (see [15], p. 632, Figure 9), the points where G and H cross the z axis are relocated to $z = \frac{1}{2}$, while the points where H_z crosses the z axis are relocated symmetrically about $z = \frac{1}{2}$, then the resulting configurations for G, H and H_z are qualitatively analogous to those shown in our Figure 7. One can only surmize, then, that Pearson's time dependent calculations are stable, but relatively inaccurate due to an accumulation of roundoff error. Such types of calculations are very common in the study of nonlinear problems.

In the study of $R = 1023$, $\Omega_1 = 1$, $\Omega_2 = -1$, Lance and Rogers assumed symmetry and reformulated the problem on the half interval $0 \leq z \leq \frac{1}{2}$. All their previous calculations were limited to $R \leqq 529$, but the use of symmetry allowed for a decrease in grid size and a corresponding increase in Reynolds number. Their results (see [12], pp. 119-120) show that the main body of the fluid is only slightly disturbed, thus contradicting the flow shown in our Figure 7, and thereby supporting Stewartson in the Batchelor-Stewartson controversy. However, Lance and Rogers failed to demonstrate that the problem they study on the interval $0 \leq z \leq \frac{1}{2}$ is, in fact, equivalent to the given problem (1.2)-(1.7) of counter rotating disks. Indeed, numerically, they should have required that the differential equations be satisfied on $z = \frac{1}{2}$, which they failed to do ([12], p. 119, eq. 5.8). Indeed, since the solution shown in our Figure 7 also satisfies the conditions (5.8) of [12], the Lance and Rogers formulation has at least two solutions, namely theirs and ours, and appears to be a weaker problem than the one originally posed. The question as to whether or not the Lance and Rogers solution satisfies the differential equations on $z = \frac{1}{2}$ does not, however, appear to be trivial.

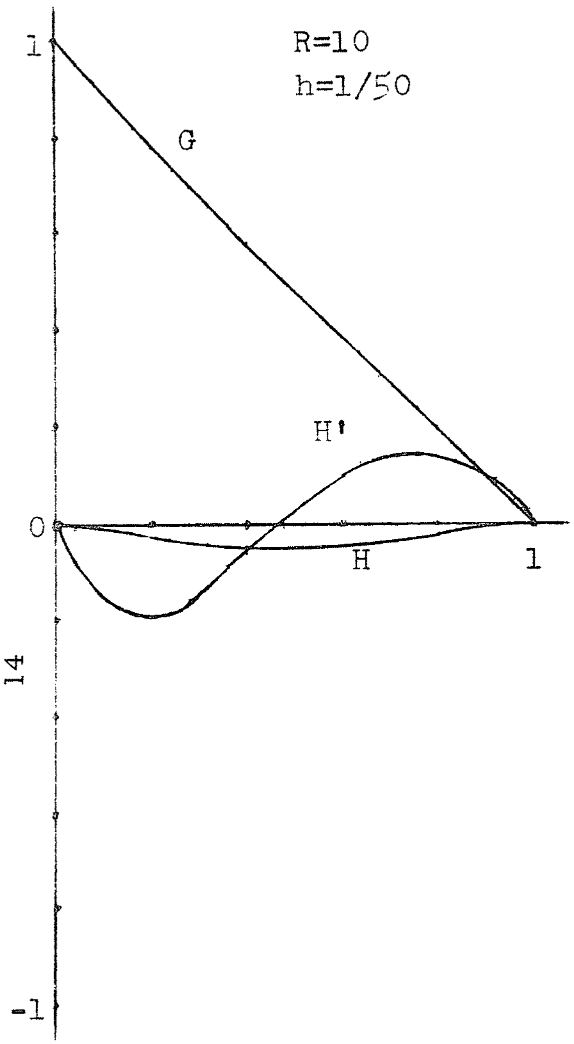

FIGURE 1

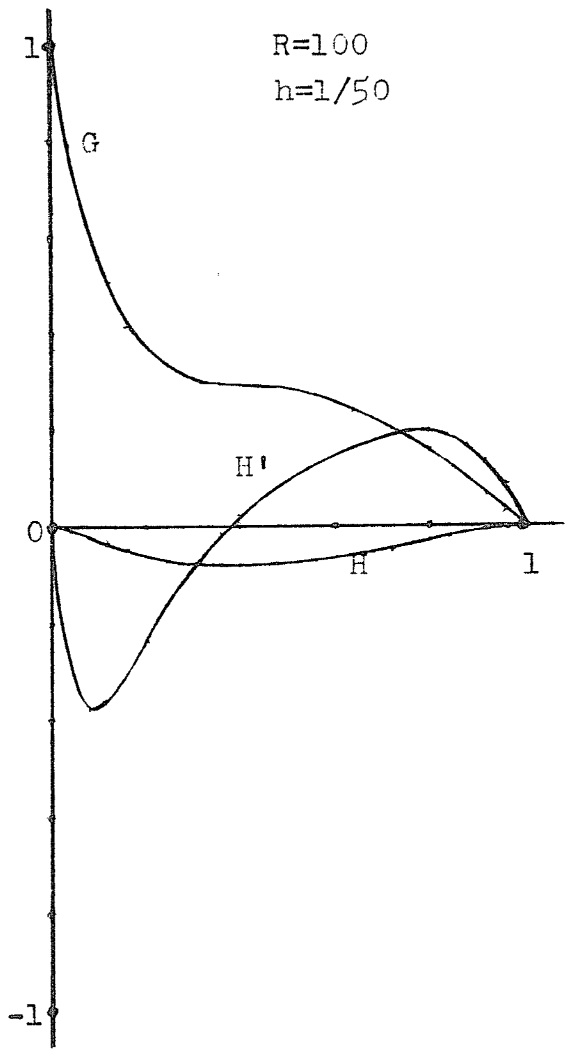

FIGURE 2

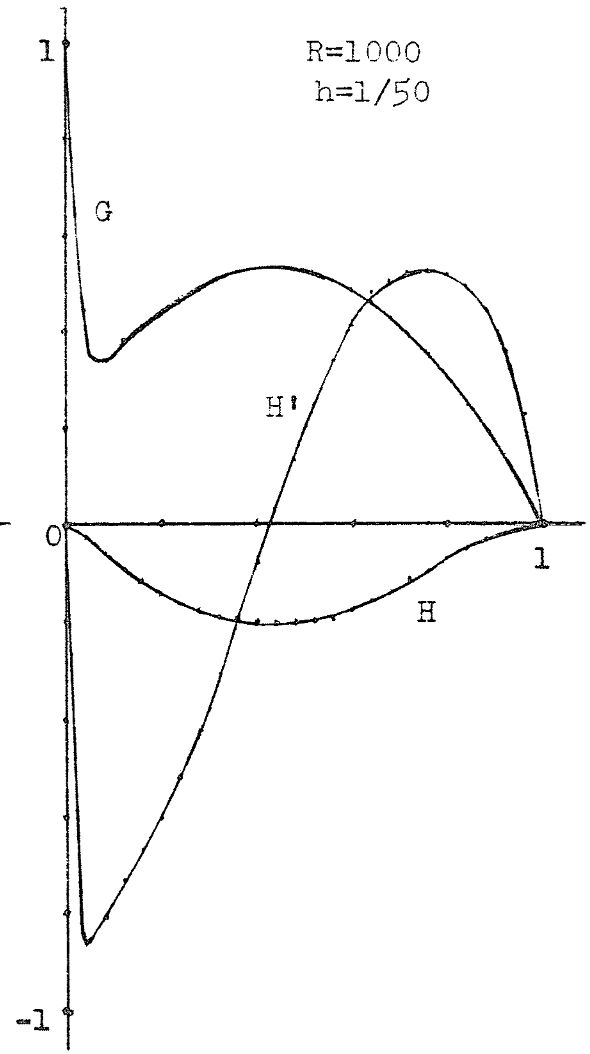

FIGURE 3

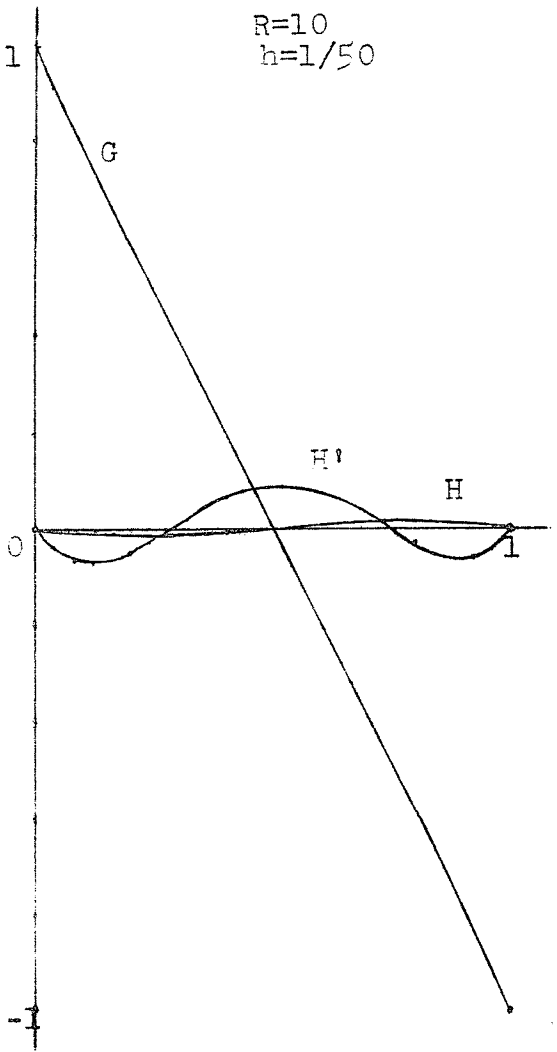

FIGURE 4

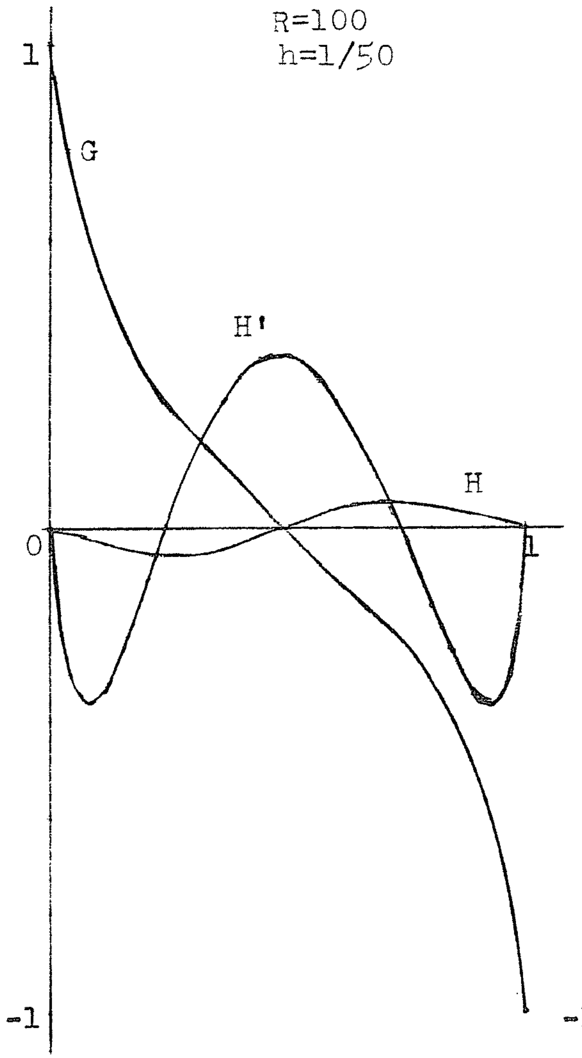

FIGURE 5

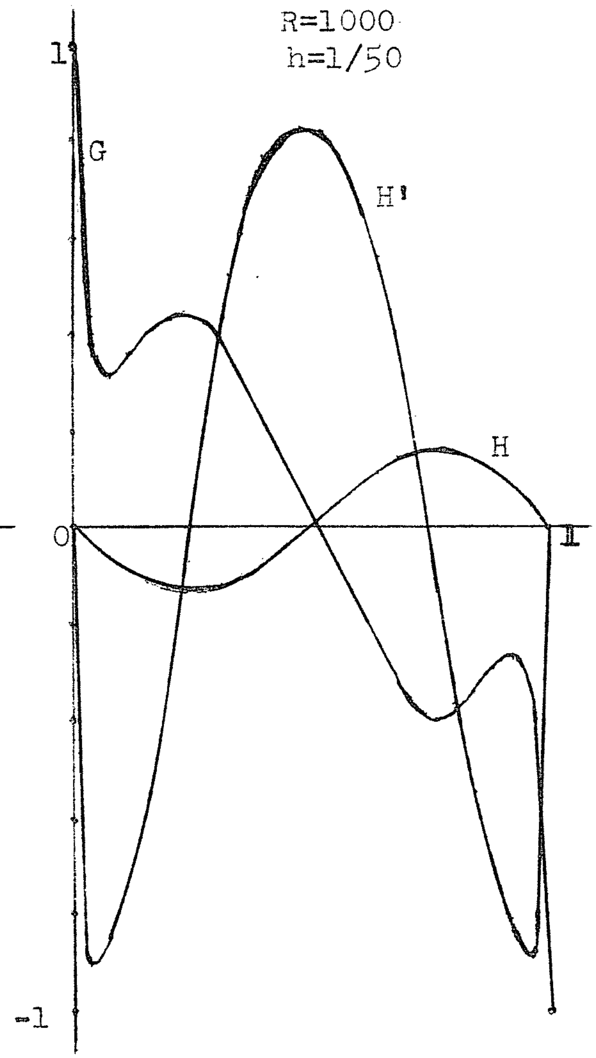

FIGURE 6

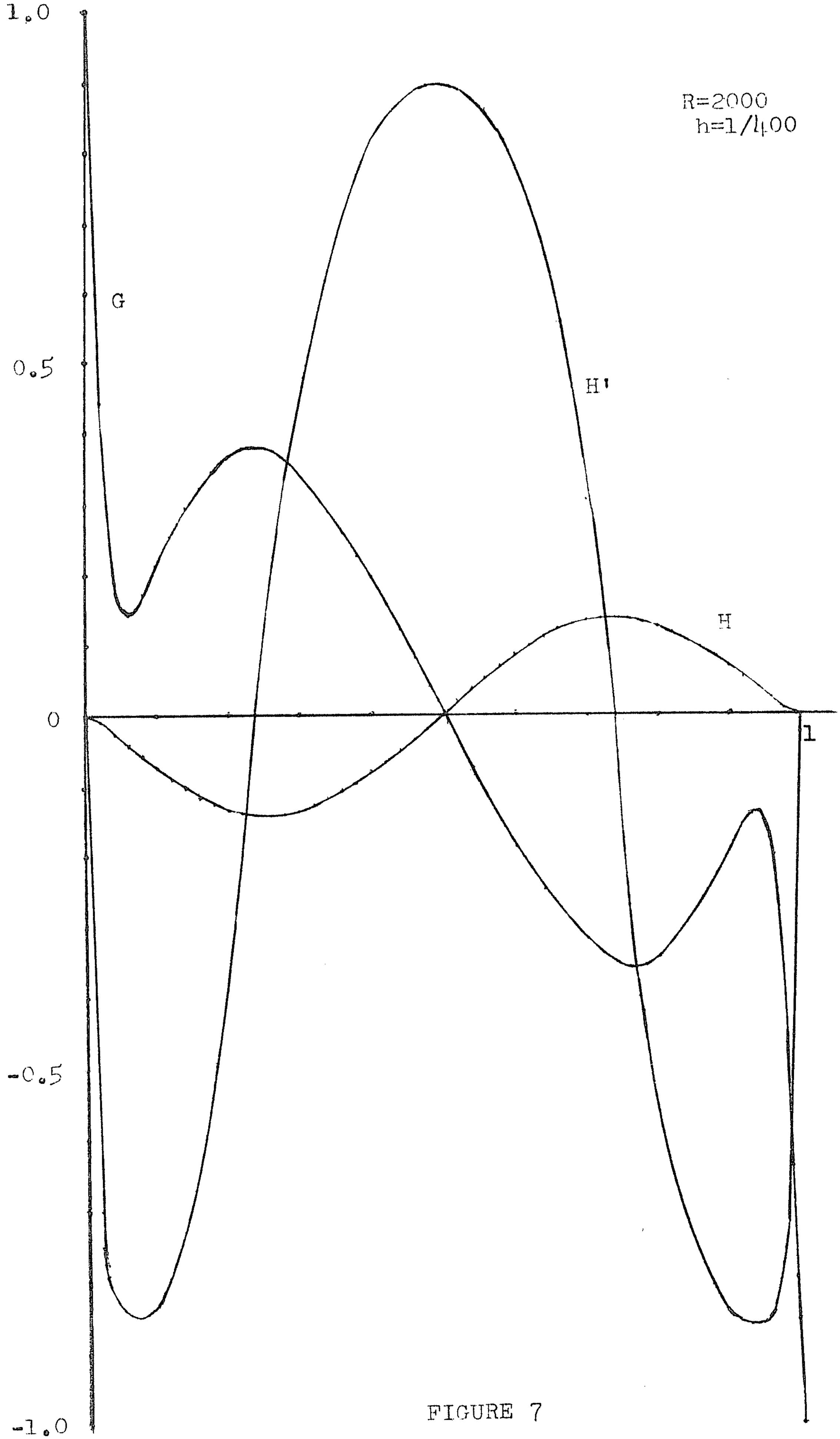

FIGURE 7

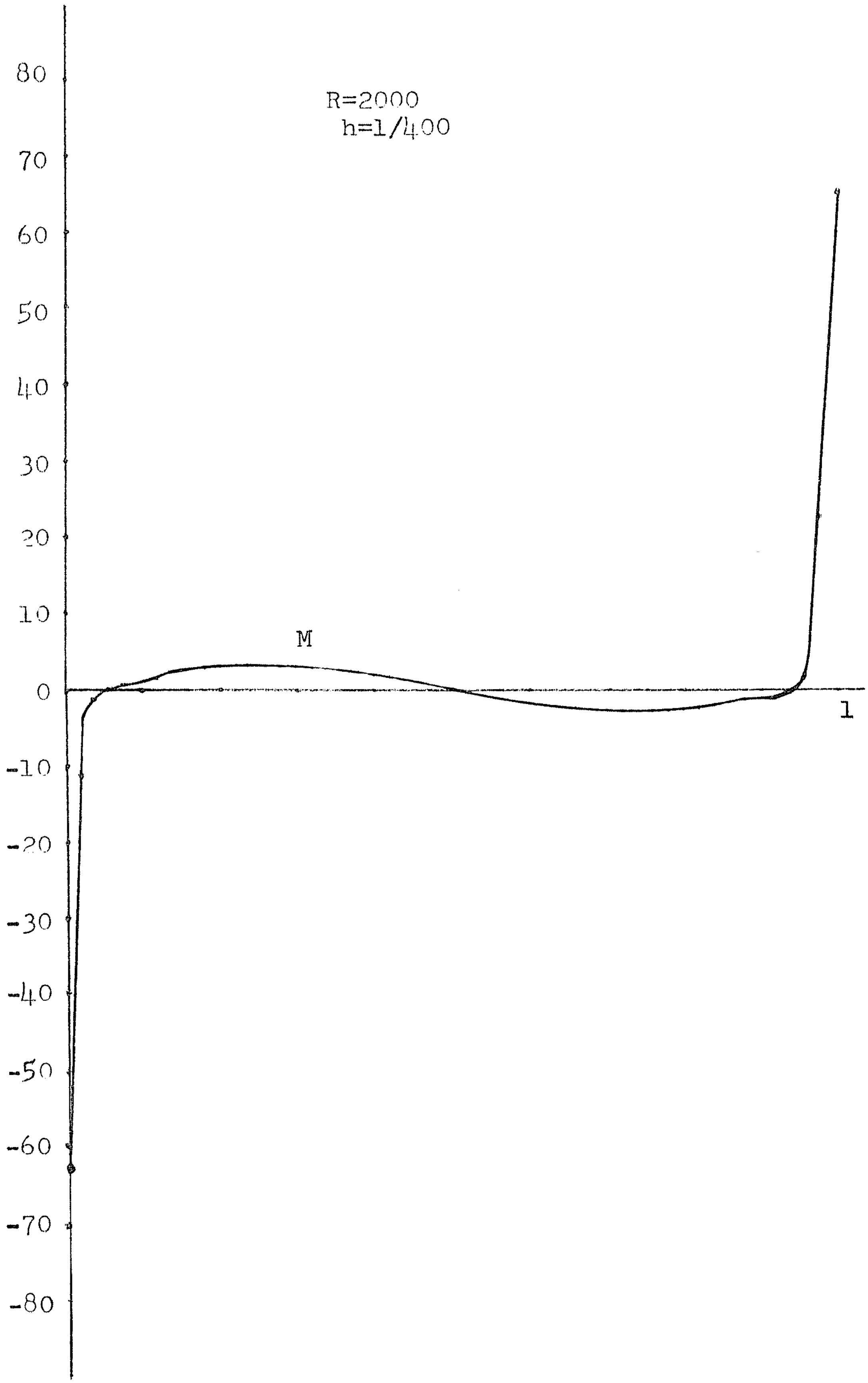

FIGURE 8

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A. Schubert's FORTRAN Program for Flow between Disks (Figure 7 Case)

```
      PARAMETER NMAX=401
      REAL MU,M(NMAX,3),MO(NMAX)
      DIMENSION OMEGA( 2),H(NMAX,3),G(NMAX,3),DXSQM(NMAX),BO(NMAX),
     *  B1(NMAX),B2(NMAX),CG(NMAX),CM(NMAX),DELTA(2),HPR(NMAX)
      DIMENSION HO(NMAX),GO(NMAX)
      COMMON H,G,M
      COMMON/ITOUT/ITOUT/NP1/NP1/NIT/NTH,NTG,NTM/INCPR/INCPR
      COMMON/CON/BO,B1,B2,CG,CM/DXSQ/DXSQ/N/N/NM1/NM1/R/R
      DATA MAXIT,MAXOUT/800,200/,IBUG,INCOUT/1,1/,INCSOR,INCRES/801,200/
      DATA INSW/0/,INCPR/4/
   99 FORMAT(11E5.5)
   98 FORMAT('OSOR ITERATION FOR H FAILED IN OUTER ITERATION' I5,
     *  '. GO TO NEXT PARAMETER CASE.')
   97 FORMAT('OSOR ITERATION FOR G FAILED IN OUTER ITERATION' I5,
     *  '. GO TO NEXT PARAMETER CASE.')
   96 FORMAT('OSOR ITERATION FOR M FAILED IN OUTER ITERATION' I5,
     *  '. GO TO NEXT PARAMETER CASE.')
   95 FORMAT('OOUTER ITERATION FAILED AFTER MAX. NO. OF ITERATIONS. PROC
     *EED TO NEXT PARAMETER CASE.')
   94 FORMAT(1H1 'DX =' F7.5,5X 'OMEGA(1) =' F5.2,5X 'OMEGA(2) =' F5.2,
     * 5X 'REYNOLDS NO. =' F7.0/1HO3X 'SMOOTHING' 5X 'OVER-RELAX.' 5X
     * 'CONVERGENCE'/5X 'FACTOR' 9X 'FACTOR' 9X 'TOLERANCE'
     * 6X 'SOR TOLERANCE'/ 1HO 1X 1HH
     *F6.3,F16.3,2F17.4/2X1HGF6.3,F16.3,2F17.4/2X1HM2F6.3,F10.3,2F17.4)
   93 FORMAT('OCONVERGENCE ATTAINED. TOTAL SOR ITERATIONS FOR H,G,M ='
     * 3I7)
   92 FORMAT('OTOTAL SOR ITERATIONS FOR H,G,M =' 3I7)
   90 FORMAT(1X F9.3,3F8.3,14F7.3)
   89 FORMAT(1H )
   88 FORMAT('OH AT SOR ITERATION' I6)
  188 FORMAT('OG AT SOR ITERATION' I6)
  288 FORMAT('OM AT SOR ITERATION' I6)
   87 FORMAT('OH-PRIME')
   84 FORMAT(13F6.3)
   83 FORMAT('OH,G,M INITIALLY'/)
C     READ INPUT PARAMETERS
      IF(INSW.NE.1) GO TO 4
      READ(5,84) (HO(I),I=1,NMAX)
      READ(5,84) (GO(I),I=1,NMAX)
      READ(5,84) (MO(I),I=1,NMAX)
    4 READ(5,99) ESH,ESG,ESM,FIN2
    5 READ(5,99) DX,OMEGA,R,EH,EG,EM,FIN
    7 READ(5,99) RHO,MU,DELTA,RH,RG,RM,END
      WRITE(6,94) DX,OMEGA,R,RHO,RH,EH,ESH,MU,RG,EG,ESG,DELTA,RM,EM,ESM
      N=1./DX+.005
      NP1=N+1
      NM1=N-1
C     INITIALIZE FUNCTIONS FOR OUTER ITERATION
C
      IF(INSW.EQ.1) GO TO 11
      DO 10 I=1,NP1
      X=FLOAT(I-1)*DX
      G(I,3)=(OMEGA(2)-OMEGA(1))*X+OMEGA(1)
```

```
      IF(X.GT. .25) GO TO 8
      H(I,3)=-1.2*X
      M(I,3)=52.*X-10.
      GO TO 10
    8 IF(X.GT. .75) GO TO 9
      H(I,3)=1.2*X-.6
      M(I,3)=-12.*X+6.
      GO TO 10
    9 H(I,3)=-1.2*X+1.2
      M(I,3)=52.*X-42.
   10 CONTINUE
      GO TO 211
   11 DO 111 I=1,NP1
      H(I,3)=HO(I)
      G(I,3)=GO(I)
  111 M(I,3)=MO(I)
  211 DO 12 J=1,2
      H(1,J)=H(1,3)
      H(NP1,J)=H(NP1,3)
      M(1,J)=M(1,3)
      M(NP1,J)=M(NP1,3)
      G(1,J)=G(1,3)
   12 G(NP1,J)=G(NP1,3)
      WRITE(6,83)
      WRITE(6,90) (H(I,3),I=1,NP1,INCPR)
      WRITE(6,90) (G(I,3),I=1,NP1,INCPR)
      WRITE(6,90) (M(I,3),I=1,NP1,INCPR)
      DXSQ=DX**2
      AH1=1.-RH
      AH2=RH/2.
      RDX=R*DX
      AG1=1.-RG
      AM1=1.-RM
      HSQ2=2./DXSQ
      NTOTH=0
      NTOTG=0
      NTOTM=0
      ITOUT=0
   16 ITOUT=ITOUT+1
      DO 17 I=2,N
      H(I,1)=H(I,3)
      G(I,1)=G(I,3)
   17 M(I,1)=M(I,3)
      M(1,1)=M(1,3)
      M(NP1,1)=M(NP1,3)
C     SOLVE FOR H-BAR BY SOR ITERATION
      NTH=0
      DO 18 I=2,N
   18 DXSQM(I)=DXSQ*M(I,1)
   20 NTH=NTH+1
      DO 22 I=2,N
   22 H(I,2)=H(I,3)
      H(2,3)=AH1*H(2,2)+RH/4.*H(3,2)
      DO 25 I=3,NM1
   25 H(I,3)=AH1*H(I,2)+AH2*(H(I-1,3)+H(I+1,2)-DXSQM(I))
      H(N,3)=AH1*H(N,2)+RH/4.*H(NM1,3)
C     TEST FOR CONVERGENCE
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      DO 27 I=2,N
      IF(.NOT.(ABS(H(I,2)-H(I,3)).LT. ESH)) GO TO 33
   27 CONTINUE
C     CONVERGENCE ATTAINED-- SMOOTH SOLUTION
      DO 30 I=2,N
   30 H(I,3)=RHO*H(I,3)+(1.-RHO)*H(I,1)
      GO TO 34
C     CONVERGENCE UNATTAINED--UPDATE FOR NEXT ITERATION UNLESS ITERATION
C    LIMIT HAS BEEN EXCEEDED.
   33 IF(MOD(NTH,INCSOR).NE.0 ) GO TO 133
      WRITE(6,88) NTH
      WRITE(6,90) (H(I,3),I=1,NP1,INCPR)
  133 IF(NTH.LT.MAXIT) GO TO 20
      WRITE(6,98) ITOUT
      GO TO 62
C     SOLVE FOR G-BAR BY SOR ITERATION
   34 DO 35 I=2,N
      BO(I)=-2.-RDX*ABS(H(I,3))
      B1(I)=(1.+RDX*AMAX1(0.,H(I,3)))
      B2(I)=(1.-RDX*AMIN1(0.,H(I,3)))
   35 CG(I)=-0.5*RDX*G(I,1)*(H(I+1,3)-H(I-1,3))
      NTG=0
   36 NTG=NTG+1
      DO 37 I=2,N
   37 G(I,2)=G(I,3)
      DO 38 I=2,N
   38 G(I,3)=AG1*G(I,2)-RG*(B1(I)*G(I-1,3)+B2(I)*G(I+1,2)-CG(I))/BO(I)
C     TEST FOR CONVERGENCE
      DO 40 I=2,N
      IF(.NOT.(ABS(G(I,2)-G(I,3)).LT.ESG)) GO TO 44
   40 CONTINUE
C     CONVERGENCE ATTAINED--SMOOTH SOLUTION
      DO 42 I=2,N
   42 G(I,3)=MU*G(I,3)+(1.-MU)*G(I,1)
      GO TO 46
C     CONVERGENCE UNATTAINED--UPDATE FOR NEXT SOR ITERATION UNLESS
C     MAX. ITERATION LIMIT HAS BEEN EXCEEDED.
   44 IF(MOD(NTG,INCSOR).NE.0) GO TO 144
      WRITE(6,188) NTG
      WRITE(6,90) (G(I,3),I=1,NP1,INCPR)
  144 IF(NTG.LT.MAXIT) GO TO 36
      WRITE(6,97) ITOUT
      GO TO 62
C     COMPUTE M-BAR AT BOUNDARY POINTS
   46 M(1,3)=DELTA(1)*HSQ2*H(2,3)+(1.-DELTA(1))*M(1,1)
      M(NP1,3)=DELTA(1)*HSQ2*H(N,3)+(1.-DELTA(1))*M(NP1,1)
C     SOLVE FOR M-BAR AT INTERIOR POINTS
      DO 48 I=2,N
   48 CM(I)=2.*RDX*G(I,3)*(G(I+1,3)-G(I-1,3))
      M(NP1,2)=M(NP1,3)
  148 NTM=0
   49 NTM=NTM+1
      DO 50 I=2,N
   50 M(I,2)=M(I,3)
      DO 52 I=2,N
   52 M(I,3)=AM1*M(I,2)-RM*(B1(I)*M(I-1,3)+B2(I)*M(I+1,2)-CM(I))/BO(I)
C     TEST FOR CONVERGENCE
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      DO 54 I=2,N
      IF(.NOT.(ABS(M(I,2)-M(I,3)).LT.ESM)) GO TO 56
   54 CONTINUE
C     CONVERGENCE ATTAINED--SMOOTH SOLUTION
      DO 55 I=2,N
   55 M(I,3)=DELTA(2)*M(I,3)+(1.-DELTA(2))*M(I,1)
      GO TO 58
C     CONVERGENCE UNATTAINED--UPDATE FOR NEXT SOR ITERATION UNLESS
C     ITERATION LIMIT HAS BEEN EXCEEDED.
   56 IF(MOD(NTM,INCSOR).NE.0) GO TO 57
      WRITE(6,288) NTM
      WRITE(6,90) (M(I,3),I=1,NP1,INCPR)
   57 IF(NTM.LT.MAXIT) GO TO 49
      WRITE(6,96) ITOUT
      GO TO 62
   58 IF(MOD(ITOUT,INCOUT).EQ.0) CALL OUTPUT
      IF(MOD(ITOUT,INCRES).EQ.0) CALL TEST
      NTOTH=NTOTH+NTH
      NTOTG=NTOTG+NTG
      NTOTM=NTOTM+NTM
C     TEST FOR CONVERGENCE OF OUTER ITERATION
      DO 59 I=2,N
      IF(.NOT.(ABS(H(I,3)-H(I,1)).LT.EH))GO TO 60
      IF(.NOT.(ABS(G(I,3)-G(I,1)).LT.EG))GO TO 60
      IF(.NOT.(ABS(M(I,3)-M(I,1)).LT.EM))GO TO 60
   59 CONTINUE
      IF(.NOT.(ABS(M(1,3)-M(1,1)).LT.EM))GO TO 60
      IF(.NOT.(ABS(M(NP1,3)-M(NP1,1)).LT.EM))GO TO 60
C     CONVERGENCE ATTAINED--OUTPUT SOLUTION
      IF(MOD(ITOUT,INCOUT).NE.0) CALL OUTPUT
C     COMPUTE H-PRIME
      DO 159 I=2,N
  159 HPR(I)=(H(I+1,3)-H(I-1,3))/(2.*DX)
      WRITE(6,87)
      WRITE(6,90) (HPR(I),I= 1,NP1,INCPR)
      IF(MOD(ITOUT,INCRES).NE.0) CALL TEST
      WRITE(6,93) NTOTH,NTOTG,NTOTM
      GO TO 62
C     CONVERGENCE UNATTAINED--OUTPUT VALUES FOR OUTER ITERATION AND
C     UPDATE FOR NEXT OUTER ITERATION UNLESS OUTER ITERATION LIMIT
C     HAS BEEN EXCEEDED.
   60 IF(ITOUT.LT.MAXOUT) GO TO 16
      WRITE(6,92) NTOTH,NTOTG,NTOTM
      WRITE(6,95)
   62 IF(ABS(END).GT.0.) GO TO 63
      GO TO 7
   63 IF(ABS(FIN).GT.0.) GO TO 64
      GO TO 5
   64 IF(ABS(FIN2).GT.0.) STOP
      GO TO 4
      END
```