Computer Sciences Department
University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

REMARKS ON SINGULAR PERTURBATIONS WITH TURNING POINTS*

by

Heinz Otto Kreiss[1] and
Seymour V. Parter[2]

Technical Report #108

January 1971

FOOTNOTES

(1) University of Uppsala; Uppsala, Sweden.

(2) University of Wisconsin, Madison, Wisconsin.

* This Research was supported by the Atomic Energy Commission (AT (30-1)-1480 with the Courant Institute) and the Office of Naval Research (Contract No. N00014-67-A-D128-0004).

1. INTRODUCTION

In this paper we consider solutions of the boundary value problem

$$\text{1.1)}\quad \varepsilon y''(x) + f(x,\varepsilon) y'(x) + g(x,\varepsilon) y(x) = 0, \qquad -a \le x \le b,$$

$$\text{1.2)}\quad y(-a) = A, \qquad y(b) = B,$$

where $a, b > 0$, $0 < \varepsilon << 1$, $f(x,\varepsilon)$ has a single simple zero in $[-a,b]$. Without loss of generality we assume $f(0,\varepsilon) = 0$ (hereafter referred to as the turning point). Many authors (Wasow [11], Cochran [2], Sibuya [10], O'Malley [7]) have studied asymptotic solutions of (1.1) as $\varepsilon \to 0+$. However, the recent work of Pearson [9] and Ackerberg and O'Malley [1] is of particular interest to us and motivated the present study.

We restrict our attention to the case where f and g are Lipschitz continuous and

$$\text{1.3)}\quad \begin{cases} f(x,\varepsilon) > 0, & -a \le x < 0 \\ f(x,\varepsilon) < 0, & 0 < x \le b \end{cases}$$

and (uniformly)

$$\text{1.4)}\quad f'(0,\varepsilon) \le -\alpha < 0, \qquad 0 \le \varepsilon \le \varepsilon_0$$

for some $\alpha > 0$ and some $\varepsilon_0 > 0$.

In the case where $f(x,\varepsilon)$, $g(x,\varepsilon)$ are analytic in (x,ε), Pearson [9] and Ackerberg and O'Malley [1] proved the following basic result: Let

$$1.5) \qquad -g(0,0)/f'(0,0) \equiv \ell .$$

Suppose $\ell \neq 0,1,2,\ldots$ and $\{y(x,\varepsilon_n)\}_{n=1}^{\infty}$ is a sequence of solutions of (1.1), (1.2) which converges (pointwise, as $\varepsilon_n \to 0+$) to a function $Y(t)$ for $t \in (-a,0) \cup (0,b)$. Then

$$1.6) \qquad Y(t) \equiv 0, \quad t \in (-a,0) \cup (0,b).$$

However, when ℓ is a nonnegative integer the situation is "cloudy". Pearson [9] seems to have ignored these cases. Ackerberg and O'Malley [1] applied the WBKJ method (which is also the basic tool of [9]) and seem to have an analysis for these cases. However, they do not make a precise statement of their hypothesis or results. And, as we shall see, their results are incomplete.

In section 2 of this report we establish certain basic estimates on the regularity of solutions $y(t,\varepsilon)$ of (1.1) and (1.2). These estimates enable us to treat the problem <u>without assuming the analyticity of $f(x,\varepsilon)$, $g(x,\varepsilon)$</u>.

In section 3 we discuss some results and examples in the exceptional case when ℓ is a non-negative integer.

As we shall see, the occurance of non-zero limit functions (called "resonance" in [1]) is an interesting and delicate phenomenon.

The basic estimates of this work are obtained by using maximum principle estimates as in [3]. However, similar results can be obtained via L^2 estimates. Indeed, related work by H. B. Keller and H. O. Kreiss [5] displays the power of the L^2 theory.

2. REGULARITY

Let $y(x,\varepsilon)$ be the solution of (1.1), (1.2). Suppose $f(x,\varepsilon)$, $g(x,\varepsilon) \in C^k[-a,b]$ as functions of x, uniformly in ε. That is, there is a constant $L > 0$ such that

$$\text{2.1)} \quad \begin{cases} \left| \left(\frac{\partial}{\partial x}\right)^j f(x,\varepsilon) \right| \le L, & 0 \le j \le k, \quad 0 \le \varepsilon \le \varepsilon_0 \\ \left| \left(\frac{\partial}{\partial x}\right)^j g(x,\varepsilon) \right| \le L, & 0 \le j \le k, \quad 0 \le \varepsilon \le \varepsilon_0 \end{cases}$$

Let

$$\text{2.2)} \qquad v_j(x,\varepsilon) \equiv \left(\frac{\partial}{\partial x}\right)^j \cdot y(x,\varepsilon), \qquad 0 \le j \le k$$

Then, a simple induction shows that

$$\text{2.3a)} \qquad \varepsilon v_j''(x) + f(x,\varepsilon) v_j'(x) + \{g(x,\varepsilon) + j f'(x,\varepsilon)\} v_j = \sum_{s=0}^{j-1} A_{js} v_s(x,\varepsilon)$$

where

$$\text{2.3b)} \qquad A_{js} = - \{ \tbinom{j}{s-1} f^{(j+1-s)}(x,\varepsilon) + \tbinom{j}{s} g^{(j-s)}(x,\varepsilon) \}.$$

We now recall some basic estimates. For any $\psi(x) \in C[\alpha,\beta]$, let

$$\text{2.4)} \qquad \|\psi\|_{\alpha,\beta} \equiv \max \{ |\psi(x)| ; \quad \alpha \le x \le \beta \}.$$

<u>Lemma 2.1</u> Let $\varphi(x) \in C^N[\alpha,\beta]$. Let $t > 0$ be given. There exist constants $C_j(t) < \infty$ $(j = 1,2,\ldots,N-1)$ such that

2.5) $$\|\varphi^{(j)}\|_{\alpha,\beta} \leq t\,\|\varphi^{(N)}\|_{\alpha,\beta} + C_j(t)\|\varphi^0\|_{\alpha,\beta}\,, \quad j = 1,2,\ldots,N-1.$$

Proof: We will carry out the proof in the case $N = 2$. The general cases follows from an elementary induction. Let x_0 be the point at which $|\varphi'(x)|$ assumes its maximum. Let $2t < (\beta-\alpha)/2$. Then there is a point $x_1 \in [\alpha,\beta]$ such that

$$|x_1 - x_0| = 2t\ .$$

Then

$$\varphi'(x_0) = \frac{\varphi(x_1) - \varphi(x_0)}{x_1 - x_0} - \frac{(x_1 - x_0)}{2}\varphi''(\xi).$$

Thus

$$\|\varphi'\|_{\alpha,\beta} = |\varphi'(x_0)| \leq \frac{\|\varphi\|}{t}_{\alpha,\beta} + t\,\|\varphi''\|_{\alpha,\beta}\ .$$

Lemma 2.2. Let $\varepsilon > 0$ and let $\varphi(x,\varepsilon)$ satisfy

2.6) $$\varepsilon\varphi'' + A(x)\varphi' = \gamma(x), \qquad \alpha \leq x \leq \beta\ ,$$

where

2.7) $$0 < A_0 \leq A(x), \qquad \alpha \leq x \leq \beta\ .$$

and

2.8) $$|\gamma(x)| \leq M_0, \qquad \alpha \leq x \leq \beta\ .$$

Then, there exists a constant M, depending only on A, $\beta-\alpha$, but not on ε, such that

2.9) $$|\varphi'(x,\varepsilon)| \leq \frac{M}{x-\alpha} \{|\varphi(\alpha,\varepsilon)|+|\varphi(\beta,\varepsilon)|+M_0\}, \quad \alpha<x\leq\beta.$$

Similiarly, if (2.8) holds and (2.7) is replaced by

$$A(x) \leq -A_0 < 0, \quad \alpha \leq x \leq \beta ,$$

then

$$|\varphi'(x,\varepsilon)| \leq \frac{M}{x+\beta} \{|\varphi(\alpha,\varepsilon)|+|\varphi(\beta,\varepsilon)|+M_0\}, \alpha \leq x < \beta.$$

Proof: See [3, theorem 2.7].

Lemma 2.3: Let ℓ be defined by (1.5). Then, in the neighborhood of the origin (say $-\Delta \leq x \leq \Delta$) the solution of the reduced equation

2.10) $$f(x,0)\, u'(x) + g(x,0) = 0$$

can be written in the form

2.11) $$\begin{cases} \lambda_1 x^{\ell} \exp \{\int_0^x \psi(t)dt\}, & -\Lambda \leq x < 0, \\ \lambda_2 x^{\ell} \exp \{\int_0^x \psi(t)dt\}, & 0 < x \leq \Lambda . \end{cases}$$

Moreover, suppose k is a natural number (i.e. a non-negative integer) with $k > \ell$ and $u(x) \in C^k(-\Delta,\Delta)$. Then $\lambda_1 = \lambda_2$ and (i) if ℓ is not a natural number, then

2.12) $$\lambda_1 = \lambda_2 = 0, \text{ and } u(x) \equiv 0;$$

(ii) if ℓ is a natural number and $u(x) \not\equiv 0$ in $[-\Delta,\Delta]$ then $u(x) \neq 0$ for $x \neq 0$ and

2.13) $$\left(\frac{d}{dx}\right)^{\ell} u(x)\Big|_{x=0} \neq 0 .$$

Proof: We rewrite (2.10) in the form

2.10') $$[xf'(0,0) + x^2\tilde{f}(x)]\,u'(x) + [g(0,0) + x\tilde{g}(x)]\,u(x) = 0.$$

Set

$$u(x) = x^{\ell}\,\omega(x).$$

Then (2.10') takes the form

$$[f'(0,0) + x^2\,\tilde{f}(x)]\omega'(x) + [\tilde{g}(x) + \tilde{f}(x)]\omega = 0.$$

Thus, (2.11) follows with

$$\psi(x) = -[\tilde{g}(x) + \tilde{f}(x)]/[f'(0,0) + x^2\tilde{f}(x)].$$

The remaining parts of the lemma follow from the representation (2.11).

For the remainder of this section we shall always assume

H.1) $f(x,\varepsilon)$ and $g(x,\varepsilon) \in C^{k+1}[-a,b]$ as a function of ψ uniformly in ε with $k > \ell$.

Lemma 2.4: Let $0 < \delta < a$. There exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and constants, $M_j(\delta)$, j 0,1,2,...k+2 such that: for all ε, $0 < \varepsilon \le \varepsilon_0$ the boundary value problem

2.14) $$\begin{cases} \varepsilon\,\omega'' + f(x,\varepsilon)\omega' + g(x,\varepsilon)\omega = 0, & -a \le x \le -\delta/2 \\ \omega(-a,\varepsilon) = \tilde{A}, \quad \omega(-\delta/2,\ \varepsilon) = 0 & \end{cases}$$

has a unique solution. Moreover

2.15) $$0 < |\omega(x,\varepsilon)| \le \tilde{A}, \qquad -a \le x \le -\delta/2$$

and

2.16) $$\left\|\frac{d^j\omega}{dx^j}\right\|_{-a+\delta,-\delta/2} \leq M_j(\delta)[\varepsilon|\widetilde{A}|\,].$$

Furthermore, the corresponding inequalities hold for the interval $\delta/2 \leq x \leq b$.

Proof: Let

$$f(x,\varepsilon) \geq 4p > 0, \qquad -a \leq x \leq -\delta/2,$$

and let

$$\varepsilon_0 = p^2/\|g\|_{-a,-\delta/2}\,.$$

Let $0 < \varepsilon \leq \varepsilon_0$ and suppose $\omega(x,\varepsilon)$ satisfies (2.14). Let

$$u(x,\varepsilon) = e^{\frac{p(x+a)}{\varepsilon}}\,\omega(x,\varepsilon).$$

Then $u(x,\varepsilon)$ satisfies

$$\begin{cases} \varepsilon u'' + (f-2p)u' - [\dfrac{f(x,\varepsilon)p - p^2}{\varepsilon} - g(x,\varepsilon)]u = 0, \\ u(-a,\varepsilon) = \widetilde{A}, \quad u(-\delta/2,\varepsilon) = 0. \end{cases}$$

However, the maximum principle shows that $u(x,\varepsilon)$ is unique. Thus $\omega(x,\varepsilon)$ is unique and (2.15) holds. Moreover, we may apply lemma 2.2 to obtain (2.16) with $j = 1$. The complete lemma follows from repeated differentiation and application of lemma 2.2.

Lemma 2.5: Let $0 < \delta < a$. There exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that for all ε, $0 < \varepsilon \leq \varepsilon_0$ and all $F(x) \in C[-a,-\delta/2]$ the boundary value problem

$$\begin{cases} \varepsilon\omega'' + f(x,\varepsilon)\omega' + g(x,\varepsilon)\omega = F(x), & -a \leq x \leq -\delta/2 \\ \omega(-a,\varepsilon) = \omega(-\delta/2,\varepsilon) = 0 \end{cases}$$

has a unique solution. Moreover, there is a constant K such that

$$2.17) \qquad \|\omega(\cdot,\varepsilon)\|_{-a,-\delta/2} \leq K\|F\|_{-a,-\delta/2}\,.$$

Proof: The unicity of the solution $\omega(x,\varepsilon)$ follows from lemma 2.4. Suppose (2.17) is false. Then there is a sequence ε_n, $F_n(x) \in C[0,1]$, $n = 1,2,\ldots$ with

$$\|F_n\|_{-a,-\delta/2} = 1, \quad \varepsilon_n \to 0+$$

and the corresponding solutions $\omega_n(x,\varepsilon_n)$ satisfy

$$\|\omega_n(\cdot,\varepsilon_n)\|_{-a,-\delta/2} \to \infty\,.$$

Let $x_n \in (0,1)$ be a point at which

$$\omega_n'(x_n,\varepsilon_n) = 0, \quad |\omega_n(x_n,\varepsilon_n)| = \|\omega_n(\cdot,\varepsilon_n)\|_{-a,-\delta/2}.$$

Let

$$Z_n(x,\varepsilon_n) = \omega_n(x,\varepsilon_n)/\|\omega(\cdot,\varepsilon_n)\|_{-a,-\delta/2}\,.$$

The functions $Z_n(x,\varepsilon_n)$ are now uniformly bounded. Thus using lemma 2.3 and the arguments of [3] (or directly using [3, theorem 4.1]), we see that for every ρ, $0 < \rho < a - \delta/2$,

2.18) $$\|Z_n(\cdot, \varepsilon_n)\|_{-a+\rho,-\delta/2} + \|Z_n'(\cdot, \varepsilon_n)\|_{-a+\rho,-\delta/2} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, for n sufficiently large

$$-a < x_n < -a + \rho .$$

Since $Z_n'(x_n, \varepsilon_n) = 0$ we may apply theorem 2.8 of [3] in the interval $[x_n, -\delta/2]$. Thus, there is a constant C such that

$$|Z_n'(x, \varepsilon_n)| \le C, \quad x_n \le x \le -\delta/2.$$

And, if $|x_n - x| \le \frac{1}{2C}$, we see that

$$|Z_n(x, \varepsilon_n)| \ge 1/2.$$

Let $\rho = 1/2\,C$. Then

$$|Z_n(-a + \rho, \varepsilon_n)| \ge 1/2$$

which contradicts (2.18).

<u>Corollary</u>: Let $0 < \varepsilon \le \varepsilon_0$. Let $\omega(x, \varepsilon)$ satisfy

$$\varepsilon\omega'' + f(x, \varepsilon)\omega' + g(x, \varepsilon)\omega = 0 \qquad -a \le x \le -\delta/2$$

$$\omega(-a, \varepsilon) = \widetilde{A}, \quad \omega(-\delta/2, \varepsilon) = \widetilde{B}.$$

Then there is a constant K_0 such that

2.19) $$\left\|\frac{d^j\omega}{dx^j}\right\|_{-a+\delta,-\delta/2} \le K_0[|\tilde{B}| + \varepsilon|\tilde{A}|], \quad j = 0,1,2,\ldots,k+1$$

Proof: In view of lemma 2.4 and the linearity of the equation we may just consider the case where $\tilde{A} = 0$. The corollary now follows from repeated differentiation and the application of lemma 2.2 and lemma 2.5.

Lemma 2.6: Let $0 < \delta < \min(a,b)$. There exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and a constant $K_0(\delta)$ such that for $0 < \varepsilon \le \varepsilon_0$ the solutions of (1.1), (1.2) belong to $C^{k+1}[-a,b]$ and satisfy

2.20) $$\|y\|_{-a,b} \le K_0[\|y\|_{-\delta,\delta} + |A| + |B|]$$

2.21) $$\left\|\frac{d^j}{dx^j} y\right\|_{-a+\delta,b-\delta} \le K_0[\|y\|_{-\delta,\delta} + \varepsilon(|A| + |B|)], \; j = 0,1,2,\ldots k+1.$$

Proof: Consider the equation 2.3a for $j = k+1$. Without loss of generality we may assume that ε and δ are so small that

$$g(x,\varepsilon) + k\, f'(x,\varepsilon) < 0, \qquad -\delta \le x \le \delta.$$

Thus, the maximum principle implies

$$\left\|\frac{d^{k+1}y}{dx^{k+1}}\right\|_{-\delta,\delta} \le C_1\left[\left|\frac{d^{k+1}}{dx^{k+1}} y(\delta,\varepsilon)\right| + \left|\frac{d^{k+1}}{dx^{k+1}} y(-\delta,\varepsilon)\right| + \sum_{j=0}^{k} \left\|\frac{d^j y}{dx^j}\right\|_{-\delta,\delta}\right].$$

The inequalities (2.20); (2.21) now follow from (2.17), (2.19) (and the corresponding results for $\delta/2 \le x \le b$) and lemma 2.1.

Lemma 2.7: Let $\varepsilon_n \to 0+$. Assume there is a constant $K_1 > 0$ such that we can find a solution $\omega_1(x,\varepsilon_n)$ of (1.1) with

2.22) $$\omega_1(-a,\varepsilon_n) = 1, \quad \|\omega_1'\|_{-a,b} \le K_1 \ .$$

Then $\ell(= -g(0,0)/f'(0,0))$ is a natural number and there is a unique solution $\hat{u}$ of the reduced equation (2.10) with

2.23) $$\lim_{n\to\infty} \|\omega_1(\cdot,\varepsilon_n) - \hat{u}\|_{-a,b} = 0$$

and

2.24) $$\hat{u}(-a) = 1, \quad \hat{u}(b) \neq 0 \ .$$

Proof: The functions $\omega_1(x,\varepsilon_n)$ form a compact set. Therefore we can find a convergent subsequence which converges to solution $\hat{u}$ of (2.10) for which $\hat{u}(-a) = 1$. Furthermore, by lemma 2.6 and (2.22) we see that

$$\|(\frac{d}{dx})^{k+1} \omega_1(\cdot,\varepsilon_n)\|_{-a/2,b/2} \le C_2$$

for some constant C_2. Thus $\hat{u} \in C^k[-a/2,b/2]$. Therefore, lemma 2.3 implies that ℓ is a natural number (since $\hat{u}(x) \not\equiv 0$) and (2.24) holds. Thus $\hat{u}$ is uniquely determined and the entire sequence $\omega_1(x,\varepsilon_n)$ converges to $\hat{u}(x)$.

Lemma 2.8: Assume that the conditions of lemma 2.7 are fulfilled and that

2.25) $$I = \int_{-a}^{b} f(x,0)dx > 0.$$

Then there is a corresponding sequence $\{\omega_2(x,\varepsilon_n)\}$ of solutions of (1.1) (with $\varepsilon = \varepsilon_n$) for which

$$2.26) \qquad \lim_{n\to\infty} \|\omega_2(x,\varepsilon_n) - \frac{1}{f(-a,0)} \exp\{-\frac{1}{\varepsilon} f(-a,0)(x+a)\}\|_{-a,b} = 0 .$$

Proof: Let $\omega_1(x,\varepsilon_n)$ be the solution of (1.1) as described in lemma 2.7. All other solutions $\omega_2(x,\varepsilon_n)$ of (1.1) are solutions of the first order equation

$$2.27) \qquad \omega_2'\omega_1 - \omega_2\omega_1' = \frac{\lambda}{\varepsilon} \exp\{-1/\varepsilon \int_{-a}^{x} f(x,\varepsilon)ds\} .$$

Let η be a constant which satisfies $0 < \eta < \min(\frac{a}{2}, \frac{b}{2})$ and choose $\lambda = 1$. Then it follows from lemma 2.7 and (2.27) that there is a solution $\omega_2(x,\varepsilon_n)$ which satisfies (2.26) on the restricted interval $-a \le x \le -\eta$. Furthermore

$$2.28) \qquad \lim_{n\to\infty} \|\frac{d^j}{dx^j}\omega_2(\cdot,\varepsilon_n)\|_{-a/2,-\eta} = 0, \quad j = 0,1,2,\dots,\ell+2.$$

Therefore lemma 2.6 and Taylor expansions give

$$2.29) \quad \begin{cases} \overline{\lim_{n\to\infty}} \|\omega_2(\cdot,\varepsilon_n)\|_{-\eta,\eta} & \le (2\eta)^{\ell+1} \overline{\lim_{n\to\infty}} \|\frac{d^{\ell+1}}{dx^{\ell+1}} \omega_2(\cdot,\varepsilon_n)\|_{-\eta,\eta} \\ & \le (2\eta)^{\ell+1} \overline{\lim_{n\to\infty}} \|\frac{d^{\ell+1}}{dx^{\ell+1}} \omega_2(\cdot,\varepsilon_n)\|_{-a/2,b/2} \\ & \le K_0(\Delta)\cdot(2\eta)^{\ell+1} \overline{\lim} \|\omega_2(\cdot,\varepsilon_n)\|_{-a/2,b} \end{cases}$$

where $\Delta = \min(a/2,b/2)$ and $K_0(\Delta)$ is independent of η.

For $x \ge \eta$ we may write this solution $\omega_2(x,\varepsilon_n)$ in the form

$$\omega_2(x,\varepsilon_n) = \omega_H(x,\varepsilon_n) + \omega_p(x,\varepsilon_n)$$

where

$$\omega_H(x,\varepsilon_n) = \frac{\omega_2(\eta,\varepsilon_n)}{\omega_1(\eta,\varepsilon_n)} \omega_1(x,\varepsilon_n)$$

and $\omega_p(x,\varepsilon_n)$ is the solution of (2.27) with $\lambda = 1$ and $\omega_p(\eta,\varepsilon) = 0$. By lemma 2.7 the function $\omega_1(x,\varepsilon_n)$ converges to the solution $\hat{u} \not\equiv 0$ of the reduced equation and therefore lemma 2.3 implies that there is a constant $C_3 > 0$ such that for all sufficiently small ε_n:

$$|\omega_1(\eta,\varepsilon_n)| \geq C_3\, \eta^{\ell} \ .$$

Furthermore, for every fixed $\eta > 0$

$$\overline{\lim_{n\to\infty}}\ \|\omega_p(\cdot,\varepsilon_n)\|_{\eta,b} = 0 \ .$$

Thus (2.28) and (2.29) imply

$$\overline{\lim_{n\to\infty}}\ \|\omega_2(\cdot,\varepsilon_n)\|_{-a/2,b} \leq \overline{\lim_{n\to\infty}}\ \|\omega_2(\cdot,\varepsilon_n)\|_{-\eta,\eta} + \overline{\lim_{n\to\infty}}\ \|\omega_2(\cdot,\varepsilon_n)\|_{\eta,b}$$

$$\leq K_0(2\eta)^{\ell+1}\ \overline{\lim_{n\to\infty}}\ \|\omega_2(\cdot,\varepsilon_n)\|_{-a,b} + \|\hat{u}\|_{-a,b} \cdot \overline{\lim_{n\to\infty}}\ \left\| \frac{\omega_2(\eta,\varepsilon_n)}{C_3\,\eta^{\ell}} \right\|_{-\eta,\eta}$$

$$\leq K_1(\eta)\ \overline{\lim}\ \|\omega_2(\cdot,\varepsilon_n)\|_{-a/2,b}$$

where

$$K_1(\eta) = 2K_0\,[(2\eta)^{\ell} + 2^{\ell} C_3^{-1}\ \|\hat{u}\|_{-a,b}] \cdot \eta \ .$$

Choosing η so small that $K_1(\eta) < 1/2$ we see that

$$\overline{\lim}\ \|\omega_2(\cdot,\varepsilon_n)\|_{-a/2,b} = 0$$

and the lemma is proven.

In exactly the same way we obtain our next result.

Lemma 2.9: Assume that the conditions of lemma 2.7 are fulfilled and that

2.30) $$I = \int_{-a}^{b} f(x,0)dx < 0.$$

Then there is a corresponding sequence of solutions $\{\omega_2(x,\varepsilon_n)\}$ of (1.1) (with $\varepsilon = \varepsilon_n$) for which

2.31) $$\text{Lim} \left\|\omega_2(\cdot,\varepsilon_n) - \frac{1}{f(b,0)\hat{u}(b)} \exp\{+\frac{1}{\varepsilon} f(b,0)(b-x)\}\right\|_{-a,b} = 0 .$$

Finally, if

2.32) $$I = \int_{-a}^{b} f(x,b)dx = 0$$

there is a corresponding sequence of solutions $\{\omega_2(x,\varepsilon_n)\}$ of (1.1) (with $\varepsilon = \varepsilon_n$) for which

2.33) $$\lim_{n\to\infty} \left\|\omega_2(\cdot,\varepsilon_n) - \frac{1}{f(-a,0)} \exp\{-\frac{1}{\varepsilon} f(-a,0)(x+a)\} - \frac{1}{f(b,0)\hat{u}(b)} \cdot \exp\{\frac{1}{\varepsilon} f(b,0)(b-x)\}\right\|_{-a,b} = 0 .$$

A consequence of the last two lemmas is

Theorem 2.1: Assume that the conditions of lemma 2.7 are fulfilled. Let

$$I = \int_{a}^{b} f(x,0)dx .$$

Then, there exists an $\bar{\varepsilon} > 0$ such that,

(i) For all A and B and all $\varepsilon_n \leq \bar{\varepsilon}$ (ε_n in the given sequence) equations

(1.1) and (1.2) have unique solution $y(x, \varepsilon_n)$,

(ii) there is a solution $u(x)$ of the reduced equation (2.10) such that

$$\lim_{n\to\infty} \| y(x,\varepsilon_n)-u \|_{-a+\delta, b-\delta} = 0,$$

(iii) if $I > 0$ then $u(b) = B$ and there is no boundary layer near $x = b$,

(iv) if $I < 0$ then $u(a) = A$ and there is no boundary layer near $x = a$,

(v) if $I = 0$ then $u(x) = \lambda \hat{u}(x)$ where

$$\lambda = \frac{A f(-a,0) - B f(b,0) \hat{u}(b)}{f(-a,0) - f(b,0) |\hat{u}(b)|^2} .$$

Proof: The general solution of (1.1) can be written in the form

$$y(x,\varepsilon_n) = \lambda \omega_1(x,\varepsilon_n) + \alpha \omega_2(x,\varepsilon_n)$$

where $\omega_2(x,\varepsilon_n)$ satisfies one of the inequalities (2.26), (2.31) or (2.33) and $\omega_1(x,\varepsilon_n) \to \hat{u}(x)$. The theorem follows without difficulties.

The results of this theorem should be compared to the claims of Acherberg and O'Malley [1]. These results are consistent with their results in cases (iii) and (iv) and yield contradictory results in case (v). However, in a private communication R. E. O'Malley has indicated the same result in case (v) when the WKBJ can be applied.

In general, it is not easy to verify the assumptions of lemma 2.7. Indeed, in general we do not know that all solutions of (1.1), (1.2) are bounded for ε sufficiently small.

Definition 2.1: A sequence $\varepsilon_n \to 0+$ will be said to satisfy Condition ZB relative to the interval $[-a,b]$ if: for any sequence of uniformly bounded solutions $y(x,\varepsilon_n)$ of (1.1) we have

$$2.34) \qquad \lim_{n\to\infty} \|y(\cdot,\varepsilon_n)\|_{-a+\delta,b-\delta} = 0 .$$

Definition 2.2: A sequence $\varepsilon_n \to 0+$ will be said to satisfy Condition Z relative to the interval $[-a,b]$ if: There exists an $\bar{\varepsilon} > 0$ such that for all choices of A and B and all $\varepsilon_n < \bar{\varepsilon}$ there exists a unique solution $y(x,\varepsilon_n;A,B)$ of (1.1) and (1.2) and there is a constant $C_4 = C_4(A,B) > 0$ such that

$$2.35) \qquad |y(x,\varepsilon_n;A,B)| \le C_4,$$

$$2.36) \qquad \lim_{n\to\infty} \|y(\cdot,\varepsilon_n;A,B)\|_{-a+\delta,b-\delta} = 0.$$

Remark: The results of lemmas 2.7 - 2.9 and theorem 2.1 as well as these two definitions have been phrased in terms of sequences of solutions rather than all solutions because of our limited knowledge of this rather delicate situation. For example, it seems possible (although we have no examples of such behavior) that there is a sequence $\varepsilon_n \to 0+$ such that the assumptions of lemma 2.7 are satisfied and another sequence $\varepsilon_{n'} \to 0+$ which satisfies condition Z relative to the interval $[-a,b]$.

Lemma 2.10: If the sequence $\varepsilon_n \to 0+$ satisfies Condition ZB relative to the interval $[-a,b]$ then it also satisfies Condition ZB relative to every larger interval.

Proof: Apply lemma 2.6.

Theorem 2.2: If $\ell \neq 0,1,2,\ldots$ then all sequences $\varepsilon_n \to 0+$ satisfy condition ZB relative to all intervals $[-a,b]$ with $0 < a$, $0 < b$.

Proof: Assume that (2.34) does not hold. Then, in view of lemma 2.6 we may assume that there is a subsequence $\varepsilon_{n'} \to 0+$ such that $y(x,\varepsilon_{n'}) \to u(x) \neq 0$, $-\Delta \leq x \leq \Delta$, where $u(x) \in C^k[-\Delta,\Delta]$ is a nontrivial solution of the reduced equation. However, this contradicts lemma 2.3.

Theorem 2.3: Suppose $\varepsilon_n \to 0$ is a sequence which satisfies condition ZB relative to the interval $[-a,b]$. Then the sequence ε_n also satisfies Condition Z relative to the interval $[-a,b]$.

Proof: If we show that all solutions of (1.1), (1.2) (with $\varepsilon = \varepsilon_n \leq \bar{\varepsilon}$) are bounded we will have established the uniqueness of solutions of (1.1), (1.2) and hence the existence of solutions of (1.1), (1.2). Thus, the theorem will be proven.

Suppose there is a subsequence $\varepsilon_{n'} \to 0+$ such that the associated solutions of (1.1), (1.2) $y(x,\varepsilon_{n'})$ are unbounded. Let

2.37) $$Z(x,\varepsilon_{n'}) = y(x,\varepsilon_{n'})/\|y(\cdot,\varepsilon_{n'})\|_{-a,b} .$$

Using lemma 2.6 we may extract a subsequence $\{Z(x,\varepsilon_{n''})\}$ which converges uniformly to a function $u(x) \in C^k[-a/2,b/2]$ which is a solution of the reduced equation (2.10). Using lemma (2.6) we see that in fact, the sequence $\{Z(x,\varepsilon_{n''})\}$ satisfies the hypothesis of lemma (2.11). However, the sequence $\{Z(x,\varepsilon_{n''})\}$ also satisfies condition ZB. Thus $u(x) \quad 0$. Now consider $Z(x, \varepsilon_{n''})$ on the intervals $[-a, -a/2]$, $[b/2, b]$. When n'' is large enough, $|Z(x, \varepsilon_{n''})|$ assumes its maximum one of these intervals. Thus

2.38) $$\|Z(\cdot, \varepsilon_{n''})\|_{-a,-a/2} + \|Z(\cdot, \varepsilon_{n''})\|_{b/2,b} \geq 1$$

while

2.39) $$|Z(-a, \varepsilon_{n''})| + |Z(-a/2, \varepsilon_{n}'')| + |Z(b/2, \varepsilon_{n''})| + |Z(b, \varepsilon_{n}'')| \to 0.$$

However, (2.38) and (2.39) are impossible in view of lemma 2.5.

Corollary: If $\ell \neq 0, 1, 2, \ldots$ then there exists an $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leq \varepsilon_0$ there exists a unique solution of (1.1) and (1.2). Moreover, there is a constant $C_5 > 0$ such that

$$\|y(x, \varepsilon)\|_{-a,b} \leq C_5 .$$

Proof: Apply theorem 2.2.

Theorem 2.4: Suppose $\{y(x, \varepsilon_n)\}$ is a sequence of solutions of (1.1), (1.2) which is unbounded on $[-a, b]$. Let $0 < \delta$, $0 < \delta'$. Then

(i) On every strictly smaller interval $[-a+\delta, b-\delta']$ there is a subsequence $\varepsilon_{n'} \to 0+$ and a sequence of solutions $\omega_1(x, \varepsilon_{n'})$ of (1.1) which satisfy the hypothesis of lemma 2.7; and,

(ii) On every strictly larger interval $[-a-\delta, b+\delta']$ there is a subsequence $\varepsilon_{n'} \to 0+$ which satisfies condition Z relative to the interval $[-a-\delta, b+\delta']$.

Proof: Let $Z(x,\varepsilon_n) = y(x,\varepsilon_n)/\|y(\cdot,\varepsilon_n)\|_{-a,b}$. Using lemma 2.6 we may extract a subsequence $\varepsilon_{n'}$ and a solution $u(x)$ of the reduced equation (2.10) such that

$$\lim_{n'\to\infty} \|Z(\cdot,\varepsilon_{n'}) - u\|_{-a+\delta, b-a} = 0.$$

The argument of Theorem 2.3 shows that

$$u(-a+\delta) \neq 0.$$

The functions

$$\omega_1(x,\varepsilon_{n'}) = \frac{Z(x,\varepsilon_{n'})}{Z(-a+\delta,\varepsilon_{n'})}$$

satisfy the assumptions of lemma 2.7 on $[-a+\delta, b-\delta]$.

On the other hand, consider any larger interval $[-a-\delta, b+\delta']$. Suppose there exists a sequence of solutions $\{\omega(x,\varepsilon_n)\}$ of (1.1) which also satisfy $y(-a-\delta,\varepsilon_n) = A_0$, $y(b+\delta,\varepsilon_n) = B_0$. If this family is unbounded on $[-a-\delta, b+\delta']$ part (i) shows that the solutions of (1.1), (1.2) on the interval $[-a,b]$ are bounded. Thus, we may assume these functions are bounded. If any subsequence $\{\omega(x,\varepsilon_{n'})\}$ were to converge to a non-zero limit solution that sequence (using lemma 2.6) would also lead to functions which satisfy the assumptions of lemma 2.7 and the original sequence $\{y(x,\varepsilon_n)\}$ is bounded on $[-a,b]$.

3. EXAMPLES

In [1] Ackerberg and O'Malley and in [8] O'Malley observed that there is a whole class of equations of the type (1.1) for which one always obtains nontrivial limit functions. Interestingly enough these are the "simplest" equations of the type (1.1). These are our first examples.

Example 1: Consider the boundary-value problem

$$\varepsilon y'' - xy' + ny = 0, \qquad -a \leq x \leq b \tag{3.1}$$

$$y(-a) = A, \qquad y(b) = B \tag{3.2}$$

where n is a natural number.

In this case the exact solution is given in terms of parabolic cylinder functions, see [1], [8]. A complete discussion of this case is given in [8]. For all $\varepsilon > 0$, $a > 0$, $b > 0$ and all n there exists a solution of the type described by (2.22) and discussed in lemma 2.7. Thus, theorem 2.1 applies.

Example 2: Suppose $\ell = -g(0,0)/f'(0,0) = 1$ and there is a family of linear functions

$$\varphi(x, \varepsilon) = \alpha(\varepsilon)x + \beta(\varepsilon) \tag{3.3}$$

which are solutions of (1.1). Suppose the coefficients $\alpha(\varepsilon)$, $\beta(\varepsilon)$

continuous in ε so that

$$\varphi(x,0) = \alpha(0)x + \beta(0)$$

will be a solution of the reduced equation. Then, of course, the functions $\varphi(x,\varepsilon)$ can be normalized to give a family of functions $\omega_1(x,\varepsilon)$ satisfying the hypothesis of lemma 2.7. Thus, theorem 2.1 applies.

As specific examples of this case consider the equations

$$\varepsilon y'' - xy' + \frac{x}{1+x} y = 0, \quad -\frac{1}{2} \le x \le \frac{1}{2},$$

$$\varepsilon y'' - x(1+x)y' + xy = 0, \quad -\frac{1}{2} \le x \le \frac{1}{2}.$$

In both of these cases

$$y(x,\varepsilon) = \sigma(1+x)$$

is a solution of both the second order equation and the reduced equation.

Example 3: Let $g(x) \equiv 0$. Then the constant function $W_1(x,\varepsilon) \equiv 1$ satisfies the hypothesis of lemma 2.7 and thus theorem 3.1 applies. It is of some interest to note that in this case one can actually "solve" the problem completely by elementary methods.

Example 4: Let $g_0(x) \in C^4[-a,\infty]$ and satisfy

$$\text{3.4)} \qquad \begin{cases} g_0(x) > 0, & -\infty < x < \alpha \\ g_0(x) = 0, & \alpha \le x \le \beta \\ g_0(x) > 0, & \beta < x < \infty \end{cases}$$

for some values α, β with $-\infty < \alpha < 0 < \beta < \infty$ and

3.5) $$\int_{\alpha}^{\beta} f(t)dt = 0.$$

We consider the equation

3.6) $$\varepsilon y'' + f(x)y' + g_0(x)\, y = 0.$$

Our discussion of this example depends upon the following elementary result.

Lemma 3.1: Let $y(x)$ be a solution of the initial value problem

3.7) $$\begin{cases} \varepsilon y'' + p_0(x)y' + p_1(x)y = 0, \quad x > x_0, \\ y(x_0) > 0, \quad y'(x_0) \leq 0 \end{cases}$$

where $p_0(x) < 0$, $p_1(x) \geq \bar{p} > 0$. Then the first zero of $y(x)$, say x_1 satisfies

$$x_0 < x_1 \leq x_0 + \sqrt{\varepsilon/\bar{p}}\ \frac{\pi}{2}$$

Proof: Since $y''(x_0) < 0$, as long as $y(x)$ is nonnegative the function $y(x)$ continues to decrease as x increases. Moreover, $y'(x)$ decreases as long as $y(\alpha)$ is nonnegative. We see that

$$y'' = -\frac{1}{\varepsilon}\,[p_0(x)y' + p_1(x)y] \leq -\frac{1}{\varepsilon}\,\bar{p}\, y.$$

Thus, as long as $y(x)$ is nonnegative and $x > x_0$ we have

$$0 \leq y(x) \leq y(x_0) \cos [\sqrt{\bar{p}/\varepsilon}\, (x-x_0)].$$

Remark: If $p_0(x) > 0$, $y(x_0) > 0$, $y'(x_0) \geq 0$, then the first zero, x_{-1}, behind x_0 satisfies

$$x_0 - \sqrt{\varepsilon/\bar{p}} \cdot \frac{\pi}{2} \leq x_{-1} \leq x_0 \, .$$

Theorem 3.1: Let

3.8) $$-a < \alpha < 0 < b \leq \beta.$$

There exists an $\varepsilon_0 > 0$ such that, for all ε, $0 < \varepsilon \leq \varepsilon_0$ there exists a solution of equation (3.6) which satisfies the conditions of lemma 2.7. Thus, theorem 2.1 applies.

Proof: We shall construct the functions $\omega_1(x, \varepsilon)$. Let $\varphi(x)$ be the solution of the reduced equation (2.10) which satisfies $\varphi(-a) = 1$. Using lemma 2.4 and lemma 2.5, we know that there is an $\bar{\varepsilon} > 0$ such that; for all ε, $0 < \varepsilon \leq \bar{\varepsilon}$ there exists a unique bounded solution $W(x, \varepsilon)$ of the boundary value problem

3.9) $$\begin{cases} \varepsilon W'' + f(x)\, W' + g_0(x)\, W = 0, & -a \leq x \leq \alpha \\ W(-a) = 1, \quad W(\alpha) = \varphi(\alpha). \end{cases}$$

Moreover, from the general theory (see [3]) it follows that

3.10) $$\lim_{\varepsilon\to 0} \|W - \varphi\|_{-a,\alpha} = 0.$$

Indeed, we also have

$$\lim_{\varepsilon\to 0} \|W' - \varphi'\|_{-a+\delta,\alpha} = 0.$$

In fact, we claim that there is an $\varepsilon_0 > 0$ such that

3.11) $$W'(\alpha, \varepsilon) < 0, \quad 0 < \varepsilon \leq \varepsilon_0$$

suppose this is false, using the remark of lemma 3.1, we see that there is a sequence $\varepsilon_n \to 0+$ such that each function $W(x, \varepsilon_n)$ has a zero in the interval $[-a, \alpha]$. Since $\varphi(x)$ has no zeros this contradicts (3.10).

Now let us extend $W(x, \varepsilon)$ into the region $\alpha \leq x \leq b \leq \beta$. Let

$$u(x) = \int_\alpha^x f(t)dt.$$

Then

3.7) $$W'(x, \varepsilon) = W'(\alpha,\varepsilon)\, e^{-\frac{1}{\varepsilon} u(x)},$$

3.8) $$W(x, \varepsilon) = \varphi(\alpha) + W'(\alpha,\varepsilon) \int_\alpha^x e^{-\frac{1}{\varepsilon} u(t)}\, dt.$$

Since $u(t) \geq 0$ in $\alpha \leq x \leq \beta$ and

$$W'(\alpha, \varepsilon) \to \varphi'(\alpha) = 0$$

we see that the assumptions of lemma 2.7 are indeed satisfied.

Theorem 3.2: Let

3.9) $$-a < \alpha < 0 < \beta < b.$$

Then all sequences $\varepsilon_n \to 0+$ satisfy Condition Z relative to the interval $[-a, b]$.

Proof: In view of theorem 2.3 we need only verify that $\{\varepsilon_n\}$ satisfies Condition ZB relative to the interval $[-a, b]$. Suppose not. Then there is a family of solutions $\{y(x, \varepsilon_n)\}$ of (1.1) which is uniformly bounded and does not converge to zero. Using lemma 2.6 we may extract a subsequence $\varepsilon_{n'} \to 0+$ such that

$$\lim_{n' \to \infty} \| y(\cdot, \varepsilon_{n'}) - Y \|_{-a+\delta, b-\delta} = 0$$

where $Y(x)$ is a nontrivial solution of the reduced equation and

$$-a + \delta < \alpha < 0 < \beta < b - \delta < b.$$

We may normalize both $Y(x)$ and $y(x, \varepsilon_n')$ so that

$$y(\alpha, \varepsilon_{n'}) = Y(\alpha) = \varphi(\alpha).$$

As in theorem 3.1 we claim that there is an $\bar{\varepsilon}$ such that

$$y'(\alpha, \varepsilon_{n'}) < 0, \qquad 0 < \varepsilon_{n'} \le \bar{\varepsilon}.$$

Moreover,

$$y'(\alpha, \varepsilon_{n'}) \rightarrow 0 \quad \text{as} \quad n' \rightarrow \infty.$$

Thus, we may consider the function $y(x, \varepsilon_n)$ in the interval $[\alpha, \beta]$. As in (3.7) we see that

$$y'(\beta, \varepsilon_{n'}) < 0.$$

Moreover, as in the proof of lemma 3.1, $y'(x, \varepsilon_n)$ decreases to the right of β as long as $y(x, \varepsilon_{n'}) \geq 0$. Let $\Delta > 0$ and let

$$\bar{p}(\Delta) = \min \{g_0(x), \ \beta + \Delta \leq x \leq b].$$

Then, applying lemma 3.1 we see that the function $y(x, \varepsilon_{n'})$ has a zero x_1 which satisfies

$$\beta < x_1 < \Lambda + \frac{\pi}{2} \sqrt{\varepsilon'_n / \bar{p}(\Delta)}$$

Thus,

$$Y(\beta) = 0$$

and

$$Y(x) \equiv 0.$$

Example 5: Suppose $g(x,\varepsilon) \in C[-a,b]$ uniformly in ε and $g(x,\varepsilon) \leq 0$. Suppose ther exist two points $x-, x+$ with

$$-a \leq x- < 0 < x+ \leq b$$

such that

$$g(x-,0) \quad g(x+,0) \neq 0.$$

Then, applying the maximum principle, we see that

(i) for each $\varepsilon > 0$ there is a unique solution $y(x,\varepsilon)$ of (1.1), (1.2). Moreover

$$|y(x,\varepsilon)| \leq \max(|A|, |B|).$$

Moreover, using the argument of [3, Thm. 3.6] we have

(ii) $$\lim_{\varepsilon \to 0+} \|y(\cdot,\varepsilon)\|_{[-a+\Delta, b-\Delta]} = 0. \quad \forall \Delta > 0.$$

Remark: This example indicates that an analysis which is based only on the behavior of $y(x,\varepsilon)$ "near" the turning point may not be adequate.

Example 6: Suppose

3.10a) $$g(x,\varepsilon) = x^2 b(x)$$

where

3.10b) $$b(x) \geq b_0 > 0.$$

Let

3.11a) $$f(x,\varepsilon) = -x\, a(x)$$

where

3.11b) $$a(x) \geq a_0 > 0$$

Then all sequences $\varepsilon_n \to 0+$ satisfy condition Z relative to every interval $[-a,b]$.

Proof: The solution of the reduced equation (2.10) is given by

$$Y(x) = Y(0) \exp \int_0^x \frac{t\, b(t)}{a(t)}\, dt\,, \tag{3.12}$$

If $Y(0) > 0$ then $Y(x)$ has a relative minimum at $x = 0$. And, if $Y(0) < 0$ then $Y(x)$ has a relative maximum at $x = 0$. Suppose $\{y(x,\varepsilon_n)\}_{n=0}^{\infty}$ is a sequence of solutions of (1.1), (1.2) such that

$$\lim_{\varepsilon_n \to 0} \| y(\cdot\,,\delta_n) - Y \|_{[-\Delta,\Delta]} = 0\,.$$

Suppose $Y(0) > 0$. Then for ε_n small enough; $y(x,\varepsilon_n) > 0$ and, in the interval $[-\Delta,\Delta]$, $y(x,\varepsilon_n)$ has an interior relative minimum. But

$$\varepsilon_n y''(x,\varepsilon_n) + f(x,\varepsilon_n)\, y'(x,\varepsilon_n) = -x^2\, b(x)\, y(x,\varepsilon_n) \le 0.$$

Applying the maximum principle, we see that $y(x,\varepsilon_n)$ cannot have an interior relative minimum. This contradiction shows that

$$Y(0) \le 0\,.$$

A similiar argument shows that

$$Y(0) \ge 0$$

and hence, using (3.12),

$$Y(x) \equiv 0.$$

Example 7: Suppose $f(x,\varepsilon) = f(x,0) = f(x) \in C^2[-a,b]$. Suppose

3.13) $$f'(x) \le -1, \qquad f'(0) = -1.$$

Suppose there exist two point $x-$, $x+$ with

3.14) $$-a < x- < 0 < x+ < b$$

with

$$f'(x-) < -1, \quad f'(x+) < -1.$$

Consider the equation

$$\begin{cases} \varepsilon y'' + f(x)y' + y = 0, & -a \le x \le b \\ y(-a) = A, & y(-b) = B \end{cases}$$

In this case

$$\ell = 1.$$

Suppose $\{y(x,\varepsilon_n)\}$ is a sequence of solutions. Suppose there is a function $Y(x) \in C(-a,b)$ such that

$$\lim_{\varepsilon_n \to 0+} \| y(\cdot,\varepsilon_n) - Y \|_{[-a+\Delta, b-\Delta]} = 0.$$

We claim that

$$Y(x) \quad 0.$$

<u>Proof:</u> As before, let $v_1(x,\varepsilon) = y'(x,\varepsilon)$. Then

$$\varepsilon v_1'' + f(x)\, v_1' + [1 + f'(x)]\, v_1 = 0 \; .$$

Since $v_1(x,\varepsilon_n)$ be bounded in any interval $[-a+\Delta, b-\Delta]$, we may apply the results of example 5 in this interval.

<u>Example 8:</u> Suppose

(a) $f'(0,\varepsilon) = -1$

(b) $g(x,\varepsilon) = g(x)$ is independent of ε

(c) $g(0) = \ell = 0$

(d) $g(x) = g(-x)$

(e) there is a $\Delta > 0$ such that $g(x) \in C^{\infty}[-\Delta, \Delta]$ and $f(x,\varepsilon) \in C^{\infty}[-\Delta,\Delta]$ as a function of x.

Then all sequences $\varepsilon_n \to 0$ satisfy condition Z relative to the interval $[-a,b]$ unless

3.15) $\quad g^{(k)}(0) = 0, \quad k = 1,2,\ldots$

3.16) $\quad v_{2j}(0,\varepsilon) = 0, \quad j = 1,2,\ldots$

Of course, if $g(x)$ is analytic in $[-a,b]$ then we require $g(x) \equiv 0$.

Proof: Let $\{y(x,\varepsilon_n)\}_{n=1}^{\infty}$ be a sequence of solutions of (1.1) which are uniformly bounded and converge to a function $Y(x) \not\equiv 0$.

Let

$$V_j = V_j(\varepsilon) = v_j(0,\varepsilon).$$

Then $Y(x) \not\equiv 0$ implies

(3.17) $\quad V_0(\varepsilon_n) \not\to 0.$

Consider (2.3a), (2.3b) with $j = 0$. Then $\forall\, \varepsilon$

$$\varepsilon V_2(\varepsilon) = 0 .$$

Thus, $V_2(\varepsilon) = 0$. Consider (2.3a), (2.3b) with $j = 2$. Then

$$\varepsilon V_4 - 2 V_2 = - g''(0) V_0.$$

Thus, (3.17) implies

(3.18) $\quad g''(0) = 0, \quad V_4(\varepsilon) = 0.$

We proceed by induction. Suppose

(3.19) $\quad g^{(2s)}(0) = 0, \quad V_{2s+2}(\varepsilon) = 0, \quad s = 1,2,\ldots,j .$

Then

$$\varepsilon V_{2j+4} - (2j+2) V_{2j+2} = - \binom{2j+2}{0} g^{(2j+2)}(0)\, V_0.$$

Or

$$\varepsilon V_{2j+4} = - g^{(2j+2)}(0)\, V_0.$$

Once more, (3.17) implies

$$g^{(2j+2)}(0) = 0, \quad V_{2j+4}(\varepsilon) = 0, \quad \forall j.$$

This completes the discussion of this example.

Before going on to other examples, we prove a basic estimate.

Lemma 3.2. Let $f'(0,\varepsilon) = -1$. Let

$$V_j = V_j(\varepsilon) = v_j(0,\varepsilon).$$

Let

3.20) $\quad \ell = n = g(0,\varepsilon).$

Let $\{y(x,\varepsilon_n)\}_{n=1}^{\infty}$ be sequence of solutions of (1.1) which are uniformly bounded and converge to a function $Y(x)$. If

3.21) $\quad 0 \le j \le n + 1 - 2k, \qquad k \ge 1,$

then

3.22) $\quad V_j(\varepsilon_s) = O(\varepsilon_s^k)$

Proof: We proceed by induction on k and j. Take $k = 1$. Suppose

$$0 \le n - 1.$$

Then

$$V_0 = -\frac{\varepsilon_s}{n} V_2 = O(\varepsilon_s).$$

Suppose that

$$V_r = O(\varepsilon_s), \quad 0 \le r \le j \le n - 2$$

Consider (2.3a), (2.3b) with $j = j + 1$. We obtain

$$\varepsilon V_{j+3}(\varepsilon_s) + [n-(j+1)] V_{j+1}(\varepsilon_s) = O(\varepsilon_s).$$

Hence

$$V_{j+1} = O(\varepsilon_s).$$

Thus, the lemma is established for $k = 1$.

Suppose the lemma has been established for k and

$$2 \leq n - 2k + 1.$$

Hence, by the inductive hypothesis

$$V_2(\varepsilon_s) = O(\varepsilon_s^k) .$$

And,

$$V_0(\varepsilon_s) = -\frac{\varepsilon_s}{n} V_2(\varepsilon_s) = O(\varepsilon_s^{k+1}) .$$

Now assume that the lemma has been established for $(k + 1)$ and all j_0 which satisfy

$$0 \leq j_0 \leq n - 2(k+1)$$

or

$$(j_0 + 1) + 2 \leq n - 2k + 1 .$$

Then

$$[n - (j_0 + 1)] V_{j_0+1} = - \varepsilon_s V_{j_0+3} + O(\varepsilon_s^{k+1})$$

$$- \varepsilon_s O(\varepsilon_s^k) + O(\varepsilon_s^{k+1})$$

and the lemma is proven.

Example 9: Consider the equation

3.23) $\varepsilon y'' - xy' + g(x)\, y = 0, \qquad -a \le x \le b.$

Under the hypothesis of lemma 3.2, if

3.24) $Y(x) \not\equiv 0$,

then

3.25) $-[g'(0)]^2 = (n + 1/2)\, g''(0).$

In particular, if

$$g(x) = n + \beta x, \qquad \beta \neq 0,$$

then (3.24) is false.

Proof: Using lemma 3.2 and (2.3a), (2.3b) with $j = n - 2$ we obtain

3.26) $V_{n-2}(\varepsilon_s) = -\frac{\varepsilon_s}{2} V_n(\varepsilon_s) + O(\varepsilon_s^2).$

Letting $j = n - 1$ and using (3.46) we obtain

3.27) $V_{n-1}(\varepsilon_s) = -\varepsilon_s [V_{n+1}(\varepsilon_s) - \frac{1}{2}\binom{n-1}{n-2} g'(0)\, V_n(\varepsilon_s)] + O(\varepsilon_s^2).$

Letting $j = n + 1$ we obtain

$$\varepsilon_s V_{n+3} - V_{n+1} = -g'(0) \binom{n+1}{n} V_n + O(\varepsilon_s)$$

or

$$3.28) \qquad V_{n+1}(\varepsilon_s) = g'(0)\,(n+1)\,V_n(\varepsilon_s) + O(\varepsilon_s).$$

Combining (3.27) and (3.28) we obtain

$$3.29) \qquad V_{n-1}(\varepsilon_s) = \varepsilon_s g'(0)\,[\tfrac{1}{2}(n-1) - (n+1)]\,V_n(\varepsilon_s) + O(\varepsilon_s^2).$$

Letting $j = n$ we have

$$\varepsilon V_{n+2}(\varepsilon_s) = -\,n\,g'(0)\,V_{n-1}(\varepsilon_s) - \frac{n(n-1)}{2}\,g''(0)\,V_{n-2} + O(\varepsilon_s^2).$$

Using (3.26) and (3.29) we obtain

$$3.30) \qquad V_{n+2}(\varepsilon_s) = \{[(n+1)n - \tfrac{1}{2}\,n(n-1)][g'(0)]^2 + \tfrac{1}{4}\,n(n-1)\,g''(0)\}\,V_n(\varepsilon_s) + O(\varepsilon_s).$$

On the other hand, letting $j = n + 2$ we obtain

$$\varepsilon V_{n+4}(\varepsilon_s) - 2V_{n+2}(\varepsilon_s) = -\,(n+2)g'(0)\,V_{n+1}(\varepsilon_s) - \tfrac{1}{2}(n+2)(n+1)g''(0)V_n(\varepsilon_s) + O(\varepsilon_s).$$

Using (3.28) this gives

$$3.31) \qquad V_{n+2}(\varepsilon_s) = \tfrac{1}{2}\,\{(n+2)(n+1)[g'(0)]^2 + \tfrac{1}{2}(n+2)(n+1)g''(0)\}V_n(\varepsilon_s) + O(\varepsilon_s).$$

However, (3.24) together with the basic representation of $Y(x)$ (See lemma 2.3) implies

$$V_n(\varepsilon_s) \not\to 0.$$

Comparing (3.30) and (3.31) we obtain (3.25).

Lemma 3.3: Consider the equation (3.23) under the additional hypotheses

(a) there is a $\Delta > 0$ such that

$$g(x) \in C^{\infty}[-\Delta, \Delta],$$

(b) $g(0) = n,$

(c) $g(x) = g(-x).$

Let $\{y(x, \varepsilon_s)\}_{s=1}^{\infty}$ be a sequence of uniformly bounded solutions of (3.23) which converge to a function $Y(x)$. Then

$$(3.32)\qquad V_{n-2j}(\varepsilon_s) = \frac{(-\varepsilon_s)^j}{2^j (j!)} V_n(\varepsilon_s) + O(\varepsilon_s^{j+1}), \quad n-2j \geq 0.$$

Proof: Using (2.3a), (2.3b) and lemma 3.2 we proceed by induction. Suppose $n-2 \geq 0$. Then

$$\varepsilon_s V_n(\varepsilon_s) + 2V_{n-2}(\varepsilon_s) = -\sum_{r=0}^{n-3} \binom{n-2}{r} g^{(n-2-r)}(0) V_r(\varepsilon_s) = O(\varepsilon_s^2).$$

Hence

$$V_{n-2}(\varepsilon_s) = -\frac{\varepsilon_s}{2} V_n(\varepsilon_s) + O(\varepsilon_s^2).$$

Suppose the lemma has been established for $j = j_0$ and $n - 2j_0 - 2 \geq 0$. Then

$$\varepsilon_s V_{n-2j_0}(\varepsilon_s) + [2j_0+2] V_{n-2j_0-2}(\varepsilon_s) = O(\varepsilon_s^{j_0+2}).$$

That is

$$V_{n-2(j_0+1)} = \frac{(-\varepsilon_s) V_{n-2j_0}}{2[j_0+1]} + O(\varepsilon_s^{(j_0+1)+1})$$

using the inductive hypothesis we obtain

$$V_{n-2(j_0+1)}(\varepsilon_s) = \frac{(-\varepsilon_s)^{j_0+1} V_n(\varepsilon_s)}{2^{j_0+1}[(j_0+1)!]} + O(\varepsilon_s^{j_0+1+1})$$

and the lemma is proven.

Lemma 3.4: Under the same hypothesis as in lemma 3.3, suppose

3.33) $$g^{(2j)}(0) = 0, \quad j = 0,1,\dots (k-1).$$

Then, for $1 \leq j \leq k$ we have

3.34) $$V_{n+2j}(\varepsilon_s) = \frac{g^{(2k)}(0)(-1)^{k+1}\varepsilon_s^{k-j}[(j-1)!]V_n(\varepsilon_s)}{2^{k+1-j}(k!)} \times \sum_{s=0}^{j-1} (-1)^s \binom{n+2s}{n-2k+2s}\binom{k}{k-s} + O(\varepsilon_s^{k+1-j}) .$$

Proof: Once more, we proceed by induction. Let $j = n$ in (2.3a), (2.3b). Using lemma 3.2 we obtain.

$$\varepsilon_s V_{n+2}(s) = -\binom{n}{n-2k} g^{(2k)}(0) V_{n-2k}(\varepsilon_s) + O(\varepsilon_s^{k+1}).$$

Using Lemma 3.3 we obtain

$$V_{n+2}(\varepsilon_s) = \frac{g^{(2k)}(0)(-1)^{k+1} \varepsilon_s^{k-1}[(1-1)!]}{2^k k!} V_n(\varepsilon_s)\binom{n}{n-2k}\binom{k}{k} + O(\varepsilon_s^k).$$

That is, the lemma is true for $j = 1$. Suppose the lemma is true for $j = 1, 2, \ldots (j_0 - 1)$ and $j_0 \le k$. Then using (2.3a) and (2.3b) with $j = n + 2(j_0 - 1)$ we obtain

3.35) $$\varepsilon_s V_{n+2j_0} = 2(j_0-1)V_{n+2(j_0-1)} - \binom{n+2(j_0-1)}{n-2k+2(j_0-1)} g^{(2k)}(0)\, V_{n-2k+2(j_0-1)}(\varepsilon_s) + O(\varepsilon_s^{k-j_0+2}).$$

Using the inductive assumption in (3.35) we obtain

$$\varepsilon_s V_{n+2j_0} = \frac{2(j_0-1)(-1)^{k+1} \varepsilon_s^{k-(j_0-1)} [(j_0-2)!]\, g^{(2k)}(0)\, V_n}{2^{k+1-(j_0-1)}(k!)} \times$$

$$\sum_{r=0}^{j_0-2} (-1)^r \binom{n+2r}{n-2k+2r}\binom{k}{k-r} - \binom{n+2(j_0-1)}{n-2k+2(j_0-1)} g^{(2k)}(0) V_{n-2k+2(j_0-1)}$$

$$+ O(\varepsilon_s^{k-j_0+2}).$$

Applying lemma 3.3 we have

3.36) $$\left\{ \begin{aligned} \varepsilon_s V_{n+2j_0} &= \frac{g^{(2k)}(0)(-1)^{k+1} \varepsilon_s^{k-j_0+1} [(j_0-1)!] V_n}{2^{k+1-j_0}(k!)} \sum_{r=0}^{j_0-2} (-1)^r \binom{n+2r}{n-2k+2r}\binom{k}{k-r} \\ &+ \left[(-1)^{j_0-1} \binom{n+2(j_0-1)}{n-2k+2(j_0-1)}\binom{k}{k-(j_0-1)}\right] \frac{g^{(2k)}(0)(-1)^{k+1} \varepsilon_s^{k-j_0+1} [(j_0-1)!] V_n}{2^{k+1-j_0}(k!)} \\ &+ O(\varepsilon_s^{k-j_0+2}). \end{aligned} \right.$$

Combining the terms on the right hand side of (3.36) and dividing by ε_s (since $k \geq j_0$) we obtain (3.34).

Corollary:

3.37) $$V_{n+2k}(\varepsilon_s) = \frac{g^{2k}(0)(-1)^{k+1}[(k-1)!]V_n(\varepsilon_s)}{2\,(k!)} \sum_{r=0}^{k-1} (-1)^r \binom{n+2r}{n-2k+2r}\binom{k}{k-r} + O(\varepsilon_s).$$

Lemma 3.5: Under the hypothesis of lemma 3.3, assume that

$$g^{(2j)}(0) = 0, \quad j = 1, 2, \ldots (k-1).$$

Then

$$V_{n+2k}(\varepsilon_s) = \frac{1}{2k}\binom{n+2k}{n} g^{(2k)}(0)\, V_n(\varepsilon_s) + O(\varepsilon_s).$$

Proof: Apply (3.2a), (3.2b) with $j = n+2k$.

Lemma 3.6 Let

3.38) $$J = \sum_{r=0}^{k} (-1)^r \binom{n+2r}{n-2k+2r}\binom{k}{k-r}.$$

Then $J > 0$.

Proof: Let ∇ denote the backward difference operator with step size 1 and let $\nabla(2)$ denote the backward difference operator with step size 2. Let k be fixed and $r-2k \geq 0$. Let

$$\varphi(r) \equiv \binom{r}{r-2k} = \frac{r!}{(2k)!\,(r-2k)!}$$

then

$$J = \nabla^k(2)\,\varphi(r)\Big|_{r=n+2k}\,.$$

As is well known, (see [6, page 6])

$$\nabla^k\,\varphi(r)\Big|_{r\geq n+k} > 0.$$

Thus, the lemma follows from the identity

$$\nabla^k(2)A(r)\Big|_{r=r_0} = \sum_{m=0}^{k}\binom{k}{m}\nabla^k A(r)\Big|_{r_0-s}\,.$$

This identity is easily established by induction.

<u>Example 10:</u> Under the hypothesis of lemma 3.3 consider the equation (3.23). Then all sequences $\varepsilon_n \to 0$ satisfy condition Z relative to the interval $[-a,b]$ unless

3.39) $$g^{(k)}(0) = 0, \qquad k = 1,2,\ldots$$

3.40) $$V_{n+2j}(\varepsilon_s) = 0, \qquad j = 1,2,\ldots$$

Of course, if $g(x)$ is analytic for $x \in [-\Delta,\Delta]$, then we require

3.41) $$g(x) = n, \qquad -\Delta \leq x \leq \Delta.$$

Proof: Let $\{y(x,\varepsilon_n)\}$ be a uniformly bounded sequence of solutions of (1.1) which converge on $[-a+\delta, b-\delta]$ to a nontrivial solution $Y(x)$ of the reduced equation 2.10. As we know

3.42) $$V_n(\varepsilon_s) \not\to 0.$$

From example 9 we know that

$$g^{(2)}(0) = 0.$$

Assume

$$g^{(2j)}(0) = 0, \quad j = 1,2,\ldots(k-1).$$

Combining lemma 3.4 with lemma 3.5 we obtain

$$\frac{g^{(2k)}(0)}{2k} V_n(\varepsilon_s) \left[\sum_{r=0}^{k-1} (-1)^r \binom{n+2r}{n+2r-2k}\binom{k}{k-r} + (-1)^k \binom{n+2k}{n}\binom{k}{0} \right] = 0.$$

Applying (3.42) and lemma 3.6 we see that

$$g^{(2k)}(0) = 0.$$

Thus we have established (3.40). Returning to (2.3a), (2.3b) with $j = n$ we obtain

$$\varepsilon V_{n+2}(\varepsilon_s) = 0, \qquad \forall \varepsilon .$$

Suppose

3.43) $$V_{n+2j}(\varepsilon) = 0, \quad j = 1, 2, \ldots j_0.$$

Then, applying (2.3a), (2.3b) we have

$$\varepsilon V_{n+2(j_0+1)} - 2j_0 V_{n+2j_0} = 0.$$

Or, using (3.43) we obtain (3.41).

Example 11: Consider the equation

3.44) $$\varepsilon y'' + f(x) y' + ny = 0, \quad -a \leq x \leq b$$

where

(a) $f'(0) = -1, \qquad f(0) = 0,$

(b) $f(x) = -f(-x)$

(c) there is a $\Delta > 0$ such that $f(x) \in C^{(\infty)}[-\Delta, \Delta]$.

Then all sequences $\varepsilon_n \to 0+$ satisfy condition Z relative to the interval $[-a, b]$ unless

3.45) $$f^{(j)}(0) = 0, \quad j \geq 2.$$

Of course, if $f(x)$ is analytic on $[-\Delta, \Delta]$, then

$$f(x) = -x, \quad -\Delta \leq x \leq \Delta.$$

Proof: The proof follows the same lines as the discussion of example 11 and is omitted.

As a special case we obtain the following result. Consider the boundary value problem

$$\begin{cases} \varepsilon y'' - x(1 + x^2)\, y' + 2y = 0, & -a \leq x \leq b \\ y(-a) = A, \quad y(b) = B \end{cases}$$

Then all sequences $\varepsilon_n \to 0$ satisfy condition Z relative to all intervals $[-a,b]$ with $a > 0$, $b > 0$.

This example is discussed by Ackerberg and O'Malley [1] who asserted the existence of nontrivial limit functions.

REFERENCES

1. R. C. Ackerberg and R. E. O'Malley, Jr., "Boundary layer problems exhibiting resonance." Studies in Appl. Math. 49, 277-295 (1970).

2. J. A. Cochran, "Problems in singular perturbation theory", Stanford University Ph.D. Thesis, 1962.

3. Fred. W. Dorr, S. V. Parter and L. F. Shampine, "Applications of the maximum principle to singular perturbation problems." To appear - see Univ. of Wisc. Computer Sciences Department Report #101. (1970).

4. F. W. Dorr and S. V. Parter: "Singular perturbations of non-linear boundary value problems with turning points," J. Math. Anal. Appl. 29, 273-293 (1970).

5. H. B. Keller and H. O. Kreiss; to appear.

6. K. S. Miller: An Introduction to the Calculus of Finite Differences and Difference Equations, Henry Holt & Company, New York, (1960).

7. R. E. O'Malley, Jr., "Topices in singular perturbations", Advances in Mathematics, 2, 365-470, 1968.

8. R. E. O'Malley, Jr., "On boundary value problems for a singularly perturbed differential equation with a turning point", SIAM J. Math. Anal. 1, 479-490, (1970).

9. C. E. Pearson, "On a differential equation of boundary layer type", J. Math. Phys. 47, 134-154, 1968.

10. Y. Sibuya, "Asymptotic solutions of a system of linear ordinary differential equations containing a parameter", Funkcial Ekvac. 4, 83-113, 1962.

11. W. Wasow, "On boundary layer problems in the theory of ordinary differential equations", New York University Ph.D. thesis, 1941.