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CONVERGENT GENERALIZED MONOTONE SPLITTING OF MATRICES[1]

by

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ABSTRACT

Let B and T be $n \times n$ real matrices and r an n-vector and consider the system $u = BTu + r$. A new sufficient condition is given for the existence of a solution and convergence of a monotone process to a solution. The monotone process is a generalization of the Collatz-Schröder procedure.

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1. INTRODUCTION

Collatz- Schröder [1] consider the system

$$(1.1) \qquad u = Tu + r$$

where T is a given $n \times n$ real matrix and r a given n-vector and prescribe the monotone iterative process

$$(1.2) \qquad \begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} T_1 & -T_2 \\ -T_2 & T_1 \end{bmatrix} \begin{bmatrix} v^i \\ w^i \end{bmatrix} + \begin{bmatrix} r \\ r \end{bmatrix} \qquad i = 0,1,2,\ldots$$

where $T = T_1 - T_2$, $T_1 \geqq 0$ and $T_2 \geqq 0$. A sufficient condition for the monotonicity and convergence of the above process is the existence of initial v^0, w^0 satisfying

$$(1.3) \qquad v^0 \leqq v^1, \quad w^1 \leqq w^0, \quad v^0 \leqq w^0 \quad ,$$

where v^1, w^1 are computed from (1.2). Condition (1.3) guarantees that (1.1) has a solution u such that

$$(1.4) \qquad v^0 \leqq v^1 \leqq \cdots \leqq v^i \leqq \cdots \leqq u \leqq \cdots\cdots \leqq w^i \leqq \cdots \leqq w^1 \leqq w^0$$

and $u = \lim_{i \to \infty} \dfrac{v^i + w^i}{2}$.

In this work we consider the system

$$(1.5) \qquad u = BTu + r$$

where B is some $n \times n$ real nonsingular matrix and prescribe the iterative process (2.3). Here however the splitting $T = T_1 - T_2$ is not monotonic with respect to the nonnegative orthant but with respect to the dual cone generated by the rows of B^{-1}, that is:

$$B^{-1}y \geqq 0 \implies T_1 y \geqq 0 \quad \text{and} \quad T_2 y \geqq 0 \; .$$

In Collatz-Schröder [1], $B^{-1} = I$. A sufficient condition for the monotonicity and convergence of the iteration (2.3) is the existence of v^0, w^0 satisfying (2.5). Condition (2.5) guarantees that (1.5) has a solution satisfying $B^{-1}v^i \leqq B^{-1}u \leqq B^{-1}w^i$ and $u = \lim_{i \to \infty} \frac{v^i + w^i}{2}$.

By using Motzkin's theorem of the alternative for linear inequalities [4,3] a sufficient condition for (2.5) can be obtained, condition (3.2). This condition insures the existence of a solution u to (1.5) and the convergence and monotonicity of the iterative process (2.3). When applied to $Au = b$, condition (3.2) gives existence results such as (3.6): If $A \geqq -I$, and $A'x \geqq 0$, $x \geqq 0$ implies that $bx = 0$, then $Au = b$ has a solution. (Here and throughout a prime denotes the transpose.)

2. THE MONOTONE SPLITTING AND ITS CONVERGENCE

We consider here the problem

$$(2.0) \qquad u = BTu + r$$

where B and T are given $n \times n$ real matrices with B nonsingular, and r is a given n-vector. We split the matrix T as follows

$$(2.1) \qquad T = T_1 - T_2$$

and require that

$$(2.2a) \qquad B^{-1}y \geqq 0 \implies T_1y \geqq 0 \text{ and } T_2y \geqq 0,$$

or equivalently [2] we require that

$$(2.2b) \qquad T_1B \geqq 0 \text{ and } T_2B \geqq 0$$

(To see the equivalence of (2.2a) and (2.2b) we note that if (2.2b) holds then $B^{-1}y \geqq 0$ implies $T_1BB^{-1}y = T_1y \geqq 0$. Conversely if (2.2a) holds then $B^{-1}B = I \geqq 0$ implies $T_1B \geqq 0$.)

We consider the iteration

$$(2.3) \qquad \begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} T_1 & -T_2 \\ -T_2 & T_1 \end{bmatrix} \begin{bmatrix} v^i \\ w^i \end{bmatrix} + \begin{bmatrix} r \\ r \end{bmatrix}$$

and begin by establishing the following result.

(2.4) Convergence and Existence Theorem Let (2.1) and (2.2) hold. If there exist v^0, w^0 in R^n such that

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^1 - v^0 \\ w^0 - w^1 \end{bmatrix} \geqq 0 \;, \quad B^{-1}(w^0 - v^0) \geqq 0 \;, \tag{2.5}$$

with v^1, w^1 computed from (2.3), then

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+1} - v^i \\ w^i - w^{i+1} \end{bmatrix} \geqq 0 \;, \quad B^{-1}(w^i - v^i) \geqq 0 \tag{2.6}$$

for $i = 0,1,2,\ldots$. In addition to system $u = BTu + r$ has a solution u such that $B^{-1}v^i \leqq B^{-1}u \leqq B^{-1}w^i$ and $u = \lim_{i \to \infty} \frac{v^i + w^i}{2}$.

Remark If we set $B = I$, then we obtain the results of Collatz-Schröder [1].

Proof We first establish (2.6) by induction. Because of (2.5), (2.6) holds for $i = 0$. Assume now that (2.6) holds for i and proceed to show that it also holds for $i + 1$.

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & -B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+2} - v^{i+1} \\ w^{i+2} - w^{i+1} \end{bmatrix}$$

$$= \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \begin{bmatrix} v^{i+1} \\ -w^{i+1} \end{bmatrix} + \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} r \\ -r \end{bmatrix} - \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+1} \\ -w^{i+1} \end{bmatrix} \quad \text{(by 2.3)}$$

$$\geqq \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \begin{bmatrix} v^{i} \\ -w^{i} \end{bmatrix} + \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} r \\ -r \end{bmatrix} - \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+1} \\ -w^{i+1} \end{bmatrix}$$

(by 2.6 and 2.2a)

$$= \quad 0 \qquad \text{(by 2.3)}$$

We also have

$$B^{-1}(w^{i+1} - v^{i+1}) = (T_1 + T_2)(w^i - v^i) \qquad \text{(by 2.3)}$$

$$\geqq 0 \qquad \text{(by 2.6 and 2.2a)}$$

Hence (2.6) holds for $i + 1$ and the induction is complete. We now have from (2.6) that

$$B^{-1}v^0 \leqq B^{-1}v^1 \leqq \cdots \leqq B^{-1}v^i \leqq \cdots\cdots \leqq B^{-1}w^i \leqq \cdots \leqq B^{-1}w^1 \leqq B^{-1}w^0 .$$

Hence the monotone sequences $\{B^{-1}v_k^i\}$ and $\{B^{-1}w_k^i\}$ have limits a_k^* and b_k^*, so that also the vector sequences $\{B^{-1}v^i\}$ and $\{B^{-1}w^i\}$

have limits a^* and b^*. From the continuity of the linear operator B we have also that the vector sequences $\{v^i\}$ and $\{w^i\}$ converge to $v = Ba^*$ and $w = Bb^*$. Hence from (2.3)

$$B^{-1}v = T_1 v - T_2 w + B^{-1} r$$

$$B^{-1}w = -T_2 v + T_1 w + B^{-1} r$$

By letting $u = \frac{v+w}{2}$ we have that

$$B^{-1}u = (T_1 - T_2)u + B^{-1}r = Tu + B^{-1}r.$$

That $B^{-1}v^i \leqq B^{-1}u \leqq B^{-1}w^i$ follows from

$$B^{-1}u = B^{-1}\left(\frac{v+w}{2}\right) = \frac{a^* + b^*}{2} . \qquad \text{Q.E.D.}$$

3. SUFFICIENT CONDITIONS FOR SOLVING $u = BTu + r$

By using Motzkin's theorem of the alternative [4,3] we give now a sufficient condition for the existence of v^0, w^0 satisfying (2.5) and hence for the existence of a solution of $u = BTu + r$ and the convergence of the iterative process (2.3).

(3.1) Convergence and Existence Theorem Theorem (2.4) holds with assumption (2.5) replaced by

$$\text{(3.2)} \quad \left\langle \begin{array}{r} (-I + T_1 B)'x + (T_2 B)'y \geqq 0 \\ (T_2 B)'x + (-I + T_1 B)'y \geqq 0 \\ x, y \geqq 0 \end{array} \right\rangle \Longrightarrow xB^{-1}r = 0$$

<u>Proof</u> We have to show that (3.2) implies (2.5). Now (3.2) implies that

$$\left\langle \begin{array}{rl} (-I + T_1 B)' x + (T_2 B)' y & \geqq 0 \\ (T_2 B)' x + (-I + T_1 B)' y & \geqq 0 \\ -xB^{-1} r + yB^{-1} r & > 0 \\ x, y & \geqq 0 \end{array} \right\rangle \text{has no solution } (x,y)$$

which implies that

$$\left\langle \begin{array}{rl} (-B^{-1} + T_1)' x + T_2' y - (B^{-1})' z & = 0 \\ -T_2' x - (-B^{-1} + T_1)' y + (B^{-1})' z & = 0 \\ (B^{-1} r)' x - (B^{-1} r)' y + \eta & = 0 \\ x, y, z & \geqq 0 \\ \eta & > 0 \end{array} \right\rangle \begin{array}{l} \text{has no solution} \\ (x, y, z, \eta) \end{array}$$

which by Motzkin's theorem is equivalent to

$$\left\langle \begin{array}{rl} (-B^{-1} + T_1) v^0 - T_2 w^0 + B^{-1} r \zeta & \geqq 0 \\ T_2 v^0 - (-B^{-1} + T_1) w^0 - B^{-1} r \zeta & \geqq 0 \\ -B^{-1} v^0 + B^{-1} w^0 & \geqq 0 \\ \zeta & > 0 \end{array} \right\rangle \begin{array}{l} \text{has a solution} \\ (v^0, w^0, \zeta) \end{array}$$

which is equivalent to

$$\left\langle \begin{array}{rl} T_1v^0 - T_2w^0 + B^{-1}r - B^{-1}v^0 & \geqq 0 \\ B^{-1}w^0 + T_2v^0 - T_1w^0 - B^{-1}r & \geqq 0 \\ B^{-1}(w^0 - v^0) & \geqq 0 \end{array} \right\rangle \text{ has a solution } (v^0, w^0)$$

which is equivalent to (2.5) having a solution. Q.E.D.

By observing that if the system $u = BTu + r$ has a solution $\bar{u}$ then the system $-u = -BTu + r$ has a solution $-\bar{u}$ we obtain the following result from Theorem (3.1) by appropriate modifications.

(3.3) <u>Convergence and Existence Theorem</u> Let (2.1), (2.2) and (3.2) hold. Then there exist v^0, w^0 such that

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^1 - v^0 \\ w^0 - w^1 \end{bmatrix} \leqq 0, \quad B^{-1}(w^0 - v^0) \leqq 0$$

where v^1, w^1 are computed from the iteration

$$\begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} T_1 & -T_2 \\ -T_2 & T_1 \end{bmatrix} \begin{bmatrix} v^i \\ w^i \end{bmatrix} - \begin{bmatrix} r \\ r \end{bmatrix}, \quad i = 0, 1, 2, \ldots$$

which produces v^i, w^i satisfying

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+1} - v^i \\ w^i - w^{i+1} \end{bmatrix} \leqq 0, \quad B^{-1}(w^i - v^i) \leqq 0.$$

In addition the system $-u = -BTu + r$ has a solution u such that $B^{-1}v^i \geqq B^{-1}u \geqq B^{-1}w^i$ and $u = \lim_{i\to\infty} \frac{v^i + w^i}{2}$.

The following convergence and existence result for $Au = b$ is obtained from Theorem (3.3) above by setting $T_2 = 0$, $A = -B^{-1} + T_1$ and $b = B^{-1}r$.

(3.4) <u>Convergence and Existence for $Au = b$</u> Consider the system $Au = b$ where A is a given $n \times n$ matrix and b is a given vector. Assume that $AB + I \geqq 0$ for some nonsingular $n \times n$ matrix B. If

$$\left\langle \begin{matrix} (AB)'x \geqq 0 \\ x \geqq 0 \end{matrix} \right\rangle \Longrightarrow bx = 0$$

then $Au = b$ has a solution u. This solution can be obtained from the iteration

$$\begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} B(T_1v^i - b) \\ B(T_1w^i - b) \end{bmatrix} \quad , \quad i = 0,1,2,\ldots$$

starting with v^0, w^0 which exist and satisfy

$$B^{-1}(v^1 - v^0) \leqq 0, \quad B^{-1}(w^0 - w^1) \leqq 0, \quad B^{-1}(w^0 - v^0) \leqq 0.$$

This iterative process produces v^i, w^i satisfying

$$B^{-1}(v^{i+1} - v^i) \leqq 0, \; B^{-1}(w^i - w^{i+1}) \leqq 0, \; B^{-1}(w^i - v^i) \leqq 0$$

$$B^{-1}v^i \geqq B^{-1}u \geqq B^{-1}w^i$$

and $u = \lim_{i \to \infty} \frac{v^i + w^i}{2}$.

(3.5) Remark If A is nonsingular, then by taking $B = -A^{-1}$ we conclude from the above that $Au = b$ has a solution for any b.

(3.6) Corollary If we take $B = I$ in (3.4) we have that for $A \geqq -I$, if $A'x \geqq 0$, $x \geqq 0$ implies $bx = 0$, then $Au = b$ has a solution, and the iteration and monotonicity relations of (3.4) simplify accordingly. Similarly if we take $B = -I$ we have that for $A \leqq I$, if $A'x \leqq 0$, $x \geqq 0$ implies $bx = 0$ then $Au = b$ has a solution.

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