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NUMERICAL STUDIES OF THE 3-BODY PROBLEM*

by

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APPENDIX: A GENERAL 3-BODY PROBLEM PROGRAM

by

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1. Introduction.

Under a given set of interactive forces, the resulting dynamical behavior of n bodies has been of interest in pure and applied science for many years (see, e.g., references [1]-[3], [5]-[10], and the additional references contained therein). Typical, important n-body problems occur in the study of the solar system under the usual assumptions that n is relatively small and that capture, but not collision, is admissable, and in the study of Brownian motion under the usual assumptions that n is relatively large and that collisions occur in accordance with an assumed probabilistic distribution.

In this first of a series of papers, we will describe and illustrate a new numerical method for <u>general</u>, plane, nondegenerate 3-body problems. The formulation will allow one to include the effects of viscosity and collision. Extension of the method to three dimensions offers no mathematical difficulty, but does require greater storage capacity for the actual computation. Application to <u>special</u> problems in celestial dynamics, molecular interaction, and free surface fluid flow will follow in subsequent papers.

2. The Numerical Method.

Consider three circular particles P_1, P_2 and P_3. For each of $i = 1,2,3$, let P_i have mass m_i and radius ρ_i. At time $t_k = k\Delta t$, $k = 0,1,2,\cdots$, let P_i, for each of $i = 1,2,3$, have center C_i at

$(x_{i,k}, y_{i,k})$, have velocity $(v_{i,k,x}, v_{i,k,y})$, and have acceleration $(a_{i,k,x}, a_{i,k,y})$. Finally, let $r_{12,k}$, $r_{13,k}$, and $r_{23,k}$ be the distances $|C_1C_2|$, $|C_1C_3|$ and $|C_2C_3|$, respectively, at time t_k. Then, for $i = 1,2,3$, it will be convenient to use the following discrete model formulas [4] for particle velocity and acceleration:

$$(2.1)\qquad v_{i,1,x} = \frac{2}{\Delta t}[x_{i,1} - x_{i,0}] - v_{i,0,x}$$

$$(2.2)\qquad v_{i,1,y} = \frac{2}{\Delta t}[y_{i,1} - y_{i,0}] - v_{i,0,y}$$

$$(2.3)\qquad v_{i,k,x} = \frac{2}{\Delta t}[x_{i,k} + (-1)^k x_{i,0} + 2\sum_{j=1}^{k-1}(-1)^j x_{i,k-j}] + (-1)^k v_{i,0,x};\ k > 1$$

$$(2.4)\qquad v_{i,k,y} = \frac{2}{\Delta t}[y_{i,k} + (-1)^k y_{i,0} + 2\sum_{j=1}^{k-1}(-1)^j y_{i,k-j}] + (-1)^k v_{i,0,y};\ k > 1$$

$$(2.5)\qquad a_{i,1,x} = \frac{2}{(\Delta t)^2}[x_{i,1} - x_{i,0} - v_{i,0,x}\Delta t]$$

$$(2.6)\qquad a_{i,1,y} = \frac{2}{(\Delta t)^2}[y_{i,1} - y_{i,0} - v_{i,0,y}\Delta t]$$

$$(2.7)\qquad a_{i,2,x} = \frac{2}{(\Delta t)^2}[x_{i,2} - 3x_{i,1} + 2x_{i,0} + v_{i,0,x}\Delta t]$$

$$(2.8)\qquad a_{i,2,y} = \frac{2}{(\Delta t)^2}[y_{i,2} - 3y_{i,1} + 2y_{i,0} + v_{i,0,y}\Delta t]$$

$$(2.9)\qquad a_{i,k,x} = \frac{2}{(\Delta t)^2}\Big\{x_{i,k} - 3x_{i,k-1} + 2(-1)^k x_{i,0} + 4\sum_{j=2}^{k-1}[(-1)^j x_{i,k-j}] + (-1)^k v_{i,0,x}\Delta t\Big\};\ k > 2$$

$$(2.10)\qquad a_{i,k,y} = \frac{2}{(\Delta t)^2}\Big\{y_{i,k} - 3y_{i,k-1} + 2(-1)^k y_{i,0} + 4\sum_{j=2}^{k-1}[(-1)^j y_{i,k-j}] + (-1)^k v_{i,0,y}\Delta t\Big\};\ k > 2.$$

We will apply Newton's general dynamical equation in the discrete form

$$(2.11)\qquad \vec{F}(t_k) = m\,\vec{a}(t_{k+1}),\ k = 0, 1, \cdots .$$

Equation (2.11) is not only computationally convenient, but it implies readily the fundamental conservation laws of classical physics [4].

For Newtonian 3-body problems, the components of $\vec{F}$ in (2.11) have factors of the form $-\frac{G}{r^2}$. For our purposes, however, it will be of great advantage to replace such terms by more general expressions of the form

$$(2.12)\qquad -\frac{G}{r^2} + \frac{H}{r^m} - \alpha\sqrt{v_x^2 + v_y^2},$$

where $G \geq 0$, $H \geq 0$, $\alpha \geq 0$ and $m > 2$. Physically, in (2.12), G is a constant of attraction, H is a constant of repulsion, and α is a damping factor. If $G > 0$ and $H > 0$, then, for large r, $-\frac{G}{r^2}$ dominates $\frac{H}{r^m}$ in (2.12), while, for very small r, $\frac{H}{r^m}$ dominates $-\frac{G}{r^2}$. For $H > 0$, the repulsive effect of the term $\frac{H}{r^m}$ in (2.12) will help to simulate particle collision without the necessity of having actual collision and will also allow us to consider repulsion, itself, as a component of force when it does exist and does have a significant effect [5]. The resulting dynamical equations, which reduce to the classical Newtonian 3-body equations when $H = \alpha = 0$, are derived simply by replacing terms of the form $-\frac{G}{r^2}$ in the usual formulas by terms of form (2.12), and are therefore

(2.13) $$a_{1,k+1,x} = \frac{m_2(x_{1,k}-x_{2,k})}{r_{12,k}}\left\{-\frac{G}{(r_{12,k})^2}+\frac{H}{(r_{12,k})^m}-\alpha[(v_{1,k,x})^2+(v_{1,k,y})^2]^{1/2}\right\}$$

$$+\frac{m_3(x_{1,k}-x_{3,k})}{r_{13,k}}\left\{-\frac{G}{(r_{13,k})^2}+\frac{H}{(r_{13,k})^m}-\alpha[(v_{1,k,x})^2+(v_{1,k,y})^2]^{1/2}\right\}$$

(2.14) $$a_{1,k+1,y} = \frac{m_2(y_{1,k}-y_{2,k})}{r_{12,k}}\left\{-\frac{G}{(r_{12,k})^2}+\frac{H}{(r_{12,k})^m}-\alpha[(v_{1,k,x})^2+(v_{1,k,y})^2]^{1/2}\right\}$$

$$+\frac{m_3(y_{1,k}-y_{3,k})}{r_{13,k}}\left\{-\frac{G}{(r_{13,k})^2}+\frac{H}{(r_{13,k})^m}-\alpha[(v_{1,k,x})^2+(v_{1,k,y})^2]^{1/2}\right\}$$

(2.15) $$a_{2,k+1,x} = \frac{m_1(x_{2,k}-x_{1,k})}{r_{12,k}}\left\{-\frac{G}{(r_{12,k})^2}+\frac{H}{(r_{12,k})^m}-\alpha[(v_{2,k,x})^2+(v_{2,k,y})^2]^{1/2}\right\}$$

$$+\frac{m_3(x_{2,k}-x_{3,k})}{r_{23,k}}\left\{-\frac{G}{(r_{23,k})^2}+\frac{H}{(r_{23,k})^m}-\alpha[(v_{2,k,x})^2+(v_{2,k,y})^2]^{1/2}\right\}$$

(2.16) $$a_{2,k+1,y} = \frac{m_1(y_{2,k}-y_{1,k})}{r_{12,k}}\left\{-\frac{G}{(r_{12,k})^2}+\frac{H}{(r_{12,k})^m}-\alpha[(v_{2,k,x})^2+(v_{2,k,y})^2]^{1/2}\right\}$$

$$+\frac{m_3(y_{2,k}-y_{3,k})}{r_{23,k}}\left\{-\frac{G}{(r_{23,k})^2}+\frac{H}{(r_{23,k})^m}-\alpha[(v_{2,k,x})^2+(v_{2,k,y})^2]^{1/2}\right\}$$

(2.17) $$a_{3,k+1,x} = \frac{m_1(x_{3,k}-x_{1,k})}{r_{13,k}}\left\{-\frac{G}{(r_{13,k})^2}+\frac{H}{(r_{13,k})^m}-\alpha[(v_{3,k,x})^2+(v_{3,k,y})^2]^{1/2}\right\}$$

$$+\frac{m_2(x_{3,k}-x_{2,k})}{r_{23,k}}\left\{-\frac{G}{(r_{23,k})^2}+\frac{H}{(r_{23,k})^m}-\alpha[(v_{3,k,x})^2+(v_{3,k,y})^2]^{1/2}\right\}$$

$$(2.18)\quad a_{3,k+1,y} = \frac{m_1(y_{3,k}-y_{1,k})}{r_{13,k}}\left\{-\frac{G}{(r_{13,k})^2} + \frac{H}{(r_{13,k})^m} - \alpha[(v_{3,k,x})^2+(v_{3,k,y})^2]^{1/2}\right\}$$

$$+ \frac{m_2(y_{3,k}-y_{2,k})}{r_{23,k}}\left\{-\frac{G}{(r_{23,k})^2} + \frac{H}{(r_{23,k})^m} - \alpha[(v_{3,k,x})^2+(v_{3,k,y})^2]^{1/2}\right\}.$$

The numerical method now can be described generally as follows. For a given set of initial data, one determines acceleration from (2.13)-(2.18) and position from (2.5)-(2.10). Specifically, the procedure to be used can be stated concisely by means of the following eight-step algorithm:

<u>Step 1</u> Select parameter values m_1, m_2, m_3, G, H, m, α.

<u>Step 2</u> Select initial data $(x_{i,0}, y_{i,0})$, $(v_{i,0,x}, v_{i,0,y})$, $i = 1,2,3$.

<u>Step 3</u> Determine $(a_{i,1,x}, a_{i,1,y})$, $i = 1,2,3$, from (2.13)-(2.18).

<u>Step 4</u> Determine $(x_{i,1}, y_{i,1})$ from (2.5) and (2.6).

<u>Step 5</u> Determine $(a_{i,2,x}, a_{i,2,y})$, $i = 1,2,3$, from (2.13)-(2.18).

<u>Step 6</u> Determine $(x_{i,2}, y_{i,2})$, $i = 1,2,3$, from (2.7)-(2.8).

<u>Step 7</u> Do Step 8 for $j = 3,4,\cdots, N$, where N is a fixed positive integer.

<u>Step 8</u> Determine $(a_{i,j,x}, a_{i,j,y})$, $i = 1,2,3$, from (2.13)-(2.18) and then determine $(x_{i,j}, y_{i,j})$, $i = 1,2,3$, from (2.9)-(2.10).

Note that if $r_{12,k}$, $r_{13,k}$, and $r_{23,k}$ are never zero, then the parameter values and initial data in Steps 1 and 2 of the above description result in a mathematical initial value problem whose solution always exists and is unique. Therefore, the only significant numerical problem of concern in actual computations is that of stability.

3. Examples.

A very large number of examples were run on the University of Wisconsin's UNIVAC 1108 using the method of Section 2. But, rather than attempt to organize all the diverse results, we will present, in detail, several typical and illustrative examples, each of which differs significantly from the others in the choice of at least one input parameter or initial value. No example required more than six minutes of computing time.

Example 1.

Consider first a particle arrangement like that shown in Figure 1(a). Specifically, let $m_1 = 10$, $m_2 = 0.1$, $m_3 = 0.1$, $G = 1$, $H = 1$, $m = 3$, $\alpha = .00981$, $x_{1,0} = 0$, $y_{1,0} = 0$, $x_{2,0} = 100$, $y_{2,0} = 0$, $x_{3,0} = -100$, $y_{3,0} = 0$, $v_{1,0,x} = 0$, $v_{1,0,y} = 0$, $v_{2,0,x} = 0$, $v_{2,0,y} = 10$, $v_{3,0,x} = 0$, $v_{3,0,y} = -10$, $\Delta t = .01$. The motion was studied for 24000 time steps. Symmetry about the origin of the initial conditions resulted in no motion for P_1 and symmetric motion about the origin for P_2 and P_3. One complete orbit of P_2 is shown in Figure 2, where its position is plotted after every 100 time steps. The orbit took 6232 time steps. During the entire time, P_2 completed almost four orbits and it intercepted the axes, in order, at (100.0,0), (0,99.91), (-100.0,0), (0,-99.41), (99.84,0), (0,99.75), (-99.67,0), (0,-100.2), (100.1,0), (0,100.4), (-100.3,0), (0,-99.68), (100.2,0), (0,99.93), (-99.69,0), (0,-100.5).

Example 2.

The data in Example 1 were changed by setting $m = 5$. The trajectory coordinates coincided with those of Example 1 to at least four decimal places.

Example 3.

The data in Example 1 were changed by setting $G = 2235$, $v_{2,0,y} = 15$, $v_{3,0,y} = -15$ and $\alpha = 0.0001$. One complete orbit of P_2 is shown in Figure 3, where its position is plotted after every 100 time steps. The orbit took only 4148 time steps, which is the only significant apparent difference between Figures 2 and 3.

Example 4.

The data in Example 1 were changed by setting $m_2 = .01$, $m_3 = 10^{-6}$, $\alpha = 0.01$, $x_{3,0} = -60$, $v_{3,0,y} = -6.085$. Trajectories for P_2 and P_3, plotted after every 100 time steps, are shown in Figure 4. The orbit for P_2 took 6201 time steps, while that for P_3 took 6148. The letters A, B, and C in Figure 4 indicate relative positions of P_2 and P_3 during their motions. The motion of P_1, though relatively small, is shown for the full 24000 time steps in Figure 5, where its position is plotted for every 250 time steps, and where the circled points indicate the 6000, 12000 and 18000 time step positions.

Example 5.

To consider a particle arrangement like that shown in Figure 1 (b), the data in Example 4 were changed by setting $v_{3,0,y} = 6.085$. The coordinates of the resulting orbit of P_2 differed by at most 10^{-3} from its orbit shown in Figure 4, while the orbit of P_3, except for a reversal of direction, was almost indistinguishable in shape from its orbit shown in Figure 4.

Example 6.

Consider next a particle arrangement like that shown in Figure 1 (c). Specifically, let us change the data in Example 1 by setting $m_2 = .01$, $m_3 = 10^{-6}$, $H = 0$, $\alpha = 0.01$, $x_{3,0} = 60$, $v_{3,0,y} = 10.01$. Though ρ_1, ρ_2, ρ_3, the respective radii of P_1, P_2, P_3 have been neglected thus far, we will for this example set $\rho_1 = 1$, $\rho_2 = .16$, $\rho_3 = .001$. Then the trajectories of P_2 and P_3 are shown for every 100 time steps in Figure 6. At step 2062, P_2 is located at (-48.78, 85.60) and P_3 is located at (-48.93, 85.60). Since the distance between the centers of the particles is 0.15, the motion is considered to be one of capture by P_2 of P_3.

Example 7.

For an arrangement like that of Figure 1 (d), let us change the data in Example 1 by setting $m_3 = 10^{-5}$, $\alpha = .01$, $x_{3,0} = 101$, $v_{3,0,y} = 10.1$. One complete orbit of P_2 is shown in Figure 7 (a), where its position is plotted after every 100 time steps. The orbit took 6202 time steps. During 24000 time steps, P_2 completed almost four orbits and it intercepted the axes, in order, at (100.0,0), (0,98.98), (-99.11,0), (0,-100.4), (98.26,0), (0,99.97), (-99.92,0), (0,-98.83), (100.1,0), (0,100.0), (-98.54,0), (0,-100.8), (100.7,0), (0,99.27), (-100.6,0), (0,99.07). The minute motion of P_1 is shown for the full 24000 time steps in Figure 7 (b), where its position is plotted for every 500 time steps. The arrow in Figure 7 (b) indicates the position of P_1 when P_2 has completed its first orbit. The motion of P_3 is one in orbit around P_2. The time for the first complete orbit was 5130 steps and the motion relative to P_2 is plotted for every 250 steps in Figure 7 (c).

Example 8.

Finally, let us consider a particle arrangement like that shown in Figure 1 (e). Specifically, set $m_1 = m_2 = m_3 = 10$, $G = 1$, $H = 1$, $m = 3$, $\alpha = .08$, $x_{1,0} = 0$, $y_{1,0} = 100$, $x_{2,0} = 100$, $y_{1,0} = 0$, $x_{3,0} = -100$, $y_{3,0} = 0$, $v_{1,0,x} = 0$, $v_{1,0,y} = -10$, $v_{2,0,x} = -10$, $v_{2,0,y} = 0$, $v_{3,0,x} = 1$, $v_{3,0,y} = 0$. The resulting motion of the system is shown for 1500 time steps in Figure 8, where each particle's position is plotted after every 50 time steps. Relative particle positions are marked by the particle number followed by a letter, so that, for example, 1A, 2A and 3A mark the relative positions of P_1, P_2 and P_3 at the same time. Though the system has a complex motion, it is interesting to note that the three particles themselves are usually relatively close to each other, as is typified by positions 1D, 2D and 3D, or by 1H, 2H and 3H. It is also of interest to note that often one or two particles slow down, while the others speed up toward it, as is evident when P_1 relocates from 1E to 1F while P_2 and P_3 relocates from 2E to 2F and 3E to 3F, respectively.

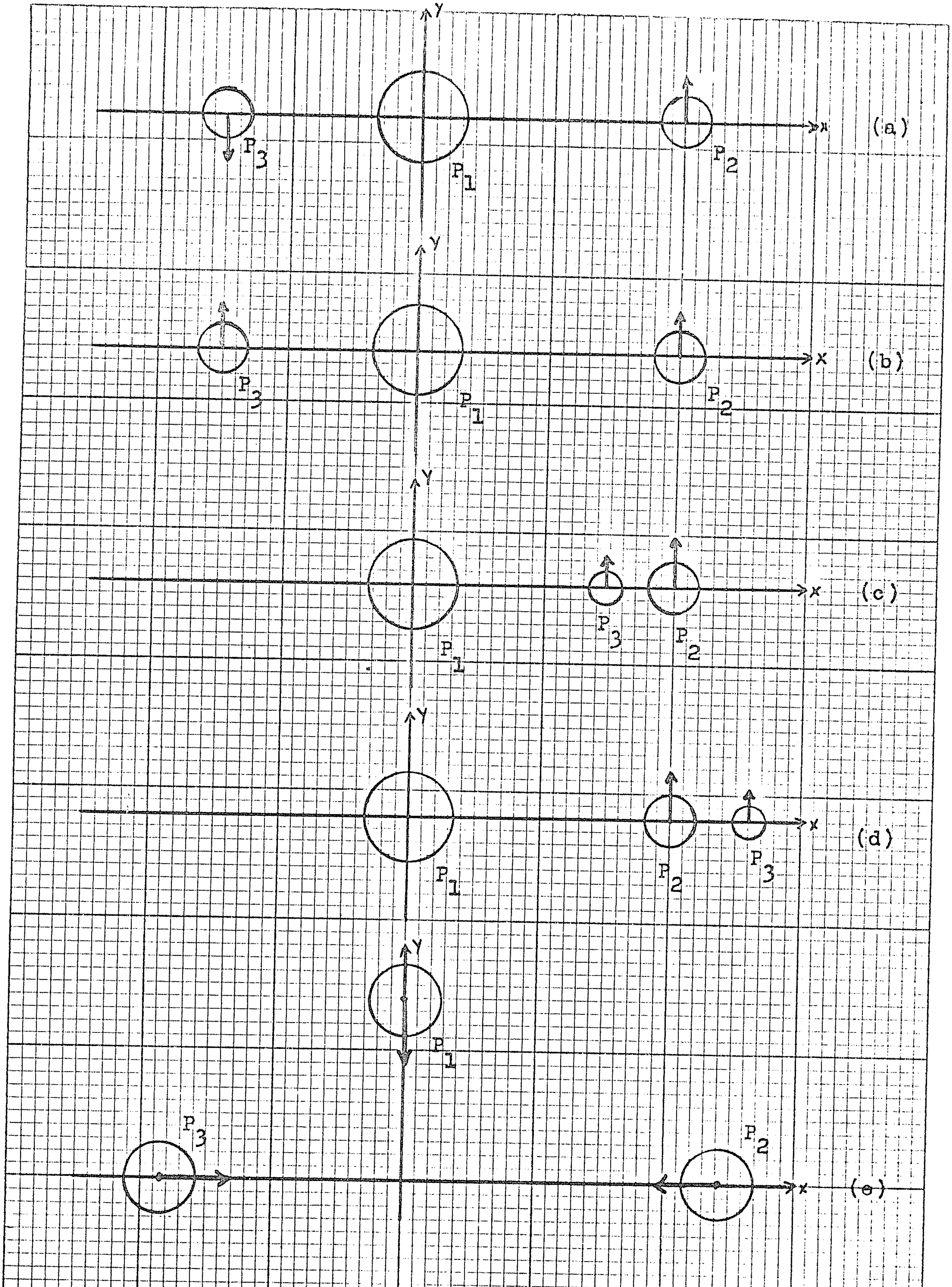

FIGURE 1

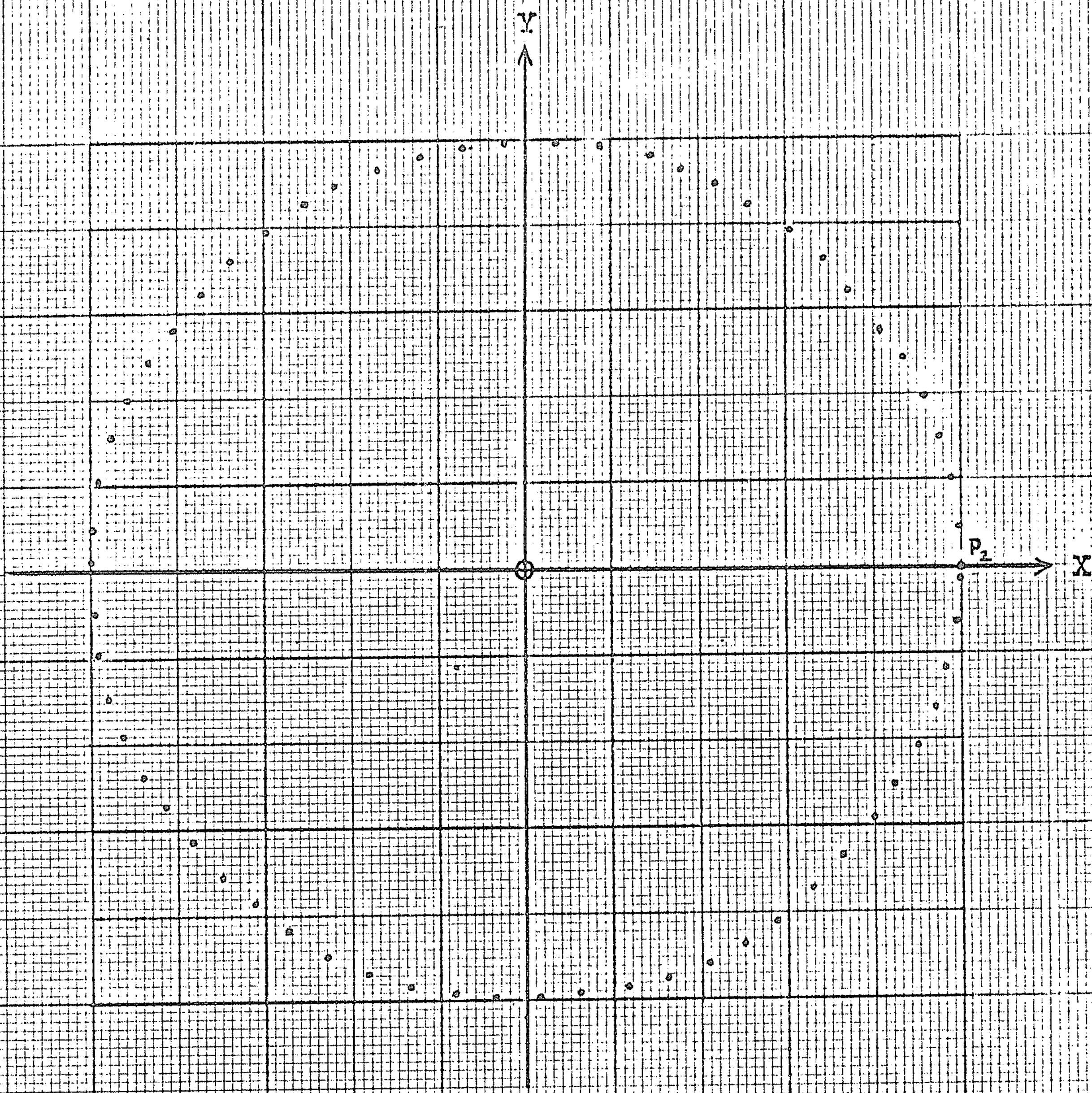

FIGURE 2

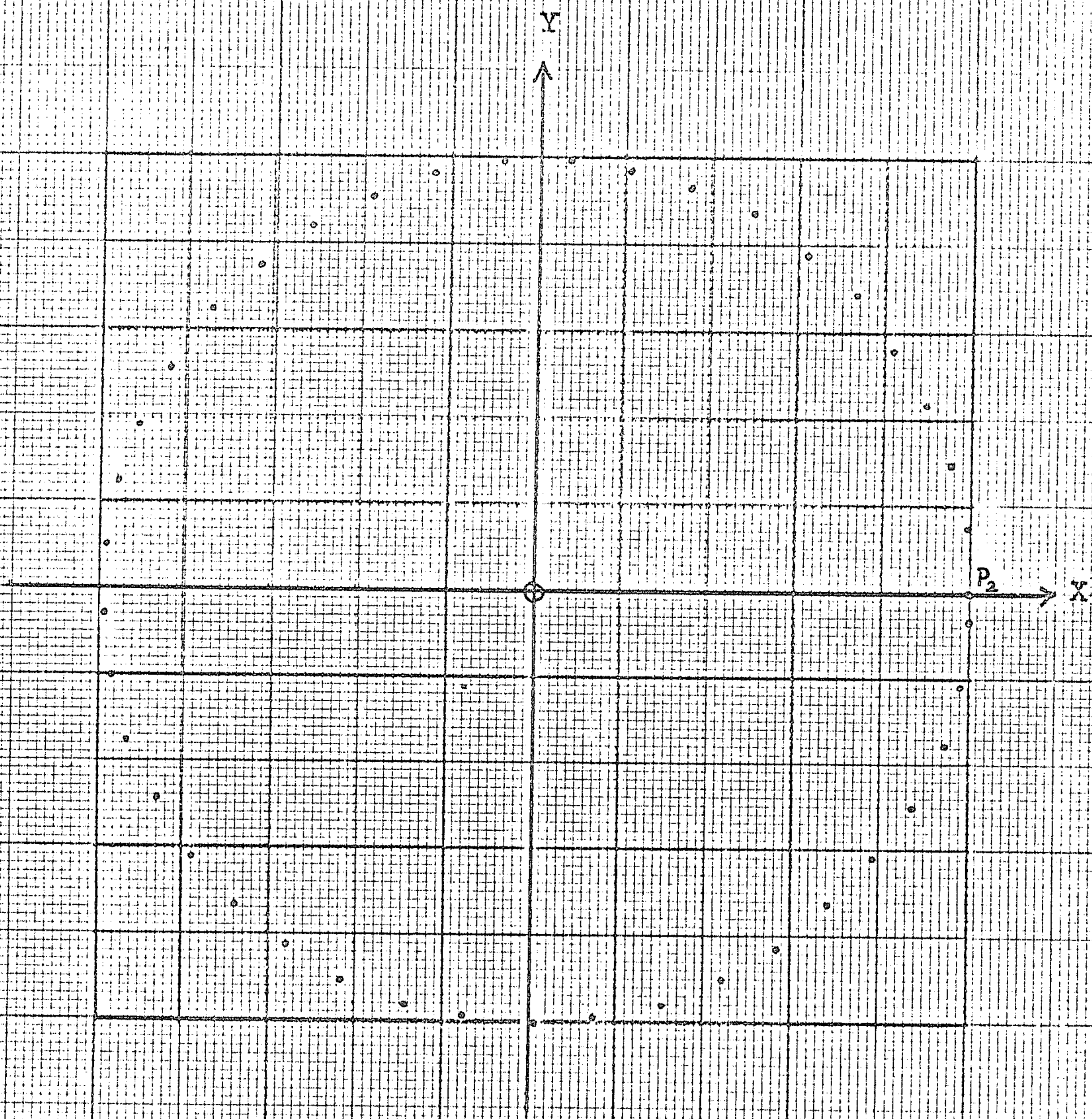

FIGURE 3

FORM C-2

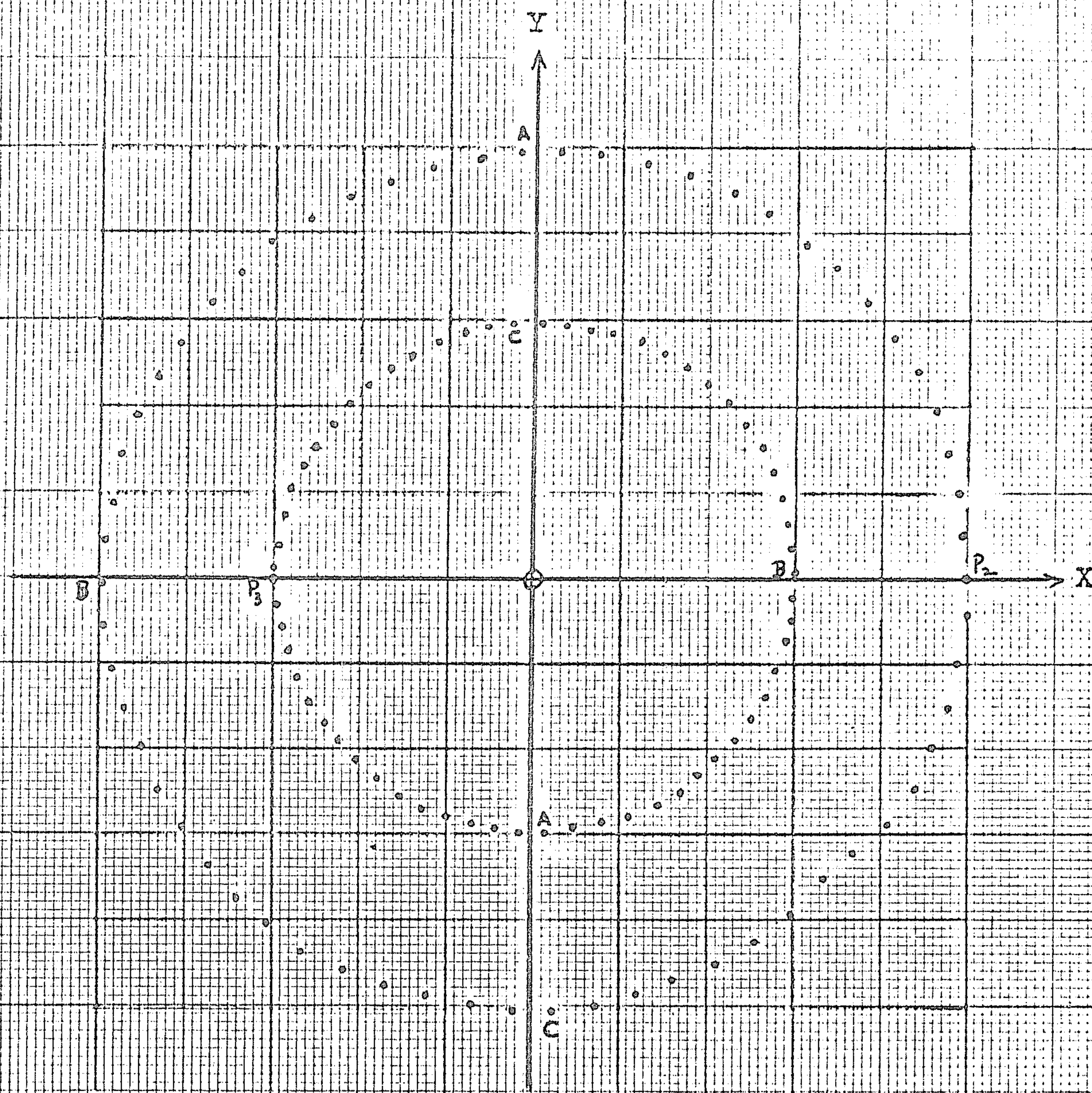

FIGURE 4

BROWN'S BOOK SHOP, INC.

FORM C3

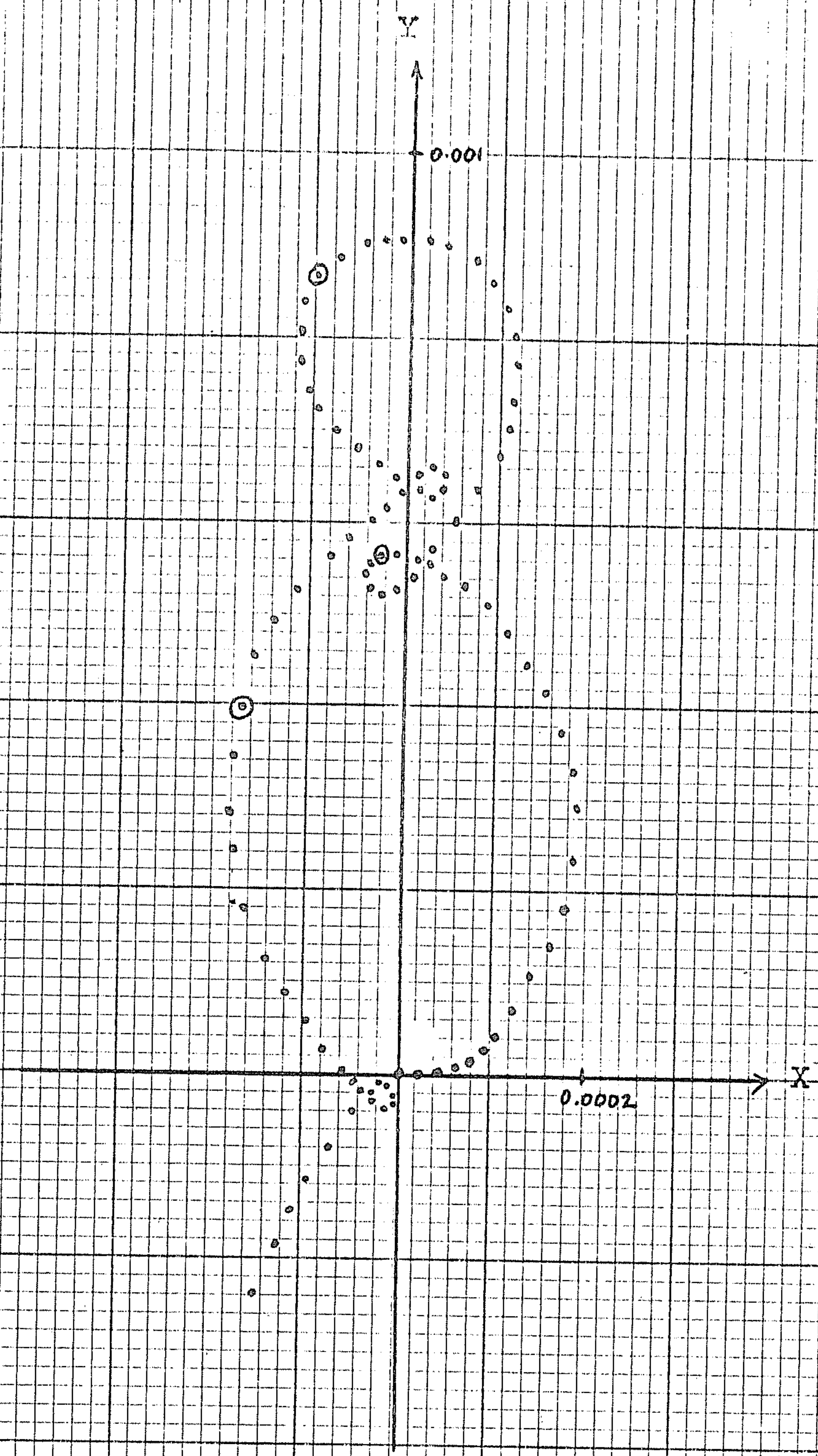

FIGURE 5

FORM C-2

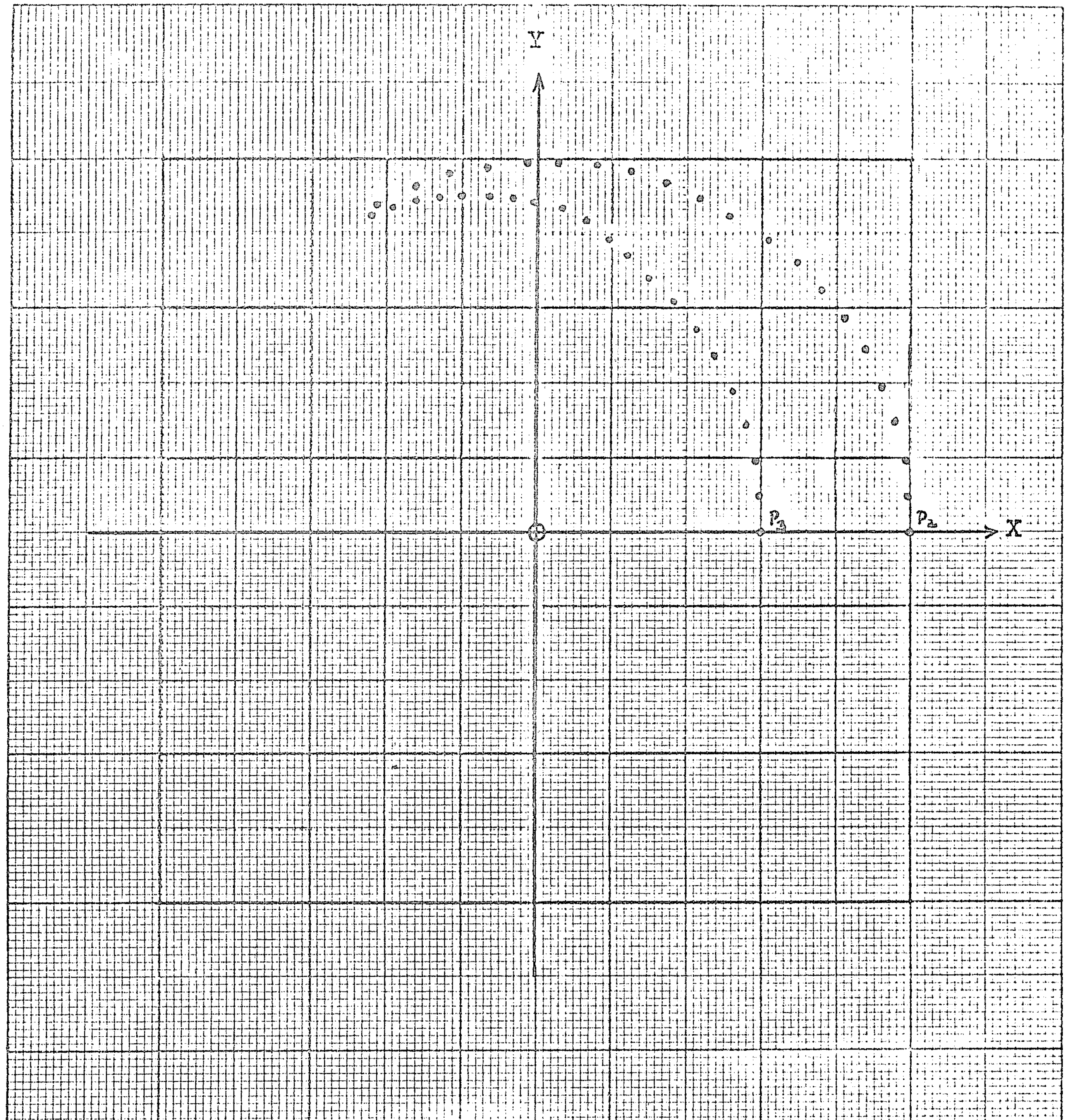

FIGURE 6

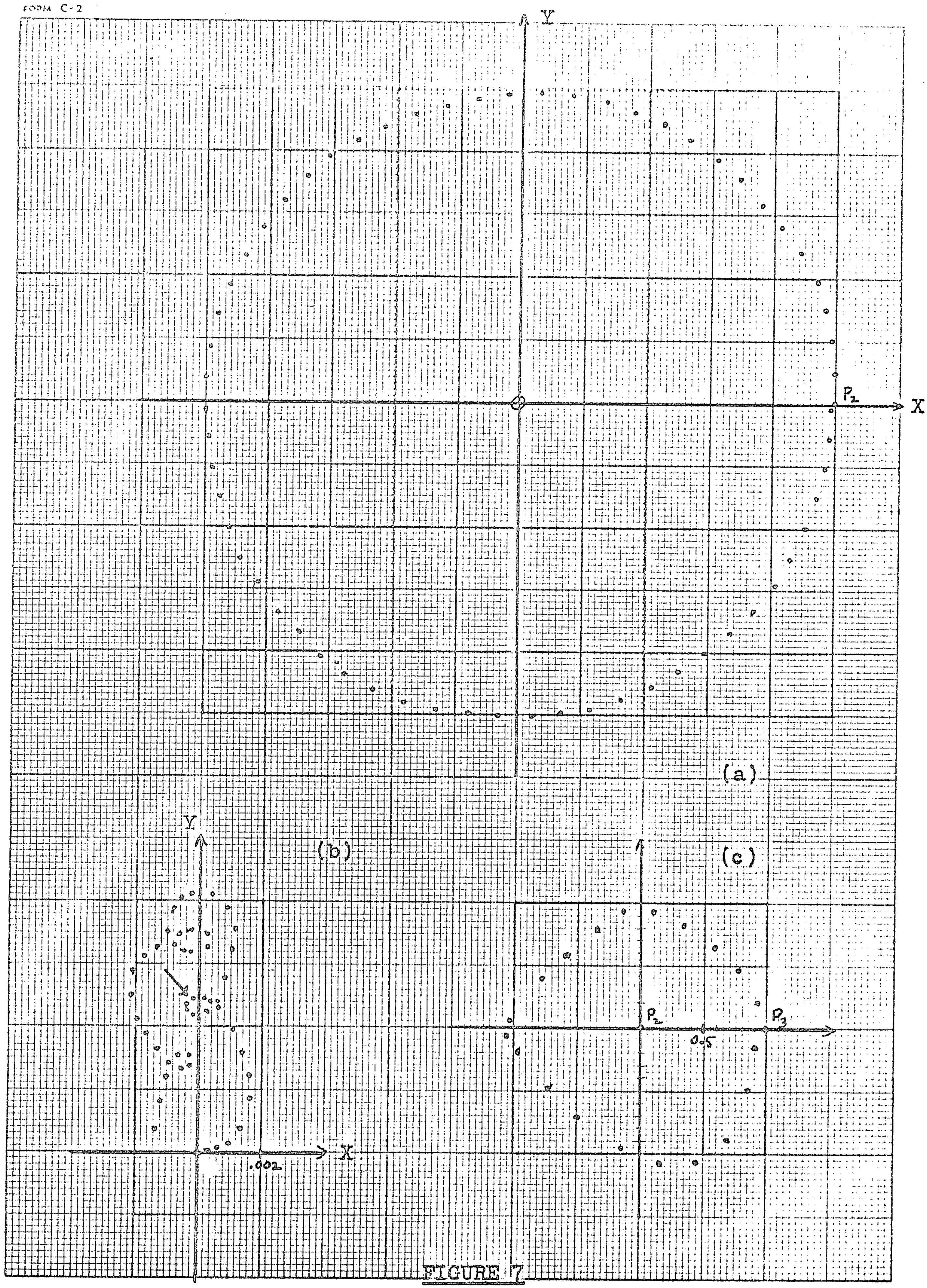

FIGURE 7

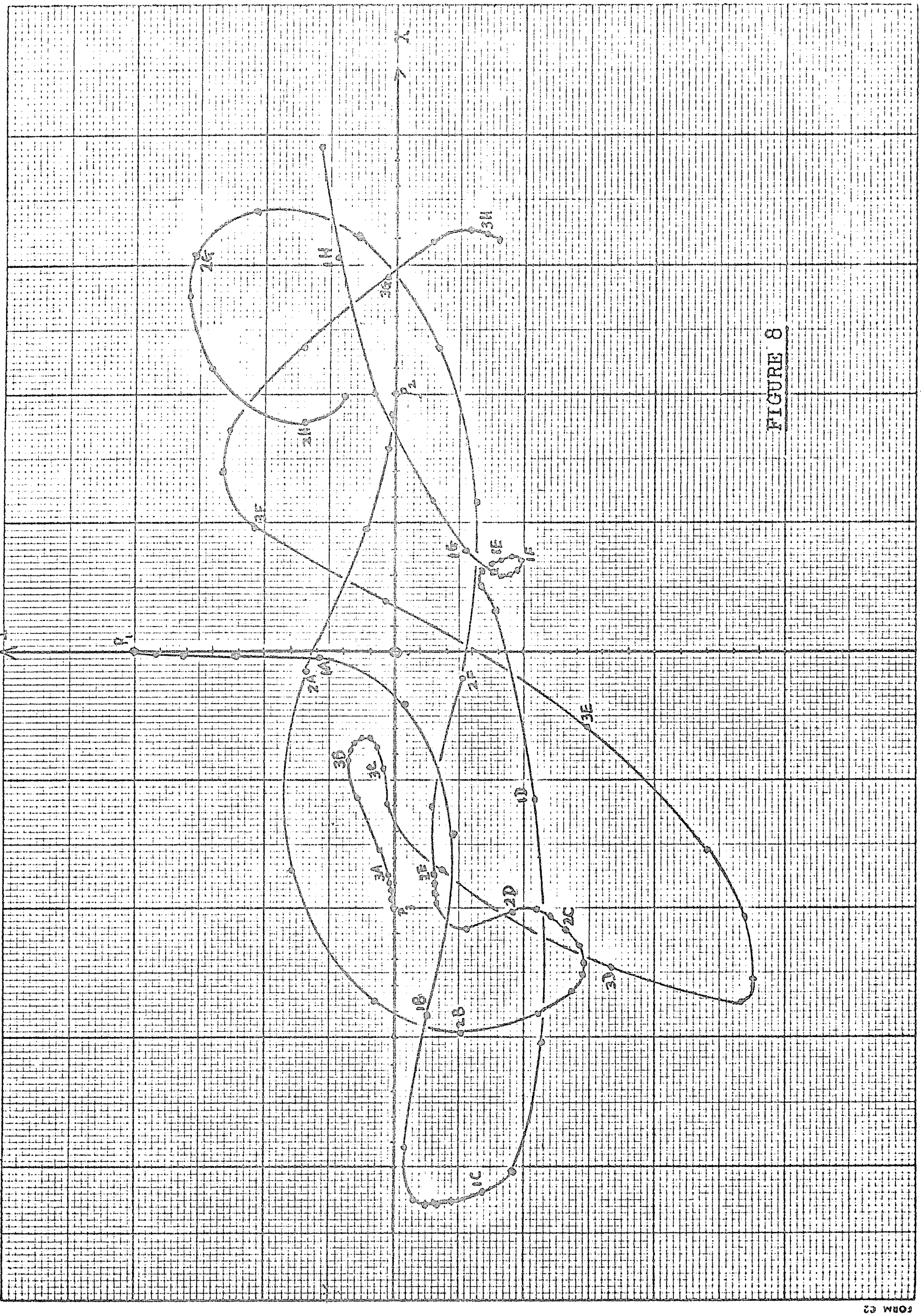
X
Y
P_1
P_2
P_3
1A
1B
1C
1D
1E
1F
1G
1H
2A
2B
2C
2D
2F
2G
2H
3A
3B
3C
3D
3E
3F
3G
3H

FIGURE 8

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Appendix: A General 3-Body Problem Program
by David Schultz

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      DIMENSION M1(9),M2(9),M3(9),X1(3,9),Y1(3,9),VOX(3,9),VOY(3,9),
     1G(9),H(9),DT(9),AA(9),AX(3),AY(3),VX(9),VY(9),X(3,10000),
     2E(3),F(3),D(3),Q(3),R(3),A(3),Y(3,10000)
      REAL M1,M2,M3
      KL=0
      NN=2000
C     READ IN THE NUMBER OF PARAMETERS
      READ 30,N1,N2,N3,N4,N5,N6,N7,N8,N9,L1,L2,L3,L4,L5,L6,L7,L8,L9,NP
30    FORMAT(20I1)
C     READ IN PARAMETERS
      READ 31,(M1(I),I=1,N1)
      READ 31,(M2(I),I=1,N2)
      READ 31,(M3(I),I=1,N3)
      READ 31,(X1(1,I),I=1,N4)
      READ 31,(X1(2,I),I=1,N5)
      READ 31,(X1(3,I),I=1,N6)
      READ 31,(Y1(1,I),I=1,N7)
      READ 31,(Y1(2,I),I=1,N8)
      READ 31,(Y1(3,I),I=1,N9)
      READ 31,(VOX(1,I),I=1,L1)
      READ 31,(VOX(2,I),I=1,L2)
      READ 31,(VOX(3,I),I=1,L3)
      READ 31,(VOY(1,I),I=1,L4)
      READ 31,(VOY(2,I),I=1,L5)
      READ 31,(VOY(3,I),I=1,L6)
      READ 31,(G(I),I=1,L7)
      READ 31,(H(I),I=1,L8)
      READ 31,(DT(I),I=1,L9)
      READ 31,(AA(I),I=1,NP)
31    FORMAT(9E8.1)
C     DO LOOP FOR EACH PARAMETER
      DO 1 I1=1,N1
      DO 1 I2=1,N2
      DO 1 I3=1,N3
      DO 1 I4=1,N4
      DO 1 I5=1,N5
      DO 1 I6=1,N6
      DO 1 I7=1,N7
      DO 1 I8=1,N8
      DO 1 I9=1,N9
      DO 1 J1=1,L1
      DO 1 J2=1,L2
      DO 1 J3=1,L3
      DO 1 J4=1,L4
      DO 1 J5=1,L5
      DO 1 J6=1,L6
      DO 1 J9=1,L7
      DO 1 K1=1,L8
      DO 1 K2=1,L9
      DO 1 K3=1,NP
C     PRINT OUT THE LIST OF PARAMETERS
C     FOR EACH CASE
      PRINT 80,M1(I1),M2(I2),M3(I3),X1(1,I4),X1(2,I5),X1(3,I6)
80    FORMAT(1H1,'M1=',E10.4,',M2=',E10.4,',M3=',E10.4,',X1=',E10.4,
     1',X2=',E10.4,',X3=',E10.4)
      PRINT 81,Y1(1,I7),Y1(2,I8),Y1(3,I9),VOX(1,J1),VOX(2,J2),VOX(3,J3)
```

```
81    FORMAT(' Y1=',E10.4,',Y2=',E10.4,',Y3=',E10.4,',VOX1=',E10.4,
     1',VOX2=',E10.4,',VOX 3=',E10.4)
      PRINT 32,VOY(1,J4),VOY(2,J5),VOY(3,J6),G(J9),H(K1),DT(K2),AA(K3)
32    FORMAT(' VOY1=',E10.4,',VOY2=',E10.4,'VOY3=',E11.5,'G=',E10.4,',H
     1 =',E10.4, ',DT=',E10.4,',AA=',E10.4)
      PRINT 22
22    FORMAT(1H ,3X,'K',5X,'X1',12X,'Y1',15X,'X2',8X,'Y2',15X,'X3',13X,'Y3')
     1'Y3')
      PRINT 21,KL,X1(1,I4),Y1(1,I7),X1(2,I5),Y1(2,I8),X1(3,I6),Y1(3,I9)
      DO 71 I=1,3
      D(I)=0
      Q(I)=0
      F(I)=0
71    E(I)=0
      NZ=0
      Z=1
C     RUN EACH CASE FOR NN TIME STEPS
      DO 91 KK=1,NN
      K=KK
      Z=-Z
      IF(K-2)6,7,7
C     FIND DISTANCE BETWEEN PARTICLES
C     FOR FIRST TIME STEP
6     R(1)=SQRT(((X1(1,I4)-X1(2,I5))**2)+((Y1(1,I7)-Y1(2,I8))**2))
      R(2)=SQRT(((X1(1,I4)-X1(3,I6))**2)+((Y1(1,I7)-Y1(3,I9))**2))
      R(3)=SQRT(((X1(2,I5)-X1(3,I6))**2)+((Y1(2,I8)-Y1(3,I9))**2))
      VX(3)=VOX(3,J3)
      VX(2)=VOX(2,J2)
      VX(1)=VOX(1,J1)
      VY(1)=VOY(1,J4)
      VY(2)=VOY(2,J5)
      VY(3)=VOY(3,J6)
      GO TO 10
7     CONTINUE
      KT=K
      K=K-1
C     FIND DISTANC: BETWEEN PARTICLES
C     FOR TIME STEPS GREATER THAN ONE
      R(1)=SQRT((( X(1,K) - X(2,K))**2) +(( Y(1,K) - Y(2,K))**2))
      R(2)=SQRT((( X(1,K) - X(3,K))**2) +(( Y(1,K) - Y(3,K))**2))
      R(3)=SQRT((( X(2,K) - X(3,K))**2) +(( Y(2,K) - Y(3,K))**2))
      K=KT
      IF(K-2)8,17,8
C     CALCULATE VELOCITIES FOR SECOND TIME STEP
17    VX(1)=(2./DT(K2))*(X(1,1)-X1(1,I4))-VOX(1,J1)
      VX(2)=(2./DT(K2))*(X(2,1)-X1(2,I5))-VOX(2,J2)
      VX(3)=(2./DT(K2))*(X(3,1)-X1(3,I6))-VOX(3,J3)
      VY(1)=(2./DT(K2))*(Y(1,1)-Y1(1,I7))-VOY(1,J4)
      VY(2)=(2./DT(K2))*(Y(2,1)-Y1(2,I8))-VOY(2,J5)
      VY(3)=(2./DT(K2))*(Y(3,1)-Y1(3,I9))-VOY(3,J6)
      GO TO 10
8     CONTINUE
      KT=K
      K=K-1
      DO 2 I=1,3
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      D(I)=-D(I)-X(I,K-1)
2     Q(I)=-Q(I)-Y(I,K-1)
C     CALCULATE VELOCITIES FOR TIME STEPS GREATER
C     THAN TWO
     VX(1)=(2./DT(K2))*(X(1,K)-Z*X1(1,I4)+2*D(1))-Z*VOX(1,J1)
     VX(2)=(2./DT(K2))*(X(2,K)-Z*X1(2,I5)+2*D(2))-Z*VOX(2,J2)
     VX(3)=(2./DT(K2))*(X(3,K)-Z*X1(3,I6)+2*D(3))-Z*VOX(3,J3)
      VY(1)=(2./DT(K2))*(Y(1,K)-Z*Y1(1,I7)+2.*Q(1))-Z*VOY(1,J4)
      VY(2)=(2./DT(K2))*(Y(2,K)-Z*Y1(2,I8)+2.*Q(2))-Z*VOY(2,J5)
      VY(3)=(2./DT(K2))*(Y(3,K)-Z*Y1(3,I9)+2.*Q(3))-Z*VOY(3,J6)
      K=KT
10    DO 3 I=1,3
C     TEST FOR DIVERGENCE
      IF(VX(I)-10000.)77,1,1
77    IF(VY(I)-10000.)3,1,1
3     A(I)=SQRT(((VX(I))**2)+((VY(I))**2))
C     CALCULATE FORCES BETWEEN PARTICLES
      B12=(-G(J9)/((R(1))**2))+(H(K1)/((R(1))**3))
      B13=(-G(J9)/((R(2))**2))+(H(K1)/((R(2))**3))
      B23=(-G(J9)/((R(3))**2))+(H(K1)/((R(3))**3))
      IF(K-2)52,40,40
52    X(1,1)=X1(1,I4)
      X(2,1)=X1(2,I5)
       X(3,1)=X1(3,I6)
      Y(1,1)=Y1(1,I7)
      Y(2,1)=Y1(2,I8)
      Y(3,1)=Y1(3,I9)
40    CONTINUE
      IF(K-2)93,92,92
92    K=K-1
      NZ=1
93    CONTINUE
C     FIND ACCELERATION OF PARTICLES
      AX(1)=(M2(I2)*(X(1,K)-X(2,K))/R(1))*(B12-AA(K3)*A(1))
     1+(M3(I3)*(X(1,K)-X(3,K))/R(2))*(B13-AA(K3)*A(1))
      AX(2)=(M1(I1)*(X(2,K)-X(1,K))/R(1))*(B12-AA(K3)*A(2))
     1+(M3(I3)*(X(2,K)-X(3,K))/R(3))*(B23-AA(K3)*A(2))
      AX(3)=(M1(I1)*(X(3,K)-X(1,K))/R(2))*(B13-AA(K3)*A(3))
     1+(M2(I2)*(X(3,K)-X(2,K))/R(3))*(B23-AA(K3)*A(3))
      AY(1)=(M2(I2)*(Y(1,K)-Y(2,K))/R(1))*(B12-AA(K3)*A(1))
     1+(M3(I3)*(Y(1,K)-Y(3,K))/R(2))*(B13-AA(K3)*A(1))
      AY(2)=(M1(I1)*(Y(2,K)-Y(1,K))/R(1))*(B12-AA(K3)*A(2))
     1+(M3(I3)*(Y(2,K)-Y(3,K))/R(3))*(B23-AA(K3)*A(2))
      AY(3)=(M1(I1)*(Y(3,K)-Y(1,K))/R(2))*(B13-AA(K3)*A(3))
     1+(M2(I2)*(Y(3,K)-Y(2,K))/R(3))*(B23-AA(K3)*A(3))
      IF(NZ-1)100,101,100
101   K=K+1
      NZ=0
100   CONTINUE
      IF(K-2)12,13,14
C     FIND POSITION AFTER FIRST TIME STEP
12    X(1,K)=X1(1,I4)+DT(K2)*VOX(1,J1)+(DT(K2)*DT(K2)/2.)*AX(1)
      X(2,K)=X1(2,I5)+DT(K2)*VOX(2,J2)+(DT(K2)*DT(K2)/2.)*AX(2)
      X(3,K)=X1(3,I6)+DT(K2)*VOX(3,J3)+(DT(K2)*DT(K2)/2.)*AX(3)
      Y(1,K)=Y1(1,I7)+DT(K2)*VOY(1,J4)+(DT(K2)*DT(K2)/2.)*AY(1)
      Y(2,K)=Y1(2,I8)+DT(K2)*VOY(2,J5)+(DT(K2)*DT(K2)/2.)*AY(2)
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      Y(3,K)=Y1(3,I9)+DT(K2)*VOY(3,J6)+(DT(K2)*DT(K2)/2.)*AY(3)
      GO TO 91
C     FIND POSITION AFTER SECOND TIME STEP
13    X(1,K)=3.*X(1,1)-2.*X1(1,I4)-DT(K2)*VOX(1,J1)+(DT(K2)*DT(K2)/2.)*
     1AX(1)
      X(2,K)=3.*X(2,1)-2.*X1(2,I5)-DT(K2)*VOX(2,J2)+(DT(K2)*DT(K2)/2.)*
     1AX(2)
      X(3,K)=3.*X(3,1)-2.*X1(3,I6)-DT(K2)*VOX(3,J3)+(DT(K2)*DT(K2)/2.)*
     1AX(3)
      Y(1,K)=3.*Y(1,1)-2.*Y1(1,I7)-DT(K2)*VOY(1,J4)+(DT(K2)*DT(K2)/2.)*
     1AY(1)
      Y(2,K)=3.*Y(2,1)-2.*Y1(2,I8)-DT(K2)*VOY(2,J5)+(DT(K2)*DT(K2)/2.)*
     1AY(2)
      Y(3,K)=3.*Y(3,1)-2.*Y1(3,I9)-DT(K2)*VOY(3,J6)+(DT(K2)*DT(K2)/2.)*
     1AY(3)
      GO TO 91
14    DO 5 I=1,3
      E(I)=-F(I)+X(I,K-2)
5     F(I)=-F(I)+Y(I,K-2)
      KT=K
      K=K-1
C     FIND POSITION AFTER ALL TIME STEPS
C     GREATER THAN TWO
      X(1,K+1)=3.*X(1,K)-2*(+Z)*X1(1,I4)-4.*E(1)-Z*DT(K2)*VOX(1,J1)
     1+(DT(K2)*DT(K2)/2.)*AX(1)
      X(2,K+1)=3.*X(2,K)-2*(+Z)*X1(2,I5)-4.*E(2)-Z*DT(K2)*VOX(2,J2)
     1+(DT(K2)*DT(K2)/2.)*AX(2)
      X(3,K+1)=3.*X(3,K)-2*(+Z)*X1(3,I6)-4.*E(3)-Z*DT(K2)*VOX(3,J3)
     1+(DT(K2)*DT(K2)/2.)*AX(3)
      Y(1,K+1)=3.*Y(1,K)-2.*Z*Y1(1,I7)- 4.*F(1)-Z*DT(K2)*VOY(1,J4)+
     1(DT(K2)*DT(K2)/2.)*AY(1)
      Y(2,K+1)=3.*Y(2,K)-2.*Z*Y1(2,I8)- 4.*F(2)-Z*DT(K2)*VOY(2,J5)+
     1(DT(K2)*DT(K2)/2.)*AY(2)
      Y(3.K+1)=3.*Y(3,K)-2.*Z*Y1(3,I9)- 4.*F(3)-Z*DT(K2)*VOY(3,J6)+
     1(DT(K2)*DT(K2)/2.)*AY(3)
      K=KT
91    CONTINUE
      DO 20 K=1,NN
20    PRINT 21, K,X(1,K),Y(1,K),X(2,K),Y(2,K),X(3,K),Y(3,K)
21    FORMAT(1X,I4,E10.4,3X,E10.4,5X,E10.4,3X,E10.4,5X,E10.4,3X,E10.4)
1     CONTINUE
      END
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