DECISION PROBLEMS FOR ω-AUTOMATA

by

L. H. Landweber

Technical Report #48

November 1968

DECISION PROBLEMS FOR ω-AUTOMATA

by

L. H. Landweber
University of Wisconsin

1. INTRODUCTION

In [4], Hartmanis and Stearns investigate properties of sets of infinite sequences which can be defined by finite automata. In this paper we consider various definitions for machines of this type, including ones introduced by Buchi [1] and McNaughton [6]. For each type of machine we classify the complexity of definable sets of sequences. More precisely let Σ^ω be the set of ω-sequences on the finite set Σ. Consider the Borel hierarchy with respect to the product topology on Σ^ω. The complexity of a subset of Σ^ω is its position in the Borel hierarchy. It is shown that increasing the complexity of requirements for a sequence to be accepted by a finite automaton, raises the level in the Borel hierarchy at which definable sets are found. Furthermore procedures are given for deciding the complexity of sets defined by a large class of machines.

In [4], Σ is taken to be $\{0,1\}$ and the usual topology on the real line is considered. We use the product topology because it is more natural when dealing with finite state machines in that it avoids the necessity of identifying infinite sequences (e.g., 100 .... equals 011 ... on the real line). Moreover the product space Σ^ω is in effect an infinite tree with

---

This research was supported in part by a grant from the University of Wisconsin Alumni Research Foundation.

paths through the tree corresponding in a 1-1 fashion with points of Σ^{ω}. We believe this analogy adds an intuitive flavor to the proofs.

In the second section we define the acceptance conditions (henceforth called output conditions) and topological properties to be considered. Section 3 gives the hierarchy results. In the fourth section we give an algorithm for deciding the complexity of sets defined by arbitrary machines. Relationships between the various machine types are then explored. In the last section we discuss reducibility relationships existing among various undecidable properties of Turing machines which accept infinite sequences.

2. DEFINITIONS

Let Σ be a finite set, the input alphabet. $\Sigma^*(\Sigma^{\omega})$ is the set of all finite (infinite) sequences on Σ. If $x, y \in \Sigma^*$, xy is the concatenation of x and y. Let $\alpha = \sigma_1 \sigma_2 \sigma_3 \ldots (\sigma_i \in \Sigma)$ be a member of Σ^{ω}. Abbreviate $\sigma_1 \sigma_2 \ldots \sigma_i$ by $\bar{\sigma}_i$ and define the partial order $\prec$ on $\Sigma^* \cup \Sigma^{\omega}$ by $\bar{\sigma}_i \prec \bar{\sigma}_j \prec \alpha$ for $i < j < \omega$. $P(S)$ is the set of all subsets of the set S. Set inclusion is indicated by $\subseteq$ and proper set inclusion by $\subset$. $c(A)$ is the cardinality of the set A.

Definition 2.1. A finite automaton (f.a.) over Σ is a system $\mathfrak{M} = \langle S, s_0, M \rangle$ where S is a finite set, the set of states, M is a function $M: S \times \Sigma \rightarrow S$ and $s_0 \in S$ is the initial state.

In the following $\mathfrak{M} = \langle S, s_0, M \rangle$ is a fixed but arbitrary f.a. .

Definition 2.2. $\overline{M}: S \times \Sigma^* \rightarrow S$ is the extension of M given by $\overline{M}(s, x\sigma) = M(\overline{M}(s,x), \sigma)$ for $\sigma \in \Sigma$, $x \in \Sigma^*$. $R_{\mathcal{M}}$ is a function, $R_{\mathcal{M}}: \Sigma^* \rightarrow S$, given by $R_{\mathcal{M}}(x) = \overline{M}(s_0, x)$, called the <u>response function of $\mathcal{M}$</u>. (To simplify the notation we omit the subscript $\mathcal{M}$ in $R_{\mathcal{M}}$.).

Definition 2.3. $R|\alpha$ is the function R restricted to $\{x \mid x \prec \alpha \in \Sigma^{\omega}\}$. Let $In(\alpha) = \{s \mid s \in S,\ c((R|\alpha)^{-1}(s)) = \omega\}$. I.e., $In(\alpha)$ is the set of states of $\mathcal{M}$ which are entered infinitely often while reading α.

To simplify the proofs we always assume that all states of $\mathcal{M}$ are accessible from the initial state. That is for all $s \in S$ there is an $x \in \Sigma^*$ such that $R(x) = s$.

To $\mathcal{M}$ we may adjoin the following conditions for acceptance of sequences $\alpha = \sigma_1 \sigma_2 \cdots$ of Σ^{ω}. The conditions are called <u>output conditions</u> or just outputs.

1. Let $D \subseteq S$. $\mathcal{M}$ accepts α with respect to D if $(\exists i) R(\overline{\sigma}_i) \in D$.

1'. " " " " " " " " " " $(\forall i) R(\overline{\sigma}_i) \in D$.

2. " " " " " " " " " " $In(\alpha) \cap D \neq \emptyset$.

2'. Let $\mathfrak{D} \subseteq P(S)$. $\mathcal{M}$ accepts α with respect to $\mathfrak{D}$ if $(\exists D \in \mathfrak{D})\ In(\alpha) \subseteq D$.

3. " " " " " " " " " " $(\exists D \in \mathfrak{D})\ In(\alpha) = D$.

Definition 2.4. An i - f.a. is a f.a. augmented by an output of type i . If $\mathcal{M}$ is an i - f.a., $T(\mathcal{M})$, <u>the set of sequences defined by</u> $\mathcal{M}$, is $\{\alpha | \alpha \in \Sigma^{\omega}, \alpha$ accepted by $\mathcal{M}\}$. (Of course the notion of acceptance is with respect to the designated set D or set of sets $\mathfrak{D}$ and the output type. To simplify the text we use just 'accept' whenever the meaning is clear.)

Definition 2.5. $A \subseteq \Sigma^{\omega}$ is <u>i-definable</u> if there is an i - f.a. which defines it.

Definition 2.6. The i - f.a. $\mathcal{M}_1$ is equivalent to the j - f.a. $\mathcal{M}_2$ if $T(\mathcal{M}_1) = T(\mathcal{M}_2)$.

1' - f.a. were studied by Hartmanis and Stearns [4]. 2 - f.a. and 3 - f.a. were introduced by Buchi [1] and McNaughton [6] respectively. In [1], non-deterministic 2 - f.a. were used to obtain a decision procedure for the restricted second-order theory of the structure $\langle N, ' \rangle$ where N is the set of natural numbers and ' is the successor unction on N . In [5], non-deterministic 3 - f.a., non-deterministic 2 - f.a. and deterministic 3 - f.a. are shown to define the same sets. This theorem can be used to simplify Buchi's decision procedure. In [2], a theorem about 3 - f.a. is used together with the results of [1] and [6] to obtain an algorithm for constructing finite automata from specifications given in the restricted second-order language of $\langle N, ' \rangle$ (see [7] for a discussion of these results.).

In [3], the hierarchy result below for 3 - f.a. is presented in a different form and is used to obtain a classification for decision problems for the restricted second-order theory of structures of the form $\langle N, ', Q \rangle$, Q a recursive predicate.

Definition 2.7. For $x \in \Sigma^*$ let $N_x = \{\alpha \mid \alpha \in \Sigma^*, \ x \leq \alpha\}$. $A \subseteq \Sigma^\omega$ is an open set of the product topology if there is a $B \subseteq \Sigma^*$ such that

$$A = \bigcup_{x \in B} N_x .$$

Hence $\{N_x \mid x \in \Sigma^*\}$ is a basis for the product topology on Σ^ω.

Definition 2.8. Let A be an open set. $B \subseteq \Sigma^*$ is a <u>basis for A</u> if $A = \bigcup_{x \in B} N_x$. B is a minimal basis for A if B is a basis and $(\forall x, y \in B)$ $[x \leq y \supset x = y]$.

Let F_0 and G_0 denote the class of subsets of Σ^ω which are both open and closed. F_1 (G_1) is the class of closed (open) sets. F_2 (G_2) is the class of sets which are denumerable unions (intersections) of closed (open) sets. F_3 (G_3) contains denumerable intersections (unions) of sets in F_2 (G_2). Similarly define $F_3, F_4, \cdots$ $(G_3, G_4, \cdots)$.

It is well known that for all i, $F_i \subset F_{i+1}$, $G_i \subset G_{i+1}$ and $F_i \cup G_i \subset F_{i+1} \cap G_{i+1}$. Also each F_i and G_i is closed under finite unions and intersections and $A \in F_i$ if and only if $A^c \in G_i$.

Definition 2.9. $A \subseteq \Sigma^{\omega}$ is a <u>Borel set</u> if it belongs to $\bigcup_i F_i = \bigcup_i G_i$. The hierarchies $F_0, F_1, \ldots$ and $G_0, G_1, \ldots$ comprise the <u>Borel hierarchy</u>.

The complexity of a subset of Σ^{ω} is given by its position in the Borel hierarchy with respect to the product topology on Σ^{ω} .

Intuitively Σ^{ω} is an infinite labelled tree where if $c(\Sigma) = n$, then each vertex has n successor vertices. Points of Σ^{ω} correspond to infinite paths of the tree. Vertices correspond to members of Σ^* . N_x $(x \in \Sigma^*)$ is the set of all paths through the vertex corresponding to x . An open set is the union of all paths through some set of vertices. The reader is urged to use this correspondence as an aid in motivating the proofs.

LEMMA 2.1. A is a member of F_0 (G_0) iff there are $x_1, \cdots, x_n$, such that

$$A = \bigcup_{i=1}^{n} N_{x_i} .$$

Equivalently $A \in F_0$ iff A has a finite minimal basis.

G_2 can be characterized as follows.

LEMMA 2.2. $A \in G_2$ iff there is a $B \subseteq \Sigma^*$ such that

$$\alpha \in A \quad \text{iff} \quad \exists\, x_1 \prec x_2 \prec \cdots, \quad x_i \in B$$
$$\text{and } x_i \prec \alpha \qquad i = 1, 2, \cdots .$$

PROOF.

1. Let $A \in G_2$. Then there are open sets $A_1 \supseteq A_2 \supseteq \cdots$ such that $A = \bigcap_i A_i$. Let B_1 be a minimal basis for A_1. Choose a minimal basis B_2 for A_2 which satisfies $B_2 \cap B_1 = \emptyset$. This is done by first picking a minimal basis $\overline{B}_2$ for A_2. If $x \in B_1 \cap \overline{B}_2$ replace x in $\overline{B}_2$ by $\{x\sigma_1, \cdots x\sigma_n \mid \Sigma = \{\sigma_1, \cdots, \sigma_n\}\}$. B_2 is the modified $\overline{B}_2$. Similarly define minimal bases $B_3, B_4, \cdots$ for $A_3, A_4, \cdots$ respectively where $B_{i+1} \cap \bigcup_{j=1}^{i} B_j = \emptyset$.

Let $B = \bigcup_i B_i$. If $\alpha \in A$, then $\alpha \in A_i$ $i = 1, 2, \cdots$. Hence there are $x_i \in B_i$ $i = 1, 2, \cdots$ such that $x_i \prec \alpha$ (B_i is a basis for A_i) and $x_i \neq x_j$ for $i \neq j$. Choose a subset $\{x_{i_j}\}$ of $\{x_i\}$ so that $x_{i_1} \prec x_{i_2} \prec \cdots$.

If $\exists\, x_1 \prec x_2 \prec \cdots \prec \alpha$, $x_j \in B_{i_j}$, then $\alpha \in \bigcap_j A_{i_j}$. Then $\alpha \in A$ because the A_i's are decreasing.

2. Let $\alpha \in A$ iff $\exists\, x_1 \prec x_2 \prec \cdots$, $x_i \in B$, $x_i \prec \alpha$ $i = 1, 2, \cdots$. Let $C_1 = B$ and define B_i $i = 1, 2, \cdots$ as follows:

$$B_1 = \{x \mid x \in C_1, \ (\forall y)[y \in C_1 \wedge y \precsim x \supset y = x]\}$$

Assume B_i has been defined. Let $C_{i+1} = C_i - B_i$ and

$$B_{i+1} = \{x \mid x \in C_{i+1}, \ (\forall y)[y \in C_{i+1} \wedge y \prec x \supset y = x]\}.$$

Then $B = \bigcup_i B_i$ and each B_i is a minimal basis for an open set A_i. It is easy to show that $A = \bigcap_i A_i$ so $A \in G_2$.

Q.E.D.

Definition 2.10. If $A \in G_2$ and B is as in Lemma 2.1, then B is a G_2-basis for A.

3. HIERARCHY RESULTS

We show that 1-definable sets (1'-definable sets) are in G_1 (F_1); 2-definable (2'-definable) sets are in G_2 (F_2) and; 3-definable sets are in $G_3 \cap F_3$.

THEOREM 3.1. Every 1-definable set is in G_1.

PROOF. Let A be 1-definable. There is a 1 - f.a. $\mathfrak{M} = \langle S, s_0, M, D \rangle$ such that $A = T(\mathfrak{M})$. Let $B = \{x \mid R(x) \in D\}$. Then $A = \bigcup_{x \in B} N_x$ so B is a basis for A and $A \in G_1$.

COROLLARY 3.2. Every 1'-definable set is in F_1.

PROOF. If A is 1'-definable, then A^c is 1-definable.

Corollary 3.2 was proved in [4] for the usual topology on the real line.

THEOREM 3.3 Every 2-definable set is in G_2.

PROOF. Let A be 2-definable. There is a 2 - f.a. $\mathfrak{M} = \langle S, s_0, M, D \rangle$ such that $A = T(\mathfrak{M})$. Let $B = \{x \mid R(x) \in D\}$. Then $A = \{\alpha \mid In(\alpha) \cap D \neq \emptyset\}$ and this is just the set of α's for which there are $x_1 \prec x_2 \prec \cdots$ such that

$x_i \in B$, $x_i \prec \alpha$ $i = 1, 2, \cdots$. Hence B is a G_2-basis for A so $A \in G_2$.

THEOREM 3.4. Every 2'-definable set is in F_2.

PROOF. Let $A = T(\mathfrak{M})$, $\mathfrak{M} = \langle S, s_0, M, \mathfrak{D} \rangle$ a 2' - f.a. . Assume $\mathfrak{D} = \{D\}$. For each $x \in \Sigma^*$ let $A_x = \{\alpha \mid x \prec \alpha, (\forall y)[x \lesssim y \prec \alpha \supset R(y) \in D]\}$. It is easy to see that A_x is closed. Then $A = \bigcup_{x \in \Sigma^*} A_x$ so A is in F_2. If $\mathfrak{D} = \{D_1, \cdots, D_n\}$, then A is a finite union of members of F_2 so A is still in F_2.

The following two theorems were proved in a different form in [3].

THEOREM 3.5. Every 3-definable set is in $G_3 \cap F_3$.

PROOF. Let $A = T(\mathfrak{M})$, $\mathfrak{M} = \langle S, s_0, M, \mathfrak{D} \rangle$ a 3 - f.a. . Assume $\mathfrak{D} = \{D\}$. Then by Theorem 3.4, $A_D = \{\alpha \mid In(\alpha) \subseteq E\}$ is in F_2. For each $E \subset D$, $A_E = \{\alpha \mid In(\alpha) \subseteq E\}$ is also in F_2. Hence $A = A_D \cap (\bigcup_{E \subset D} A_E)^c$ is in the Boolean algebra over F_2 and therefore $A \in F_3 \cap G_3$. If $\mathfrak{D} = \{D_1, \cdots, D_n\}$, then A is a union of members of $F_3 \cap G_3$ so A is in $F_3 \cap G_3$.

In the following assume $\Sigma = \{0, 1\}$. This will simplify the notation.

Definition 3.1. Let $A^\dagger$ consist of those members of Σ^ω in which a finite number of 1's occur. I.e., $A^\dagger = \{\alpha \mid c\{x \mid x \in \Sigma^*, x1 \prec \alpha\} < \omega\}$.

LEMMA 3.1. $A^{\dagger}$ is in F_2 but not in G_2.

PROOF. Assume $A^{\dagger} \in G_2$ with G_2-basis B. Obtain a contradiction by constructing an $\alpha \in \Sigma^{\omega}$ which contains an infinite number of elements of B as initial segments but which is not in $A^{\dagger}$.

Choose n_1 such that $o^{n_1} \in B$. n_1 exists because $o^{\omega} \in A^{\dagger}$. Choose n_2 such that $o^{n_1} 1 o^{n_2} \in B$. n_2 exists because $o^{n_1} 1 o^{\omega} \in A^{\dagger}$. Choose n_3, $n_4, \cdots$ similarly. Let $\alpha = o^{n_1} 1 o^{n_2} 1 o^{n_3} 1 \cdots$. $c\{x \mid x \in B, x \prec \alpha\} = \omega$, but $\alpha \notin A^{\dagger}$. Hence B is not a G_2-basis for $A^{\dagger}$ and $A^{\dagger} \notin G_2$.

$$A^{\dagger^c} \in G_2 \text{ with } G_2\text{-basis } \{x1 \mid x \in \Sigma^*\}.$$

LEMMA 3.2. Let $A^{\neq} = \{\alpha \mid o \prec \alpha, \ \alpha \in A^{\dagger}\} \cup \{\alpha \mid 1 \prec \alpha, \ \alpha \in A^{\dagger^c}\}$. $A^{\neq}$ is in neither G_2 nor F_2.

PROOF. Similar to proof of Lemma 3.1.

It is easy to show

THEOREM 3.6. $A^{\dagger}$ is 2'-definable, $A^{\dagger^c}$ is 2-definable and $A^{\neq}$ is 3-definable.

Theorem 3.6 demonstrates that Theorems 3.3-3.5 give the best possible characterization of 2-, 2'- and 3-definability. In the next section we give procedures for deciding the type set defined by an arbitrary i - f.a. We also show that 2 - f.a. and 2' - f.a. differ from 3 - f.a. only on $G_3 \cap F_3$.

4. ALGORITHMS FOR DETERMINING COMPLEXITY

In this section we give an effective procedure for determining the complexity (with respect to the Borel hierarchy) of a set defined by an arbitrary 3 - f.a. The complexity of sets defined by other types of f.a. can be calculated by first constructing an equivalent 3 - f.a. and then applying the given decision method.

In the following let $\mathfrak{M} = \langle S, s_0, M, \mathfrak{D} \rangle$ be a fixed but arbitrary 3 - f.a.

Definition 4.1. For $x, y \in \Sigma^*$ $x \prec y$, let $\mathfrak{R}(x,y) = \{R(z) \mid x \preceq z \preceq y\}$. For $s \in S$ let $Ac(s) = \{q \mid q \in S, (\exists x)\overline{M}(s,x) = q\}$. Call $Ac(s)$ the <u>set of states accessible from s</u>.

Definition 4.2. For $q \in S$ let $\mathfrak{H}q = \{\mathfrak{R}(x,y) \mid R(x)=R(y)=q, \ x, y \in \Sigma^*\}$

THEOREM 4.1. $T(\mathfrak{M})$ is open iff every non-empty $\mathfrak{H}s$ satisfies,

a) $\mathfrak{H}s \cap \mathfrak{D} = \emptyset$

or b) for all $q \in Ac(s)$, $\mathfrak{H}q \subseteq \mathfrak{D}$.

PROOF.

1. Assume $T(\mathfrak{M}) \in G_1$ and $\mathfrak{H}s \cap \mathfrak{D}$ is non-empty. Then there is an $x \in \Sigma^*$ such that $R(x) = s$ and $N_x \subseteq T(\mathfrak{M})$. Let $q \in Ac(s)$ and $D \in \mathfrak{H}q$. Prove that $D \in \mathfrak{D}$. By the definition of $\mathfrak{H}q$, Ac, there is a β of the form $x\,y\,z^{\omega}$ where $R(x\,y) = q$ and for all i, $\mathfrak{R}(x\,y\,z^i, x\,y\,z^{i+1}) = D$. Since $N_x \subseteq T(\mathfrak{M})$, $D \in \mathfrak{D}$.

2. Assume a) or b) is true for any non-empty $\mathcal{H}s$. Let $B = \{x \mid \mathcal{H}(R(x)) \cap \quad \neq \emptyset\}$. Prove that B is a basis for $T(\mathcal{M})$ so $T(\mathcal{M}) \in G_1$. Assume $T(\mathcal{M}) \neq \emptyset$ (the empty set is open). Let $\alpha \in T(\mathcal{M})$ so $D = In(\alpha) \in \mathfrak{D}$. Choose a $y \prec \alpha$ such that $R(y) \in D$. Then $D \in \mathcal{H}(R(y)) \cap \mathfrak{D}$ so $\mathcal{H}(R(y)) \cap \mathfrak{D}$ is non-empty and $y \in B$.

If $\alpha \in N_y$, $y \in B$, then $\mathcal{H}(R(y)) \cap \mathfrak{D} \neq \emptyset$ so by b) for all $q \in Ac(R(y))$, $\mathcal{H}q \subseteq \mathfrak{D}$. Therefore $In(\alpha) \in \mathfrak{D}$ so $\alpha \in T(\mathcal{M})$ and B is a basis.

Q.E.D.

THEOREM 4.2. $T(\mathcal{M})$ is in G_2 iff for all $s \in S$, $D \in \mathfrak{D} \cap \mathcal{H}s$ and $E \in \mathcal{H}s$ implies $D \cup E \in \mathfrak{D}$.

PROOF.

1. A sume $T(\mathcal{M}) \in G_2$ with G_2-basis B. Let $s \in S$, $D \in \mathfrak{D} \cap \mathcal{H}s$ and $E \in \mathcal{H}s$. Prove $D \cup E \in \mathfrak{D}$. This is done by defining an $\alpha \in T(\mathcal{M})$ for which $In(\alpha) = D \cup E$. Choose x, y_1, w_1, z_1, to satisfy $R(x) = R(xy_1) = R(xy_1w_1) = s$, $\mathfrak{R}(x, xy_1) = D$, $\mathfrak{R}(xy_1, xy_1w_1) = E$, and $z_1 \in B$, $z_1 \prec xy_1$. y_1, w_1 and z_1 exist because $D, E \in \mathcal{H}s$, $D \in \mathfrak{D}$ and B is a G_2 - basis for D. Choose y_2, w_2, z_2 such that $R(xy_1w_1y_2) = R(xy_1w_1y_2w_2) = s$, $\mathfrak{R}(xy_1w_1, xy_1w_1y_2) = D$, $\mathfrak{R}(xy_1w_1y_2, xy_1w_1y_2w_2) = E$ and z_2 satisfies $z_1 \prec z_2 \prec xy_1w_1y_2$, $z_2 \in B$. Similarly choose y_i, w_i, z_i $i = 3, 4, \cdots$. Let $\alpha = xy_1w_1y_2w_2 \cdots$. $\alpha \in T(\mathcal{M})$ because $z_i \in B$, $z_i \prec \alpha$ $i = 1, 2, \cdots$ and B is a G_2 basis for $T(\mathcal{M})$. But $In(\alpha) = D \cup E$ so $D \cup E \in \mathfrak{D}$.

2. Assume for all $s \in S$, $D \in \mathfrak{D} \cap \mathfrak{H}s$, $E \in \mathfrak{H}s$, implies $D \cup E \in \mathfrak{D}$. For $s \in S$ let

$$B_1^s = \{x \mid R(x) = s, \quad (\forall y \prec x)\ R(y) \neq s\}$$

$$B_{i+1}^s = \{x \mid R(x) = s, \quad (\exists y \in B_i^s)\ [y \prec x \wedge \mathfrak{R}(y,x) \in \mathfrak{D} \wedge$$
$$(\forall w)\ [y \prec w \prec x \supset R(w) \neq s \vee \mathfrak{R}(y,w) \notin \mathfrak{D}\ ]]\}$$

Let $B = \bigcup_{s \in S} \bigcup_i B_i^s$. Show B is a G_2 basis for $T(\mathfrak{M})$.

a) Assume there are $x_1 \prec x_2 \prec \cdots$ such that $x_i \prec \alpha$, $x_i \in B$ for $i = 1, 2, \cdots$. Prove $\alpha \in T(\mathfrak{M})$. Choose a subset $y_1 \prec y_2 \prec \cdots$ of $\{x_i\}$ such that $\{y_i\} \subseteq \bigcup_i B_i^s$ for some s. Let $z_1 \prec z_2 \prec \cdots$ be a subset of the $\{y_i\}$ such that $\mathfrak{R}(z_i, z_{i+1}) = In(\alpha)$. But by the definition of the B_i^s sets $\mathfrak{R}(z_i, z_{i+1}) = In(\alpha)$ is a union of sets in $\mathfrak{D} \cap \mathfrak{H}s$ so $In(\alpha) \in \mathfrak{D}$ by the hypothesis and $\alpha \in T(\mathfrak{M})$.

b) Assume $\alpha \in T(\mathfrak{M})$ and prove there are $x_1 \prec x_2 \prec \cdots$ such that $x_i \prec \alpha$, $x_i \in B$ for $i = 1, 2, \cdots$. Let $D = In(\alpha)$ and fix $s \in D$. Then there is an $x_1 \in B_1^s$, $x_1 \prec \alpha$. Assume $x_i \prec \alpha$, $x_i \in \bigcup_j B_j^s$ has been determined and choose x_{i+1} as follows: Let z and y be such that $x_i \prec y \prec z$, $R(z) = R(y) = s$ and $\mathfrak{R}(y, z) = D \in \mathfrak{D}$. Since $\mathfrak{R}(x_i, y)$ and $\mathfrak{R}(y, z)$ are in $\mathfrak{H}s$, the hypothesis implies $\mathfrak{R}(x, z) \in \mathfrak{D}$. Hence by the definition of the B_j^s sets some $\bar{z}$, $x_i \prec \bar{z} \preceq z$ is in $\bigcup_j B_j^s$. Let x_{i+1} be $\bar{z}$.

Q.E.D.

It is easy to see that

LEMMA 4.1. There is an effective procedure for obtaining the $\mathfrak{H}$s and Ac(s) sets from $\mathfrak{M}$.

LEMMA 4.2. If $\mathfrak{M} = \langle S, s_0, M, \mathfrak{D} \rangle$ then $T(\mathfrak{M})^c = T(\langle S, s_0, M, P(S) - \mathfrak{D} \rangle)$.

LEMMA 4.3. Given a 1-, 1'-, 2- or 2'-f.a., an equivalent 3 - f.a. can be effectively obtained.

Theorems 4-1 - 4.2 and Lemmas 4.1 - 4.2 imply

THEOREM 4.3. There is an effective procedure for deciding complexity of $T(\mathfrak{M})$, with respect to the Borel hierarchy, for any 3 - f.a. $\mathfrak{M}$. I.e., we can decide whether $T(\mathfrak{M})$ is in G_1, F_1, G_2, F_2 or $G_3 \cap F_3$.

By Theorem 4.3 and Lemma 4.3

THEOREM 4.4. There is an effective procedure for deciding the complexity of $T(\mathfrak{M})$ for any i - f.a. .

The next two theorems obtain some relationships between the machine types. They show that 2 - and 2'- f.a. differ from 3 - f.a. only on $G_3 \cap F_3$.

THEOREM 4.5. If $A \in G_2$ is 3-definable, then it is 2-definable.

PROOF. Let $T(\mathfrak{M}) \in G_2$, $\mathfrak{M} = \langle S, s_0, M, \mathfrak{D} \rangle$ a 3 - f.a. . A 2 - f.a. $\mathfrak{M}^*$ satisfying $T(\mathfrak{M}) = T(\mathfrak{M}^*)$ is defined as follows:

For each $s \in S$, let $\mathfrak{M}_s$ be a f.a. which for any input sequence α satisfies.

a) $\mathfrak{M}s$ enters a designated state ϵ the first time $\mathfrak{M}$ would enter s in reading α.

b) $\mathfrak{M}s$ reenters ϵ each time and only at such times that the set of states entered by $\mathfrak{M}$ in reading α, since the previous time $\mathfrak{M}s$ was in ϵ, is in $\mathfrak{D}$.

$\mathfrak{M}^*$ is $\langle \mathfrak{M}s_1 \times \cdots \times \mathfrak{M}s_n, \mathfrak{D}^* \rangle$ where $S = \{s_1, \cdots, s_n\}$, $\times$ is the usual product operation on machines and

$$D^* = \{(d_1, \cdots, d_n) \mid d_i \text{ a state of } \mathfrak{M}s_i,\ (\exists j)(d_j = \epsilon)\}$$

is the output condition.

Note that $\mathfrak{M}$ is built into each $\mathfrak{M}s$. It is clear that a finite automaton can be designed to satisfy a) and b). In the following $\text{In}(\alpha)$ and $\mathfrak{H}$s always refer to $\mathfrak{M}$.

1. $T(\mathfrak{M}) \subseteq T(\mathfrak{M}^*)$. Let $\alpha \in T(\mathfrak{M})$ and $\text{In}(\alpha) \in \mathfrak{D}$. Choose $s \in \text{In}(\alpha)$. $\mathfrak{M}s$ enters ϵ the first time $\mathfrak{M}$ enters s, while reading α. This occurs because $s \in \text{In}(\alpha)$.

Assume $\mathfrak{M}s$ enters ϵ for the nth time at time t and let $E_1, E_2, \cdots$ be the sets of states $\mathfrak{M}$ enters between successively entering s after time t ($E_i \neq \emptyset$ $i = 1, 2, \ldots$ since $s \in \text{In}(\alpha)$). There is a finite sequence

$E_j, E_{j+1}, \cdots, E_{j+k}$ such that $In(\alpha) = \bigcup_{\ell=1}^{k} E_{j+\ell}$. But then since $\mathcal{H}s \cap \mathcal{D} \neq \emptyset$, Theorem 4.2 implies that $\bigcup_{\ell=1}^{j+k} E_\ell \in \mathcal{D}$. Hence $\mathcal{M}_s$ enters ϵ an n+1 st time. This proves that if $\alpha \in T(\mathcal{M})$, then some $\mathcal{M}_s$ enters ϵ infinitely often so $\alpha \in T(\mathcal{M}^*)$.

2. $T(\mathcal{M}^*) \subseteq T(\mathcal{M})$. Let $\alpha \in T(\mathcal{M}^*)$ so there is an s such that s enters ϵ infinitely often while reading α . Let $E_i (i = 1, 2, \cdots)$ be the set of states entered by $\mathcal{M}$ between the ith and i + 1 st times $\mathcal{M}$ s enters ϵ . Then $E_i \in \mathcal{H}s \cap \mathcal{D}$ $(i = 1, 2, \cdots)$ by the definition of $\mathcal{M}$ s. In(α) must equal to a finite union of E_i's but since $\mathcal{H}s \cap \mathcal{D} \neq \emptyset$, Theorem 4.2 implies that any finite union of E_i's is in $\mathcal{D}$. Hence $\alpha \in T(\mathcal{M})$.

Q.E.D.

THEOREM 4.6. A is 2-definable iff A^c is 2'-definable.

PROOF. Let $\mathcal{M} = \langle S, s_0, M, D \rangle$ 2-define A . The 2' - f.a. $\langle S, D_0, M, P(S-D) \rangle$ 2' - defines A^c. If $\mathcal{M}$ 2'-defines A first obtain a 3 - f.a. $\mathcal{M}_1$ which defines A . Then modify $\mathcal{M}_1$ to a 3 - f.a. $\mathcal{M}_2$ which 3-defines A^c. $A^c \in G_2$ so by Theorem 4.5 there is a 2-f.a. which defines it.

COROLLARY 4.7. If $A \in F_2$ is 3-definable, then A is 2'-definable.

5. UNDECIDABLE PROBLEMS

In this section we consider Turing machines which define sets of sequences. The model employed is the one-tape on-line Turing machine augmented by the various output conditions of section 2. The problem of classifying the Borel complexity of sets defined by these machines is shown to be unsolvable. We also explore how various decision problems for these machines are related with respect to Turing reducibility.

Definition 5.1. A one-tape on-line Turing machine (T.M.) over the input alphabet Σ and work tape alphabet W is a system $\mathfrak{M} = \langle S, D_0, M \rangle$ with a single 2-way infinite work tape, a one way infinite input tape and a single head for each tape. S is a finite set of states. s_o is a distinguished member of S called the initial state and

$$M: S \times \Sigma \times W \rightarrow S \times W \times \{0, 1, -1\} \times \{0, 1\}$$

where $M(s, \sigma, w) = (s', w', \epsilon_1, \epsilon_2)$ means that if $\mathfrak{M}$ is in state s reading σ and w on its input tape and output tape respectively, then $\mathfrak{M}$: 1. enters s'; 2. writes w' over w on its work tape; 3. shifts the work tape head left, right or not at all depending on whether ϵ_1 is $1, -1$ or 0 and; 4. shifts the input tape head left if ϵ_2 is 1 and not at all if ϵ_2 is 0.

An i-T.M. is a Turing machine augmented by an output of type i (as in definition of an i-f.a.). Definitions 2.4 and 2.5 with T.M. replacing f.a. define 'Turing machine $\mathcal{M}$ i-defines $A \subseteq \Sigma^{\omega}$' and 'A is i-definable by a Turing machine'. In [3], it is shown that every 3-T.M. defines a set in $F_3 \cap G_3$. In fact the method of proof in Theorems 3.1-3.5 is immediately applicable to the class of T.M.'s and indeed to any class of machines augmented by the corresponding output type.

We assume a recursive indexing of the set of all 3-T.M.'s. $\mathcal{M}_x$ is the x-th 3-T.M. under this indexing. Let P(c) stand for the problem of determining whether an arbitrary 3-T.M. defines a set in $c \subseteq P(\Sigma^{\omega})$. Let τ be the class of Turing machines which have a single two-way infinite work-tape and no input tape. For $\mathfrak{T} \in \tau$, $T(\mathfrak{T})$ is the set of finite sequences over Σ on which $\mathfrak{T}$ halts. $\underline{P}_{\phi}$ is the problem of deciding, for an arbitrary $\mathfrak{T} \in \tau$, whether $T(\mathfrak{T})$ is empty. Of course $\underline{P}_{\phi}$ is undecidable. We reduce $\underline{P}_{\phi}$ to problems on 3-T.M.'s to show the latter undecidable. Note that problems on τ are always underlined to distinguish them from problems on 3-T.M.

THEOREM 5.1. $P(G_i)$ is undecidable for i=0,1,2.

PROOF.

1. $P(G_0)$: Let $\mathfrak{T} \in \tau$. $\mathcal{M}_{\mathfrak{T}}$ is a 3-T.M. satisfying: For any input tape $\alpha = \alpha_1\alpha_2\cdots$, $\mathcal{M}_{\mathfrak{T}}$ a) attempts to find a tape in $T(\mathfrak{T})$; b) if a

tape satisfying a) is found, $\mathfrak{M}_{\mathfrak{T}}$ accepts α if and only if α is $000\cdots$; c) if no tape satisfying a) is found, $\mathfrak{M}_{\mathfrak{T}}$ does not accept α (i.e., continues forever to generate members of Σ^* and check if they are in $T(\mathfrak{T})$). Now $\mathfrak{M}_{\mathfrak{T}}$ defines a set in G_0 if and only if $T(\mathfrak{T})$ is empty, so if $P(G_0)$ could be decided so could $\underline{P}_\phi$. Hence $P(G_0)$ is undecidable. The same proof shows $P(G_1)$ is undecidable.

2. $P(G_2)$: Modify b) in 1. to read: if an input tape satisfying a) is found, $\mathfrak{M}_{\mathfrak{T}}$ accepts α if and only if $\alpha \in A^+$ (as in Lemma 3.1).

COROLLARY 5.2. $P(F_i)$ is undecidable for $i=0,1,2$.

Hartmanis and Hopcroft [5] have investigated the relationship of undecidable problems for various types of machines with respect to Turing reducibility. The following theorems compare problems for 3-T.M.'s with problems, on extensions of τ.

τ^* is the class obtained from τ by allowing machines to have an oracle which, given an index x for a machine in τ, decides whether $x \in T(\mathfrak{T}_x)$. If $\underline{P}$ is a problem on τ, then $\underline{P}^*$ is the corresponding problem for τ^*. $P_1 \leq P_2$ means that problem P_1 is (Turing) reducible to P_2. $P_1 \equiv P_2$ if $P_1 \leq P_2$ and $P_2 \leq P_1$. $P_1 < P_2$ if $P_1 \leq P_2$ but $P_2 \not\leq P_1$.

The proof of the next theorem employs the notion of a valid computation of $\mathfrak{T} \in \tau^*$. We define 'valid computation' informally and refer the reader to [5] for a more precise definition.

Definition 5.2. Let $\mathfrak{T} \in \tau^*$. Let $\alpha_0 \# \alpha_1 \# \cdots \# \alpha_n$ be a finite sequence satisfying 1. α_i is an instantaneous description of $\mathfrak{T}$, $0 \leq i \leq n$; 2. α_0 (α_n) is an initial (terminal) instantaneous description of $\mathfrak{T}$; 3. α_{i+1} follows from α_i by the execution of a single step by $\mathfrak{T}$. A valid computation is obtained as follows: If α_{i+1} follows from α_i as a result of a 'yes' answer from the oracle (i.e., the oracle decides that some x is in $T(\mathfrak{T}_x)$), then insert before α_{i+1} a history of the computation of $\mathfrak{T}_x$ which shows $x \in T(\mathfrak{T}_x)$.

Definition 5.3. $\underline{P}_{\mathfrak{F}}$, $\underline{P}_{\Sigma^*}$ and $\underline{P}_R$ are respectively the problems of deciding for an arbitrary $\mathfrak{T} \in \tau$ whether $T(\mathfrak{T})$ is finite, all of Σ^* or recrusive.

THEOREM 5.3. $P(F_i) \equiv P(G_i)$ $i = 0,1,2$.

PROOF. Obvious.

THEOREM 5.4. $\underline{P}_{\mathfrak{F}}^* \leq P(G_i)$ $i = 0,1$.

PROOF.

1. Let $\mathfrak{T} \in \tau^*$. $\mathfrak{M}$ is a 3-T.M. satisfying: $\mathfrak{M}$ accepts $\alpha \in \Sigma^\omega$ if there is an $x\$ \prec \alpha$ such that x is a valid computation of $\mathfrak{T}$ and $\$ \in \Sigma$ does not appear in x. $\mathfrak{M}$ operates by first finding x and then checking whether it is a valid computation. The only difficulty occurs in verifying a 'no' answer by the oracle $(x \notin T(\mathfrak{T}_x))$. To do this $\mathfrak{M}$ simulates $\mathfrak{T}_x$ on x. If x is ever found to be in $\mathfrak{T}_x$, then $\mathfrak{M}$ rejects α. Moreover if in any of its checking $\mathfrak{M}$ discovers that x is not a valid computation, then $\mathfrak{M}$ rejects α.

Otherwise $\mathcal{M}$ cycles in a set of states which result in α being accepted. If $T(\mathfrak{T})$ is finite, there are a finite number of valid computations $x_1, \cdots, x_n$ and $T(\mathcal{M}) = \bigcup_{i=1}^{n} N_{x_i}$ is in G_0. If $T(\mathfrak{T})$ is infinite, there are an infinite number of valid computations $x_1, x_2, \cdots$ and $T(\mathcal{M}) = \bigcup_i N_{x_i}$. Note that if x and y are different valid computations, then $x \not\prec y$ and $y \not\prec y$. It is easy to show that a set of the form $\bigcup_{i=1}^{\infty} N_{x_i}$, $x_i \not\prec x_j$ for $i \neq j$ is not in G_0. Hence $T(\mathfrak{T})$ is finite if and only if $T(\mathcal{M}) \in G_0$.

2. The definitions of 3-T.M., G_0 and G_1 imply $P(G_0) \leq P(G_1)$. Hence $\underline{P}^*_{\mathfrak{T}} \leq P(G_1)$.

COROLLARY 5.5. $\underline{P}^*_{\mathfrak{T}} \leq P(F_2 - F_1)$

PROOF. Modify $\mathcal{M}$ in the proof of Theorem 5.4 to accept α if it is of the form $x \$ 00 \cdots$ where x is a valid computation of $\mathfrak{T}$. Then $T(\mathfrak{T})$ finite implies $T(\mathcal{M}) \in F_1$ and $T(\mathfrak{T})$ infinite implies $T(\mathcal{M}) \in F_2 - F_1$.

THEOREM 5.6. $\underline{P} \leq P_\phi$ (emptiness problem for 3-T.M.)

PROOF. Given $\mathfrak{T} \in \tau$ obtain a 3-T.M. which for any input α: 1. checks each $x \in \Sigma^*$ for membership in $T(\mathfrak{T})$ (of course $\mathfrak{T}$ will be checking more than one x at a given time); whenever an $x \in T(\mathfrak{T})$ is found, $\mathcal{M}$ traverses $D \in_o \mathcal{D}$; 2. while checking x's, $\mathcal{M}$ stays in a subset of D. $T(\mathcal{M})$ is Σ^ω if $T(\mathfrak{T})$ is infinite and $T(\mathcal{M})$ is empty if $T(\mathfrak{T})$ is finite.

It is well known [5,8] that

$$\underline{P}_{\phi} < \underline{P}_{\mathcal{J}} \equiv \underline{P}_{\Sigma^*} < \underline{P}_{\mathcal{J}}^{\,*} \equiv \underline{P}_{R}.$$

Our results show that problems such as $P(F_0)$ are at least as difficult as $\underline{P}_R$. We have not been able to prove that $\underline{P}_{\mathcal{J}}^* < P(F_0)$. A general open problem is that of characterizing the degrees of unsolvability of problems on 3-T.M. as well as their relationship to problems on τ.

REFERENCES

1. Buchi, J. R. On a decision procedure in restricted second-order arithmetic. Proc. Int. Cong. Logic, Method. and Philos. Sci. 1960, Stanford Univ. Press, 1962.

2. Buchi, J. R. and Landweber, L. H. Solving sequential conditions by finite state strategies. Purdue University Report CSD TR 14 (Sept. 1967).

3. Buchi, J. R. and Landweber, L. H. Definability in the monadic second-order theory of successor. Purdue University Report CSD TR 15 (Sept. 1967).

4. Hartmanis J. and Stearns, R. E. Sets of numbers defined by finite automata. Amer. Math. Monthly 74 (May 1967), 539-542.

5. Hartmanis J. and Hopcroft S. E. Structure of undecidable problems in automata theory, unpublished.

6. McNaughton, R. Testing and generating infinite sequences by a finite automation. Inf. and Control 9 (1966), 521-530.

7. Landweber, L. H. Synthesis algorithms for sequential machines. To appear in Proceedings of IFIP Congress 68.

8. Rogers Jr., H. <u>Theory of Recrusive Functions and Effective Computability</u>. McGraw-Hill, New York. 1967.