CHARACTERIZATIONS OF REAL MATRICES OF MONOTONE KIND

by

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An m by n real matrix A is said to be of <u>monotone kind</u> if

(1) $$Ax \geqq 0 \implies x \geqq 0 .$$

Collatz [2] treats square matrices of monotone kind and shows that for such matrices the above implication is equivalent to: A^{-1} exists and $A^{-1} \geqq 0$.[3] Matrices of monotone kind have useful applications in numerical analysis [2, 7].

It is the purpose of this note to generalize Collatz's result to rectangular matrices, and also to show that, for the general rectangular case, a matrix of monotone kind can be further characterized as one for which the convex conical hull of the rows contains the nonnegative orthant.

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[3] That is, each element of A^{-1} is nonnegative.

(For an m by n matrix A, the convex conical hull of the rows of A is defined as

$$K(A) = \{z \mid z = A^T u,\ u \geqq 0\}.$$

The nonnegative orthant E^n_+ is defined by

$$E^n_+ = \{x \mid x \in E^n,\ x \geqq 0\},$$

where E^n is the n-dimensional real Euclidean space.)

Theorem 1. Let A be an m by n real matrix. Then the following two statements are equivalent:

(2) A has a nonnegative left inverse. In other words, there exists an n by m matrix $Y \geqq 0$ such that $YA = I$.

(3) $$K(A) \supset E^n_+$$

Proof. Clearly (2) holds if and only if each row I_i of the identity matrix I of order n is a nonnegative linear combination of the rows of A. But this is equivalent to the statement that each unit vector is contained in K(A), which is the case if and only if (3) holds. Q.E.D.

Of course, if A is square, either (2) or (3) is equivalent to A being nonsingular and $Y = A^{-1}$ being nonnegative.

It can be shown by elementary arguments that (1) and (2) are equivalent for a square matrix A, and that (2) implies (1) for a general rectangular

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It can be shown by elementary arguments that (1) and (2) are equivalent for a square matrix A , and that (2) implies (1) for a general rectangular

matrix A . The proof that (1) implies (2) for a general rectangular A seems to require the use of either the duality theory of linear programming or a theorem of the alternative for linear inequalities, such as Motzkin's theorem [4, 5,8]. (Theorems of the alternative may be considered a consequence of the separation theorem for convex sets [1].)

<u>Theorem 2.</u> For any m by n real matrix A , (1) and (2) are equivalent.

<u>Proof.</u> If (2) holds, then $Ax \geqq 0$ implies that $x = YAx \geqq 0$, and (1) is established.

If (1) holds, then A must be of rank n . For, $Ax = 0$ implies that $Ax \geqq 0$ and $A(-x) \geqq 0$, and hence by (1), $x = 0$, and the rank of A is $n \leqq m$.

Thus if (1) holds and A is square $(m = n)$, it is nonsingular, and (1) together with $AA^{-1} = I \geqq 0$ imply that $A^{-1} \geqq 0$.

For $m \geqq n$ a different argument is required. We note that $Ax \geqq 0$, $I_i x < 0$ has <u>no</u> solution for each $i = 1, \ldots, n$. By Motzkin's theorem [4, 5, 8] it follows that $yA = I_i$, $y \geqq 0$ has a solution for each i , and (2) follows. Q.E.D.

An alternate proof that (1) implies (2) may be based on the duality theory of linear programming [6] instead of on Motzkin's theorem. If (1) holds then

$$\underset{x}{\text{minimum}} \quad \{I_i x \mid Ax \geqq 0\} = 0 \quad \text{for each } i = 1, \ldots, n .$$

By the duality theory of linear programming [6]

$$\underset{y}{\text{maximum}} \quad \{0\,y \mid yA = I_i\,, \quad y \geqq 0\} = 0 \quad \text{for each } i = 1, \ldots, n,$$

where the zero denotes an m vector of zeros. Hence for each $i = 1, \ldots, n$, $yA = I_i$, $y \geqq 0$, has a solution. This establishes (2).

Remark. For square matrices, because $(A^{-1})^T = (A^T)^{-1}$, it follows from (2) above that any of the statements (1), (2) or (3) above is equivalent to any of the three statements below:

(1°) $A^T y \geqq 0 \implies y \geqq 0$.

(2°) $(A^T)^{-1}$ exists and $(A^T)^{-1} \geqq 0$.

(3°) $K(A^T) \supset E^n_+$.

Rectangular Matrices of Monotone Kind with Respect to Another Matrix: Let A be an m by n real matrix and let B be a k by n real matrix. Then the following are equivalent:

(1") $Ax \geqq 0 \implies Bx \geqq 0$

(2") $YA = B\,, \quad Y \geqq 0$

(3") $K(A) \supset K(B)$

The equivalence of the above three statements is established by replacing I by B or B^T in the proofs of Theorems 1 and 2 (omitting in the latter case, the demonstration that A is of full rank and the special argument for non-singular A).

Finally it should be remarked that if we define the polar cone of the rows of a matrix A as

$$P(A) = \{x \mid Ax \geqq 0\} ,$$

then (1") above can be stated as

$$(1") \qquad P(A) \subset P(B) .$$

The equivalence of (1") and (3") follows then directly from the duality theorem for polyhedral convex cones of Goldman and Tucker [3, lemma 2].

Example. Consider the following m by 2 matrix ($m \geqq 2$)

$$A = \begin{bmatrix} r_1 \cos \theta_1 & r_1 \sin \theta_1 \\ \vdots & \vdots \\ r_m \cos \theta_m & r_m \sin \theta_m \end{bmatrix} ,$$

where $r_i \geqq 0$, $-\pi \leqq \theta_i \leqq \pi$, for $i = 1, \ldots, m$. Our necessary and sufficient condition (3) (that A be of monotone kind (1) or have a nonnegative left inverse (2)) becomes this: there exist i, j, $i \neq j$, such that for all $k \neq i$, $k \neq j$ ($1 \leqq i, j, k \leqq m$) we have that

$$r_i > 0, \quad r_j > 0 , \quad \theta_j \leqq \theta_k \leqq \theta_i$$

$$\frac{\pi}{2} \leqq \theta_i - \theta_j < \pi , \quad -\frac{\pi}{2} < \theta_j \leqq 0 , \quad \frac{\pi}{2} \leqq \theta_i < \pi .$$

If A is a 2 by 2 matrix, then $i = 1$ or 2, $j = 1$ or 2, $i \neq j$, and the above condition is necessary and sufficient for A^{-1} to exist and $A^{-1} \geqq 0$. We have then

$$A^{-1} = \frac{1}{\sin(\theta_2 - \theta_1)} \begin{bmatrix} \dfrac{\sin\theta_2}{r_1} & \dfrac{-\sin\theta_1}{r_2} \\ \\ \dfrac{-\cos\theta_2}{r_1} & \dfrac{\cos\theta_1}{r_2} \end{bmatrix} \geqq 0 .$$

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