CS 559: Computer Graphics

Homework 1

This homework must be done individually. Submission date is Tuesday, February 3, 2004, in class.

Encouragement: Some of these questions may look a little difficult; if they do, it would be worth reviewing how dot products, cross products and determinants are formed. An excellent reference for this material is the textbook or the appendices of "Real Time Rendering" by Moller and Haines.

Question 1: Vectors are extremely important to computer graphics. They are used to represent both locations in space (points) and directions. Assume you have three points in space, represented by $\mathbf{a} = (a_x, a_y, a_z)$, $\mathbf{b} = (b_x, b_y, b_z)$ and $\mathbf{c} = (c_x, c_y, c_z)$.

a. How do you find the direction vector $\mathbf{v}$ that points *from* $\mathbf{a}$ *toward* $\mathbf{b}$?

b. How is the length, $\|\mathbf{v}\|$, of $\mathbf{v}$ computed?

c. A *unit vector*, $\hat{\mathbf{v}}$, in the direction $\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$ but with length 1. How do you compute $\hat{\mathbf{v}}$? Computing $\hat{\mathbf{v}}$ is also referred to as *normalizing* $\mathbf{v}$.

d. If you wanted to rapidly determine which point, $\mathbf{b}$ or $\mathbf{c}$, was closer to $\mathbf{a}$, which quantities would you compare? Why? (Hint: We are not concerned about the actual distances, only about which point is closer.)

Question 2: Consider two vectors in 3D, $\mathbf{a}$ and $\mathbf{b}$.

a. How is the dot product $\mathbf{a} \cdot \mathbf{b}$ computed? The dot product is more generally called the inner product.

b. What is the relationship between $\mathbf{a} \cdot \mathbf{b}$ and the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

c. For this part of the question, assume that $\mathbf{a}$ and $\mathbf{b}$ are *unit vectors* – that is, their length is 1. What is the value of $\mathbf{a} \cdot \mathbf{b}$ if:

   (i) $\mathbf{a}$ and $\mathbf{b}$ point in the same direction?

   (ii) $\mathbf{a}$ and $\mathbf{b}$ point in opposite directions?

   (iii) $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* (at right angles)?

d. How can you write $\|\mathbf{a}\|$ in terms of a dot product?

Question 3: Consider two vectors in 3D, $\mathbf{a}$ and $\mathbf{b}$.

a. How is the cross product vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ computed?

b. What is the geometric relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$?

c. What is the relationship between $\mathbf{c}$ and the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

d. For this part of the question, assume that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. What is the length of $\mathbf{c}$ if:

   (i) $\mathbf{a}$ and $\mathbf{b}$ point in the same direction?

   (ii) $\mathbf{a}$ and $\mathbf{b}$ are orthogonal?

More over . . .

Question 4: Consider three points in 2D, (x_1, y_1), (x_2, y_2) and (x_3, y_3).

a. Show that the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

is proportional to the area of the triangle whose corners are the three points. (Hint: The area of a triangle in terms of its corners is $A = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$.)

b. What is the value of the determinant if the points lie on a straight line? (Hint: What is the area of the triangle?)

c. The *implicit* equation of a line in the plane is $ax + by + c = 0$, where (x, y) is a point on the line and a, b and c are constant for a given line. Given two points on the plane, (x_1, y_1) and (x_2, y_2), show how to find the values of a, b, c for the line that passes through those two points. (Hint: Every point (x, y) on the line must be collinear with (x_1, y_1) and (x_2, y_2). So use the result from part (b).)

Question 5: Let $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ be three non-collinear points in space. A plane can always be defined to pass through three such points.

a. Consider the cross product $\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)$. What does the vector n mean geometrically?

b. Let $\mathbf{p}$ be any point on the plane formed by $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$. What is the geometric relationship between $\mathbf{n}$ and $\mathbf{p} - \mathbf{p}_0$? (Hint: All points on the plane satisfy the implicit equation $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$.)

c. Typically, the equation of a plane is written as $ax + by + cz + d = 0$, where $\mathbf{p} = (x, y, z)$. What are the values of a, b, c and d in terms of $\mathbf{n} = (n_x, n_y, n_z)$ and $\mathbf{p}_0 = (p_x, p_y, p_z)$? (Hint: You can find the answer by expanding the implicit equation above.)