CS 559: Computer Graphics

Homework 1

This homework must be done individually. Submission date is Tuesday, February 5, 2002, in class.

Pep-talk: Some of these questions may look a little difficult; if they do, it would be worth reviewing how dot products and cross products are formed, how matrices are multiplied with vectors and with each other, and how determinants are formed.

Question 1: How is the length, $\|\mathbf{a}\|$, of a 3D vector $\mathbf{a} = (a_x, a_y, a_z)$ computed?

Question 2: Consider two vectors in 3D, $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$.

a. How is the dot product $\mathbf{a} \cdot \mathbf{b}$ computed? The dot product is more generally called the inner product.

b. What is the relationship between $\mathbf{a} \cdot \mathbf{b}$ and the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

c. For this part of the question, assume that $\mathbf{a}$ and $\mathbf{b}$ are *unit vectors* – that is, their length is 1. What is the value of $\mathbf{a} \cdot \mathbf{b}$ if:

   (i) $\mathbf{a}$ and $\mathbf{b}$ point in the same direction?

   (ii) $\mathbf{a}$ and $\mathbf{b}$ point in opposite directions?

   (iii) $\mathbf{a}$ and $\mathbf{b}$ are at right angles (or orthogonal)?

d. How can you write $\|\mathbf{a}\|$ in terms of a dot product?

Question 3: Consider two vectors in 3D, $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$.

a. How is the cross product vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ computed?

b. What is the geometric relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$?

c. What is the relationship between $\mathbf{c}$ and the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

d. For this part of the question, assume that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. What is the length of $\mathbf{c}$ if:

   (i) $\mathbf{a}$ and $\mathbf{b}$ point in the same direction?

   (ii) $\mathbf{a}$ and $\mathbf{b}$ are at right angles?

e. Hence, under what circumstances is $\mathbf{c}$ ill-defined?

More over …

Question 4: Consider three points in 2D, (x_1, y_1), (x_2, y_2) and (x_3, y_3).

a. Show that the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

is proportional to the area of the triangle whose corners are the three points. Hint: The area of a triangle in terms of its corners is $A = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)]$.

b. What is the value of the determinant if the points lie on a straight line? Hint: What is the area of the triangle?

c. The equation of a line in the plane is $ax + by + c = 0$, where (x, y) is a point on the line and a, b and c are constant for a given line. Given two points on the plane, (x_1, y_1) and (x_2, y_2), show how to find the values of a, b, c for the line that passes through those two points. Hint: Every point (x, y) on the line must be collinear with (x_1, y_1) and (x_2, y_2). So use the result from part (b).

Question 5: Let $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ be three non-collinear points in space. A plane can always be defined to pass through three such points.

a. Consider the cross product $\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)$. What does the vector n mean geometrically?

b. Let $\mathbf{p}$ be any point on the plane formed by $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$. What is the geometric relationship between $\mathbf{n}$ and $\mathbf{p} - \mathbf{p}_0$?

c. Hence, why is the equation of the plane $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$.

d. Typically, the equation of a plane is written as $ax + by + cz + d = 0$, where $\mathbf{p} = (x, y, z)$. What are the values of a, b, c and d in terms of $\mathbf{n} = (n_x, n_y, n_z)$ and $\mathbf{p}_0 = (p_x, p_y, p_z)$?