CS 559: Computer Graphics

Homework 1

This homework must be done individually. Submission date is Tuesday, September 17, 2002, in class.

Encouragement: Some of these questions may look a little difficult; if they do, it would be worth reviewing how dot products, cross products and determinants are formed.

Question 1: Vectors are extremely important to computer graphics. In fact, you can't do graphics without them. Assume you have three points in space, $\mathbf{a} = (a_x, a_y, a_z)$, $\mathbf{b} = (b_x, b_y, b_z)$ and $\mathbf{c} = (c_x, c_y, c_z)$.

a. How do you find the vector $\mathbf{v}$ that points *from* $\mathbf{a}$ *toward* $\mathbf{b}$?

b. How is the length, $\|\mathbf{v}\|$, of $\mathbf{v}$ computed?

c. If you wanted to determine which point, $\mathbf{b}$ or $\mathbf{c}$, was closer to $\mathbf{a}$, which quantities would you compare? Why? (Hint: We are not concerned about the actual distances, only about which point is closer.)

Question 2: Consider two vectors in 3D, $\mathbf{a}$ and $\mathbf{b}$.

a. How is the dot product $\mathbf{a} \cdot \mathbf{b}$ computed? The dot product is more generally called the inner product.

b. What is the relationship between $\mathbf{a} \cdot \mathbf{b}$ and the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

c. For this part of the question, assume that $\mathbf{a}$ and $\mathbf{b}$ are *unit vectors* – that is, their length is 1. What is the value of $\mathbf{a} \cdot \mathbf{b}$ if:

   (i) $\mathbf{a}$ and $\mathbf{b}$ point in the same direction?

   (ii) $\mathbf{a}$ and $\mathbf{b}$ point in opposite directions?

   (iii) $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* (at right angles)?

d. How can you write $\|\mathbf{a}\|$ in terms of a dot product?

Question 3: Consider two vectors in 3D, $\mathbf{a}$ and $\mathbf{b}$.

a. How is the cross product vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ computed?

b. What is the geometric relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$?

c. What is the relationship between $\mathbf{c}$ and the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

d. For this part of the question, assume that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. What is the length of $\mathbf{c}$ if:

   (i) $\mathbf{a}$ and $\mathbf{b}$ point in the same direction?

   (ii) $\mathbf{a}$ and $\mathbf{b}$ are orthogonal?

More over ...

Question 4: Consider three points in 2D, (x_1, y_1), (x_2, y_2) and (x_3, y_3).

a. Show that the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

is proportional to the area of the triangle whose corners are the three points. (Hint: The area of a triangle in terms of its corners is $A = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$.)

b. What is the value of the determinant if the points lie on a straight line? (Hint: What is the area of the triangle?)

c. The equation of a line in the plane is $ax + by + c = 0$, where (x, y) is a point on the line and a, b and c are constant for a given line. Given two points on the plane, (x_1, y_1) and (x_2, y_2), show how to find the values of a, b, c for the line that passes through those two points. (Hint: Every point (x, y) on the line must be collinear with (x_1, y_1) and (x_2, y_2). So use the result from part (b).)

Question 5: Let $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ be three non-collinear points in space. A plane can always be defined to pass through three such points.

a. Consider the cross product $\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)$. What does the vector n mean geometrically?

b. Let $\mathbf{p}$ be any point on the plane formed by $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$. What is the geometric relationship between $\mathbf{n}$ and $\mathbf{p} - \mathbf{p}_0$? (Hint: All points on the plane satisfy the *implicit* equation $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$.)

c. Typically, the equation of a plane is written as $ax + by + cz + d = 0$, where $\mathbf{p} = (x, y, z)$. What are the values of a, b, c and d in terms of $\mathbf{n} = (n_x, n_y, n_z)$ and $\mathbf{p}_0 = (p_x, p_y, p_z)$? (Hint: You can find the answer by expanding the implicit equation above.)