Numerical Analysis Depth Examination Spring 1996

You should answer five out of the eight problems below.

·•1•· You are to approximate

$$\int_{-1}^{1} \frac{e^{-x^2}}{\sqrt{|x|(1-x)(1+x)}}\mathrm{d}x$$

based on *four* evaluations of the integrand. What are the four points of your choice? Explain.

·•2•· Eudoxos proposed the following algorithm for the generation of increasingly accurate rational approximations to the length of the diagonal of the unit square: Starting with $p_0 = 1 = q_0$, generate p_{n+1} as $p_n + q_n$, and q_{n+1} as $p_{n+1} + p_n$, $n = 0, 1, 2, \ldots$. Then q_n/p_n converges to $\sqrt{2}$.

(a) Explain Eudoxos' method in terms of the power method.

(b) Explain how you would estimate the n for which q_n/p_n is within 10^{-6} of $\sqrt{2}$.

·•3•· Consider the boundary-value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 + y_1 \\ y_1 y_4 + x \end{bmatrix}, \qquad 0 \le x \le 1,$$

subject to the boundary conditions

$$y_1(0) = y_3(0) = 0, \quad y_1(1) = y_2(1) = 0.$$

Discuss the "shooting" method for the numerical solution of this problem.

·•4•· Consider the initial-value problem

$$y' = 2(1-x^2)y, \quad y(0) = \frac{1}{2}, \quad 0 \le x \le 4$$

and the numerical method

$$\left\{ \begin{array}{rcl} Y_n &=& Y_{n-2} + 2h[2(1-x_{n-1}^2)]\,Y_{n-1} \\ x_j &=& jh, \quad h = \dfrac{1}{40} \\ Y(0) &=& \dfrac{1}{2}, \quad Y_1 \;=\; \dfrac{1}{2} + h \end{array} \right\}$$

(a) Discuss the convergence or non-convergence of this method.

(b) Discuss the behavior of the error.

·• 5 •· Consider the Crank-Nicolson scheme applied to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in \mathbb{R}$ and $t > 0$. Determine conditions on $\Delta t/(\Delta x)^2$ such that the solution v_m^n satisfies

$$\max_m |v_m^{n+1}| < \max_m |v_m^n|$$

for $n > 0$ and all initial data with compact support.

·• 6 •· Show that any finite difference scheme of the form

$$v_m^{n+1} = \sum_{j=0}^{L} a_j v_{m+j}^n$$

that is consistent with

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is unstable.

·• 7 •· Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Assuming that A is invertible (while C evidently is not invertible), what conclusion can you draw from this concerning $\|A^{-1}\|_p$ for $p = 1, 2, \infty$? How? Why?

·• 8 •· (a) Determine the norm of the linear functional $\lambda : \Pi_2 \to \mathbb{R} : p \mapsto Dp(0)$ of differentiation at 0, where Π_2 is the normed linear space of all polynomials of degree ≤ 2 with norm the 2-norm on the interval $[-1\,..\,1]$.

(b) How would you go about determining $\|\lambda\|$ if the norm on Π_2 were the max-norm on $[-1\,..\,1]$ instead?