Numerical Analysis   Depth Examination   Spring 1995

You should answer five out of the eight problems below.

·• 1 •· A system $Ax = b$ of equations is given, where $b := [1, 1]^T$, and

$$A := \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix},$$

and α is a real parameter. We try to solve this problem by using iterative methods, with an initial guess $x_0 = [1, 1]^T$.

(a) State a necessary and sufficient condition on α that is equivalent to the convergence of the Jacobi iterative method.

(b) Discuss the following statement: "For this example, the Gauss-Seidel method converges faster than the Jacobi method."

(c) How do your answers in (a) and (b) change, if the initial guess x_0 is changed to $[0, 0]^T$?

(d) Prove your statement in (a). In your proof, you may use any convergence theorem of iterative methods that you are aware of. If you use such a theorem, state that theorem within your proof.

·• 2 •· This question deals with best approximation by Π_n (polynomials of degree $\le n$) to continuous functions in the uniform (=max) norm.

(a) Given a continuous f on the interval $[a\,..\,b]$, state and prove a condition on $p \in \Pi_n$ that guarantees it to be the best approximation to f. Your condition should be in terms of the oscillatory behaviour of the error $f - p$.

(b) Determine a minimal $b(n)$ such that the best approximation from Π_n to $f : x \mapsto \sin x$ on the interval $[0\,..\,b(n)]$ is the zero function.

(c) Weierstraß' Theorem asserts that every continuous function can be approximated arbitrarily well by polynomials. How could such an assertion co-exist with the observation made in (b) above?

·• 3 •·

(a) State Taylor's formula with integral remainder (for the Taylor expansion, truncated after terms of degree $< k$, of some function f at some point a) and interpret it as a pointwise error formula for the Hermite interpolant from $\Pi_{<k}$ to f at a.

(b) Where in the error formula in (a) does there appear a right inverse for the linear map $f \mapsto D^k f$ of k-fold differentiation, and what is it?

(c) Give an abstract version of the above formula, involving some bounded linear projector P (instead of Hermite interpolation) on some normed linear space X, and some bounded linear map $E : X \to Y$ with $\operatorname{ran} P = \ker E$ (instead of D^k).

·• 4 •·

(a) Define what it means for a linear map, from some normed linear space to some normed linear space, to be *bounded below*, and relate it to the property of the map having a bounded inverse.

(b) Give an example of a bounded linear map which is bounded below yet not invertible.

(c) Give an example of a bounded linear map which is invertible yet not bounded below.

(d) State the basic theorem in Functional Analysis which implies that a bounded linear map, from some Banach space to some Banach space, is boundedly invertible if and only if it is 1-1 and onto.

·• 5 •· Consider the Crank-Nicolson scheme applied to $u_t = u_{xx}$ for $x \in [0..1]$ and $t > 0$. Determine conditions on $\Delta t/(\Delta x)^2$ such that if solutions v_m^n and w_m^n satisfy

$$v_m^n \le w_m^n$$

for $n = 0$ and at the boundaries for $n \ge 0$, then

$$v_m^n \le w_m^n$$

in the interior as well.

(For v_m^n and w_m^n, the superscript is the time level and the subscript is the index of the spatial grid point.)

·• 6 •· Consider the finite difference scheme

$$v_m^{n+1} = \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right) - \frac{\lambda}{1+\lambda^2}\left(v_{m+1}^n - v_{m-1}^n\right)$$

for the equation $u_t + u_x = 0$ where $\lambda = \Delta t/\Delta x$ and v_m^n is an approximation to $u(t_n, x_m)$.

(a) Show that this scheme is stable for all values of λ.

(b) For the initial value problem, discuss the convergence of the solution of the scheme to the solution of the differential equation.

·• 7 •· Consider the initial-value problem

$$\begin{aligned} y' &= f(t,y), \qquad 0 \le t \le T, \\ y(0) &= y_0. \end{aligned}$$

For its numerical solution, consider a Predictor-Corrector numerical method of the following form:

$$\begin{aligned} \widetilde{Y} &= \sum_{j=1}^{k} a_j Y_{n-j} + h \sum_{j=1}^{k} b_j\, f(t_{n-j}, Y_{n-j}) \\ Y_n &= \sum_{j=1}^{k} \alpha_j Y_{n-j} + h\beta_0 f(t_n, \widetilde{Y}) + h \sum_{j=1}^{k} \beta_j\, f(t_{n-j}, Y_{n-j}), \end{aligned}$$

with $Y_0, Y_1, \ldots, Y_{k-1}$ converging to y_0 as $h \to 0$.

Discuss the conditions on the coefficients $(a_j, b_j, \alpha_j, \beta_j : j = 1, \ldots, k)$ and β_0 which are necessary and sufficient for the stable convergence of this method.

·• 8 •· Consider a boundary-value problem of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = f(u_1, u_2, u_3, u_4) \qquad 0 \le t \le 1,$$

$$u_1(0) = 0 = u_2(0), \qquad u_1(1) = 0 = u_2(1).$$

Describe the application of the "Shooting Method" to the numerical solution of this problem.