Numerical Analysis

A student taking the exam for "breadth" must answer three (3) questions. A student taking the exam for "depth" must answer six (6) questions, with at least three (3) chosen from problems 6-10. In either case, the answers to three questions must be turned in after 2 hours and these will be the only questions looked at for the "breadth" grade.

1.

(a) Define a Householder matrix.

(b) Prove that a Householder matrix H is (i) orthogonal (ii) symmetric and (iii) idempotent (i.e., $H^2 = I$).

(c) Prove or disprove the following claim: every matrix which is orthogonal, symmetric and idempotent is a Householder matrix.

(d) How do you efficiently multiply two Householder matrices?

(e) Is the product of two Householder matrices (i) orthogonal? (ii) symmetric? (iii) idempotent? Explain.

2.

(a) Define the Gauss-Seidel method for solving the $n \times n$ linear system $Ax = b$.

(b) Determine whether the Gauss-Seidel method converges for the system of equations

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

3.

You are given a polynomial p in Newton form

$$p(t) = \sum_{i=0}^{n} a_i \prod_{j=1}^{i} (t - c_j).$$

You are asked to find the coefficients $\{a'_0, ..., a'_n\}$ in the representation of the above p with respect to another (given) set of centers $\{c'_1, ..., c'_n\}$, that is, you need to find $\{a'_0, ..., a'_n\}$ for which

$$p(t) = \sum_{i=0}^{n} a'_i \prod_{j=1}^{i} (t - c'_j). \qquad (*)$$

(a) Suggest an algorithm that computes the required $\{a'_i\}_i$ and involves $O(n^2)$ arithmetic operations. Prove that your algorithm does the job, i.e., that the output $\{a'_0, ..., a'_n\}$ of your algorithm satisfies (*).

(b) Discuss two different applications of this algorithm.

4.

(a) Prove that Simpson's rule is exact for cubic polynomials and conclude from this that the error in the composite Simpson's rule is $O(h^4)$ when applied to any sufficiently smooth function.

(b) Prove that the composite Simpson's rule converges for all continuous functions.

5.

Let $\|\cdot\|$ be the uniform (i.e., max) norm over some given interval $[a, b]$. Suppose that f is continuous on $[a, b]$, p is a polynomial of degree at most n, and $x_0 < x_1 < ... < x_{n+1}$ are points in the interval such that

$$(-1)^j(f(x_j) - p(x_j)) = \|f - p\|, \quad j = 0, ..., n + 1.$$

Prove that for any polynomial q of degree at most n,

$$\|f - p\| \leq \|f - q\|.$$

6. You are given the initial value problem

$$\begin{array}{l} y_1'(t) = y_2(t) \\ y_2'(t) = \sin y_1(t) \end{array}, \quad \begin{array}{l} y_1(0) = 0 \\ y_2(0) = 1 \end{array}, \qquad 0 \leq t \leq 1.$$

(a) Write in detail the Euler method for this problem.
(b) Prove that this method converges to the solution of the above problem.

7. Consider the following two initial-value problems:

$$\begin{aligned} y' &= -y^3 - 2x^2 y =: f(x, y), \qquad 0 \leq x \leq 5 \\ y(0) &= 1 \end{aligned}$$

$$\begin{aligned} y' &= y^3 + 2x^2 y =: f(x, y), \qquad 0 \leq x \leq 5 \\ y(0) &= 1 \end{aligned}$$

Discuss the numerical solution of these problems using the linear two-step method

$$\begin{aligned} Y_{n+1} &= Y_{n-1} + 2hf(X_n, Y_n), \\ Y_0 &= 1, \\ h &= 1/20. \end{aligned}$$

In particular, discuss (i) convergence, as $h \to 0$, (ii) validity of the approximation for this fixed h (i.e., 1/20), (iii) behavior of the error.

8.

(a) State (nontrivial) sufficient conditions for the convergence of fixed-point iteration in a metric space.

(b) Use (a) to prove that there is exactly one $C^{(1)}$-function x on the interval $[0,1]$ which satisfies

$$
\begin{aligned}
&x'(t) = \sqrt{|x(t)|} \quad \text{for } 0 \le t \le 1, \\
&x(0) = 1.
\end{aligned}
$$

(c) Would your argument still work if the interval were changed to $[0,5]$?

9. Consider the boundary value problem

$$
(A) \qquad
\begin{aligned}
&u'' + \pi^2 u = 1, \quad 0 < x < 1, \\
&u(0) = u(1) = 0,
\end{aligned}
$$

and the numerical approximation

$$
(B) \qquad
\begin{aligned}
&U_{n+1} + (\pi^2 h^2 - 2)U_n + U_{n-1} = h^2, \quad n = 1, 2, ..., M, \\
&U_0 = 0,\ U_{M+1} = 0, \quad h = \frac{1}{M+1}.
\end{aligned}
$$

(a) Discuss the existence and uniqueness of solutions to Problem (A).

(b) Discuss the existence and uniqueness of solutions to Problem (B), for $0 < h < .1$.

(c) Discuss the behavior of the solutions of Problem (B) as $h \to 0$. Here, "behavior" refers to aspects such as boundedness, positivity etc.

10. Consider an initial value problem whose PDE is

$$u_t - u_{txx} - b u_{xx} = 0,$$

and the scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - \frac{\delta^2 u_m^{n+1} - \delta^2 u_m^n}{\Delta t} - \frac{b}{2}(\delta^2 u_m^{n+1} + \delta^2 u_m^n) = 0,$$

with

$$\delta^2 u_m := \frac{u_{m+1} - 2u_m + u_{m-1}}{\Delta x^2}.$$

Discuss the stability and accuracy of the scheme.