Numerical Analysis Spring 1991

A student taking the exam for "breadth" must answer three (3) questions to be handed in after two (2) hours. A student taking the exam for "depth" must answer six (6) questions, with at least three (3) chosen from problems 6–10.

1. Give the largest possible interval I you can determine so that, for any $a \in I$, the iteration

$$x_{n+1} := a x_n (1 - x_n), n = 0, 1, 2...$$

converges for any $x_0 \in [0 \,..\, 1]$.

2. Let A be a $n \times n$ real matrix whose eigenvalues have only simple algebraic multiplicities. In order to find an eigenvector of A, we apply the power method,

$$x^{k+1} = A x^k,$$

where x^0 is some initial guess. By definition, the method converges to a vector v, if there exist scalars $\alpha_0, \alpha_1, ...$ such that

$$\frac{\|x_k - \alpha_k v\|}{\|x_k\|} \to 0 \text{ as } k \to \infty,$$

for some norm, $\|\cdot\|$.

We tried this power method with two different initial guesses, none of them was an eigenvector. For the first initial value, the method converged, but it failed to converge for the second.

(a) What can be said about n, the order of A?

(b) What can you say about the spectrum of A and about the two initial guesses that had been used?

3.

(a) Let F be a linear space of functions defined on $[a \,..\, b]$. Assume $\dim F = n + 1$. Let $a \leq x_0 < x_1 < x_2 \ldots < x_n \leq b$. Suppose that the only $f \in F$ which satisfies $f(x_i) = 0, \quad i = 0, ..., n$ is $f \equiv 0$. Prove that for any numbers $y_0, \ldots, y_n$ there exists a unique $f \in F$ such that

$$f(x_i) = y_i \quad i = 0, \ldots, n.$$

(b) Suppose now that

$$F = \text{span}\{1, x^2, x^4, \ldots, x^{2n}\}.$$

Prove that for any $0 < x_0 < x_1 < \cdots < x_n$ and any numbers $y_0, \ldots, y_n$ there exists a unique $f \in F$ s.t.

$$f(x_i) = y_i \quad i = 0, \ldots, n$$

4. For a linear system of the form $(A_1 + A_2)x = b$, show that the iterative method

$$(I + A_1)\hat{x} = (I - A_2)x^k + b$$
$$(I + A_2)x^{k+1} = (I - A_1)\hat{x} + b$$

converges if and only if all the eigenvalues of

$$(I + A_1 A_2)^{-1}(A_1 + A_2)$$

have positive real part.

5. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 & 0 \end{bmatrix}$$

Let λ be any eigenvalue of A. Show that there is only one linearly independent eigenvector X associated with λ.

6. Show that any finite difference scheme of the form

$$v_n^{n+1} = \sum_{j=0}^{L} a_j v_{n+j}^n,$$

that is consistent with

$$u_t + u_x = 0,$$

is unstable.

7. Give an example of a 1-dimensional subspace M of $C[0\,..\,1]$ and some $f \in C[0\,..\,1]$ which has several uniform best approximations from M. Can you give such an example when, instead of the uniform norm

$$\|f - m\|_\infty := \max_{t \in [0..1]} |f(t) - m(t)|$$

you use the mean-square norm

$$\|f - m\|_2 := \left(\int_0^1 |f(t) - m(t)|^2 \right)^{1/2}$$

to measure the distance between f and $m \in M$?

8. (a) Prove: For every inner product $(f, g) \mapsto \langle f, g \rangle$ on the space Π_k of (univariate) polynomials of degree $\leq k$, there exists a unique sequence $v_0, v_1, ..., v_k$ in Π_k with the property that, for each j, (a) v_j is of exact degree j, with leading coefficient 1; and (b) v_j is perpendicular to Π_{j-1}.

(b) Prove: The polynomial sequence $v_0, v_1, ..., v_k$ determined in (a) satisfies a three-term recurrence relation, i.e., for every $j = 0, 1, ..., k-1$ there exist numbers b_j and c_j so that

$$v_{j+1}(t) = (t - b_j)v_j(t) - c_j v_{j-1}(t), \ \forall t.$$

9. Consider the two initial-values problems

(1)

$$y' = (1 + \frac{1}{2}\sin x)y = f(y,x), \quad 0 \le x \le 10$$
$$y'(0) = 1$$

(2)

$$y' = -(2 + \cos x)y = f(y,x), \quad 0 \le x \le 10$$
$$y(0) = 1$$

For each problem discuss the effectiveness of the numerical method

$$Y_n = Y_{n-2} + 2hf(Y_{n-1}, x_{n-1})$$
$$Y_0 = 1, \quad Y_1 = 1 + hf(Y_0, 0)$$

In particular, does the method exhibit different behavior for these different problems?

10. Consider the two-point boundary value problem

$$(1) \begin{cases} y''(x) - y(x) = \sin x, & 0 < x < 1 \\ y(0) = y(1) = 0 \end{cases}$$

and the numerical method given by

$$(2) \begin{cases} \frac{Y_{k-1} - 2Y_k + Y_{k+1}}{h^2} - Y_k = \sin x_k, & 1 \le k \le N \\ Y_0 = Y_{n+1} = 0 \\ h = \frac{1}{N+1} \end{cases}$$

(a) What is the best bound M, such that

$$|Y_k| \le M.$$

(b) Discuss the convergence of the solutions of this scheme Y_k to the solution $y(x_k)$.