Numerical Analysis Spring 1989

A student taking the exam for "breadth" must answer three (3) questions to be handed in after two (2) hours. A student taking the exam for "depth" must answer six (6) questions, with at least three (3) chosen from problems 6–10.

I. For a twice differentiable function g and an initial point $x_0 \in \mathbb{R}$, consider the method

$$x_{k+1} = g(x_k) \qquad k = 0, 1, \ldots \tag{1}$$

which is aimed to solve a certain problem in Numerical Analysis.

(a) State and prove a sufficient condition for the sequence of numbers $x_0, x_1 \ldots$ to converge. Determine the rate of convergence which follows from this condition.

(b) In view of the condition you have stated in (a), dicuss the convergence of (1) with $g(x) = \arctan x$.

(c) Prove that if $|g'(x)| > 1$ everywhere, the algorithm diverges regardless of the choice of x_0.

(d) An alternative method to (1) is

$$x_{k+1} = x_k - \frac{g(x_k) - x_k}{g'(x_k) - 1} \tag{2}$$

What is the problem (1) and (2) are meant to solve? Discuss the convergence rate of (2).

II. Consider the tridiagonal matrix A, of order N with elements

$$\begin{aligned} a_{ij} =& 0 \quad \text{unless} \quad |i-j| \le 1 \\ a_{i,i-1} =& 1, \quad a_{i,i+1} = 1, \quad a_{ii} = 5 \end{aligned}$$

What is the best bound you can prove for

$$\|A^{-1}\|_\infty?$$

III.

(i) Describe and analyze Jacobi iteration for the linear system

$$\begin{aligned} x + 10y &= 3 \\ x/20 + y &= 4 \end{aligned}$$

(ii) What can you conclude from the fact that the 1-norm of the iteration matrix is much larger than 1?

(iii) Would Gauss-Seidel iteration for this system converge?

IV. Consider the two point boundary value problem

$$(1) \qquad \begin{cases} y''(x) - y(x) = \sin x, & 0 < x < 1 \\ y(0) = y(1) = 0 \end{cases}$$

and the numerical method given by

$$(2) \qquad \begin{cases} \frac{Y_{k-l} - 2Y_k + Y_{k+1}}{h^2} - Y_k = \sin x_k, & 1 \leq k \leq N \\ Y_0 = Y_{N+1} = 0 \\ h = \frac{1}{N+1} \end{cases}$$

(a) What is the best bound M, such that

$$|Y_k| \leq M.$$

(b) Discuss the convergence of the solutions of this scheme Y_k to the solution $y(x_k)$.

V.

(i) Describe the Gram-Schmidt method for the construction of an orthonormal basis from a basis for some linear subspace of $\mathbb{R}^n$.

(ii) Describe also the Modified Gram-Schmidt method and discuss the reason behind the modification.

(iii) How could a program for constructing the QR factorization of a matrix be used to solve the same problem?

VI. For fixed real distinct constants $\lambda_1, \ldots, \lambda_n$, let E be the space of all functions of the form

$$(1) \qquad f(x) = \sum_{j=1}^{n} a_j e^{\lambda_j x} \qquad a_1, \ldots, a_n \in \mathbb{R}.$$

(a) Prove that if $n = 2$, then the only function in E that vanishes at two different points x_1, x_2 is $f = 0$.

(b) Prove that for every set of data points

$$(x_1, y_1), \ldots, (x_n, y_n)$$

there exists a unique $f \in E$ satisfying

$$f(x_j) = y_j \qquad j = 1, \ldots, n.$$

Here you may use (without proof) the following generalization of (a):

"The only $f \in E$ that vanishes at $x_1, \ldots, x_n$ is $f = 0$."

(c) Show that the existence and uniqueness theorem for interpolation by polynomials follows from the result you were asked to prove in (b).

VII. Consider the Initial-Value Problem

$$y' = f(y,t), \quad y(0) = y_0 \tag{1}$$

and the linear Multistep method

$$Y_n + Y_{n-1} - 2Y_{n-2} = h\Big[\frac{5}{2}f_n(Y_{n-1},t_{n-1}) + \frac{1}{2}f(Y_{n-2},t_{n-2})\Big] \tag{2a}$$

$$Y_0 = y_0 \tag{2b}$$

$$Y_1 = y_0 + hf\,(y_0,t_0) \tag{2c}$$

Discuss the consistency, stability and convergence of this method.

VIII. Quadrature is the task of evaluating as best one can expressions of the form

$$\lambda f := \int_a^b w(x)f(x)\,dx$$

for given f.

Take, specifically, $[a,b] = [0,10]$, $w = 1$, and suppose that f is Lipschitz continuous on $[0,10]$, with Lipschitz constant 2, and that $f(x) = x$ for $x \in \{0,1,\ldots,10\}$.

(i) If that is all you know about f, what is the best you can say about the number λf?

(ii) If you know, in addition, that f is differentiable and that $Df(x) = 1$ for $x \in \{0,1,\ldots,10\}$, what is now the best you can say about the number λf?

IX. Explain why one cannot have a consistent, explicit, unconditionally stable finite difference scheme for a hyperbolic partial differential equation.

X. Analyze the following scheme for the equation $u_t + au_x = f$.

$$\begin{aligned}\tilde{v}_m^{n+1} &= v_m^n - \frac{1}{2}a\lambda(v_{m+1}^n - v_{m-1}^n) + kf_m^n\\ v_m^{n+1} &= \frac{1}{4}(\tilde{v}_{m+1}^{n+1} + 2\tilde{v}_m^{n+1} + \tilde{v}_{m-1}^{n+1}).\end{aligned}$$

Give the order of accuracy and determine when it is stable.