Numerical Analysis
Depth Examination
Fall 97

You should answer five out of the eight problems below.

•1• You are given a real square matrix, A. You are told that the eigenvalues of this matrix are all algebraically simple. You intend to find an eigenvector of A by using the power method:

$$x^{new} = Ax^{old}.$$

The above power method is considered 'convergent' if, for every positive tolerance ε, we obtain, after sufficiently many iterations, an x^{new} so that, for some eigenvector v of A, $\|x^{new} - v\| \le \varepsilon \|v\|$.

(a) Explain why the above definition of convergence should be preferred over the simpler one, where we merely require that $\|x^{new} - v\| \le \varepsilon$.
(b) Is the following statement true? "The power method for the above A will converge regardless of the choice of the initial (non-zero) vector, provided that A is symmetric positive definite." Explain your answer.
(c) Will you change your answer to (b) if A is not necessarily positive definite?

•2•

(a) Give the definition of the norm of a linear functional λ on the normed linear space X.
(b) Compute the norm of the linear functional $\lambda : f \mapsto (f(1) - f(-1))$ on the normed linear space $C([-1, 1])$ (of all continuous functions on the interval $[-1, 1]$ with the max-norm).
(c) Explain why one might prefer a quadrature rule with positive weights to a quadrature rule of the same order which also has some negative weights.

•3• There exist continuous functions f on the interval $[0, 1]$ whose best uniform approximation from Π_3 is the zero function. Here Π_3 is the space of all polynomials of degree ≤ 3 (in one variable).

(a) Give an example of such a function f.
(b) Would the zero function still be the best approximation to the f you chose in (a) if we replace Π_3 by Π_k for $k = 2, 4$? Explain. (Note that you are asked here two separate questions: one for $k = 2$ and one for $k = 4$).
(c) Would the zero function still be the best approximation to the f you chose in (a) if we replace $[0, 1]$ by $[0, 1/2]$? Explain.

•4• Consider the Crank-Nicolson scheme applied to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in \mathbb{R}$ and $t > 0$. Determine conditions on $\Delta t/(\Delta x)^2$ such that the scheme is stable in the L_2-norm.

•**5**• Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \qquad F(\mathbf{u}, x) = \begin{bmatrix} f_1(\mathbf{u}, x) \\ f_2(\mathbf{u}, x) \\ f_3(\mathbf{u}, x) \\ f_4(\mathbf{u}, x) \end{bmatrix}.$$

Consider the Boundary Value Problem

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = F(\mathbf{u}, x), \qquad 0 \le x \le 1,$$
$$u_1(0) = 0,\ u_2(0) = 1,\ u_2(1) = 0,\ u_3(1) = 0.$$

Describe and discuss the shooting method for this problem.

•**6**• Consider the Initial Value Problem

$$\begin{aligned} y'(x) &= -(1-x)y(x), \qquad 0 \le x \le 4, \\ y(0) &= 1, \end{aligned}$$

and the two numerical methods

A)
$$\begin{cases} Y_{k+1} = Y_k - h\{(1-x_k)Y_k\}, \\ Y_0 = 1; \end{cases}$$

B)
$$\begin{cases} Y_{k+1} = Y_{k-1} - 2h\{(1-x_k)Y_k\}, \\ Y_0 = 1, \quad Y_1 = 1-h. \end{cases}$$

Describe the behavior of the error and explain this behavior.

•**7**• Show that any finite difference scheme of the form

$$v_m^{n+1} = \sum_{j=0}^{L} a_j v_{m+j}^n$$

that is consistent with

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is unstable.

•**8**• Let F be a real linear space of dimension n consisting of real-valued functions defined on the interval $[0,1]$. Let T be a subset of $[0,1]$ consisting of n points. Given a function g defined on $[0,1]$, we consider the existence and uniqueness of an interpolant on T from F to g, i.e., a function $f \in F$ such that $f(t) = g(t)$, $t \in T$. It was disclosed to you that there exists *some* function g_0 for which the above interpolation problem has a unique solution. Prove or disprove the following statements:

(a) For every function g defined on $[0,1]$ there is a unique solution $f \in F$ for the above interpolation problem.

(b) If T' is obtained by adjoining to T a point t_0 from $[0,1]$ but not in T, then an interpolant from F on T' to any function g on $[0,1]$ may exist, but is never unique.

(c) If T' is obtained by adjoining to T a point t_0 from $[0,1]$ but not in T, then an interpolant from F on T' to any function g on $[0,1]$ must be unique if it exists.

(d) The assertion in (a) is correct, and, moreover, it is correct if we replace the set T by any other set in $[0,1]$ of the *same cardinality*.