Numerical Analysis
Depth Examination
Fall 96

You should answer five out of the eight problems below.

•1•
(a) Define "a Householder matrix". Give an example of a 3×3 Householder matrix whose entries are all non-zero.
(b) Prove that every Householder matrix H is symmetric, orthogonal, and self-invertible (i.e., $H^2 = I$).
(c) Discuss *briefly* two numerical algorithms that employ Householder matrices.
(d) Prove or disprove: every symmetric, orthogonal, self-invertible matrix is a Householder matrix.

•2• In this question, we consider numerical approximations to the integral

$$\int_0^1 f(t)\,dt.$$

(a) Derive an error formula for the composite trapezoid rule (using a uniform partition).
(b) Derive from the composite trapezoid rule another rule by a one-step application of Richardson's extrapolation. What is the order of the new rule?
(c) Compare the composite trapezoid rule to the composite midpoint rule (in terms of the error estimates, and in any other terms you choose).

•3• Many approximations Pf to a given element f of a given real linear space X are of the form $Pf = V(\Lambda V)^{-1}\Lambda f$, with Λ a linear map from X to $\mathbb{R}^n$, and V a linear map from $\mathbb{R}^n$ to X.
(a) Give three examples of such approximations in common use, identifying in each case X, V and Λ.
(b) Prove that any such P is a linear projector.
(c) Prove that, if X is a normed linear space, then P is a bounded linear projector if and only if the 'rows' of Λ, i.e., the linear functionals λ_i, $i = 1, \ldots, n$ defined by $\Lambda f =: (\lambda_i f : i = 1, \ldots, n)$, are bounded.

•4• Let $a := (a_n : n = 1, 2, 3, \ldots)$ be a sequence of bounded linear functionals on the normed linear space X and K be some subset of X.
(a) Explain the difference between uniform convergence and pointwise convergence of such a sequence on K.
(b) Give an example of interest in Numerical Analysis in which such a sequence a converges only pointwise and not uniformly on the unit ball of X.
(c) Provide a proof or a counterexample (but not both) for the statement: "If a converges pointwise on X, then it converges uniformly on totally bounded subsets of X."

•5• You are given the boundary value problem

$$u^{(iv)} + uu' = x^2, \quad -1 \le x \le 1,$$

$$u(1) = u(-1) = u'(1) = u'(-1) = 0$$

(where u is the unknown function, and x the independent argument).
Discuss the numerical solution of this problem using the shooting method.

•**6**• You are given the initial-value problem

$$y'(x) = (1 - x^2)y(x), \quad 0 \le x \le 2,$$
$$y(0) = 1,$$

as well as the numerical method

$$Y_{k+1} = Y_{k-1} + 2h(1 - x_k^2)Y_k,$$
$$Y_0 = 1,\ Y_1 = 1 + h,\ x_k = kh.$$

(a) Discuss the convergence or non-convergence of the method.
(b) Discuss the behaviour of the error.

•**7**• Consider the Crank-Nicolson scheme applied to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in \mathbb{R}$ and $t > 0$. Determine conditions on $\Delta t/(\Delta x)^2$ such that the solution v_m^n satisfies

$$\max_m |v_m^{n+1}| \le \max_m |v_m^n|$$

for $n \ge 0$ and all initial data with compact support.

•**8**• Consider the finite difference scheme

$$v_m^{n+1} = \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right) - \frac{\lambda}{2(1+\lambda^2)}\left(v_{m+1}^n - v_{m-1}^n\right)$$

for the equation $u_t + u_x = 0$ where $\lambda = \Delta t/\Delta x$ and v_m^n is an approximation to $u(t_n, x_m)$.
(a) Show that this scheme is stable for all values of λ.
(b) For the initial value problem, determine conditions on Δt and Δx so that the solution of the scheme converges to the solution of the differential equation.