Numerical Analysis
Depth Examination
Fall 94

You should answer six out of the ten problems below. At least three of these questions should be selected from problems (6-10).

1. Let u, v be two n-dimensional real vectors. Consider the following problem:
"Find $c_0 \in \mathbb{R}$ such that

$$\|c_0 v - u\|_2 \leq \|cv - u\|_2, \quad \forall c \in \mathbb{R}."$$

(a) State a condition on the pair (u, v) which is equivalent to the existence of a unique solution c_0 to the above problem. Assuming your condition holds, find an expression for c_0 in terms of the pair (u, v).
(b) Prove the necessity and sufficiency of your condition.
(c) Discuss the above problem in the case when your condition fails to hold.

2.
(a) Determine all possible values for $f(0)$ given that f is continuously differentiable, with $\max_{-1 \leq x \leq 1} |Df(x)| \leq 2$, and $f(-1) = 1$, $f(1) = 0$.
(b) What are all possible values if you know instead that $\int_{-1}^{1} |Df(x)|^2\, dx \leq 2$, and $f(-1) = 1$, $f(1) = 0$?
Feel free to use the fact that broken line interpolation to given data minimizes $\int |Df|^2$ over all possible interpolants.

3. A is an $n \times n$ real symmetric matrix, $[a, b]$ is an interval.
(a) Describe a finite algorithm that determines the number of eigenvalues (counting multiplicities) that A has in the interval $[a, b]$. State basic theorems from linear algebra which, in your opinion, are the key in the success of your algorithm.
(b) Apply your above algorithm in order to find the number of eigenvalues the matrix

$$\begin{pmatrix} -9 & 3 & -3 \\ 3 & 2 & -5 \\ -3 & -5 & 13 \end{pmatrix}$$

has in the interval $[-6, 0]$.

4. In this question, we consider uniform approximation to a continuous function f on an interval $[a, b]$ from the space Π_n of all polynomials of degree $\leq n$.
(a) Assuming f is even and is monotonely decreasing on $[0, \infty)$, find the best uniform approximation to f from Π_1 on the interval $[-a, a]$. Justify your choice.
(b) Prove that for every n, there exists a sufficiently large a such that the best approximation to the function $f(t) := \cos t$ on $[-a, a]$ from Π_n is the zero polynomial.
(c) Prove or disprove the following statement: if, for some $K > 0$, 0 is the best approximation to $\sin t$ from Π_n on $[0, K]$, then 0 is also the best approximation from Π_n on that interval to $\cos t$.

5. Consider the iteration $x_j \to x_{j+1}$ defined by

$$y = x_j - \omega A x_j + \omega b$$

$$x_{j+1} + \omega A x_{j+1} = y + \omega b,$$

where $\omega > 0$ and A is a *symmetric* matrix with eigenvalues $1 = \lambda_1 \leq \ldots \leq \lambda_n = 2$. For which values of ω does the iteration converge and which choice yields the fastest rate of convergence?

6. Consider the Initial-Value Problem:

$$y' = \frac{1-y^2}{1+y^2}, \quad y(0) = 0, \ 0 \leq t \leq 2.$$

Consider also the numerical method for solving it:

$$Y_{k+2} = Y_k + 2h\frac{1-Y_{k+1}^2}{1+Y_{k+1}^2}, \quad (k+2)h \leq 2,$$

$$Y_0 = 0, \quad Y_1 = h.$$

Discuss the behaviour of the solution $(Y_k)_k$ along the following lines:
(a) Convergence to the true solution?
(b) Order of convergence?
(c) Stability of the process?

7. Show that any finite difference scheme of the form

$$v_m^{n+1} = \sum_{j=0}^{L} a_j v_{m+j}^n$$

that is consistent with

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is unstable.

8. (a) Describe how you would go about determining the condition (number) of the power basis for polynomials of degree $\leq k$, as a subspace of the Banach space of continuous functions on the interval $[-1, 1]$.
(b) Apply your general description to the special case $k = 2$.

9. Consider the boundary value problem

$$y^{(4)} + y = f, \quad 0 \leq x \leq 1,$$

$$y(0) = 0, \quad y(1) = 0,$$

$$y'(0) = 1, \quad y'(1) = 1.$$

Describe the "shooting method" as applied to this problem. You may assume that you have in hand an effective algorithm for the solution of initial value problems.

10. (a) Discuss the uniform convergence of the Fourier series for a 2π-periodic continuous function on $\mathbb{R}$, with special emphasis on its possible failure to converge.
(b) Explain why, in spite of the discussion in (a), Fourier series is used routinely and successfully for the approximation and (approximate) representation of 2π-periodic functions.
(c) Contrast the use of Fourier series with the use of power series for the approximate representation of 2π-periodic functions, on the interval $[-\pi, \pi]$ say.