Numerical Analysis

A student taking the exam for "breadth" must answer three (3) questions. A student taking the exam for "depth" must answer six (6) questions, with at least three (3) chosen from problems 6-10. In either case, the answers to three questions must be turned in after 2 hours and these will be the only questions looked at for the "breadth" grade.

1.

(a) Discuss the bisection method for finding an eigenvalue of a real-valued symmetric matrix.

(b) You are given a real-valued matrix A and want to find a real eigenvalue λ in a neighborhood of 7. Suggest a way to find λ.

(c) Why is it important to assume in (b) that λ is real?

2.

(a) Given a smooth function f, analyze the error of the midpoint rule

$$\int_a^b f(x)\,dx \approx (b-a)f\left(\frac{a+b}{2}\right).$$

(b) You have applied a composite midpoint rule in order to estimate $\int_a^b f(x)\,dx$. In order to improve the accuracy of your approximation, you consider the following:

1) Use the trapezoid rule on the same set of intervals.
2) Use the Simpson rule on the same set of intervals.
3) Use the midpoint rule on a subdivision of the current set of intervals.

What would be your preferred choice? Explain your choice in terms of computational effort vs. the accuracy obtained.

3. Consider the linear system

$$\begin{pmatrix} 10 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Estimate the number of iterations necessary to get an (absolute) error of 10^{-5} by using the Jacobi method, assuming that the initial vector is the zero vector.

4. Let f be the absolute value function, i.e., $f(x) := |x|$. Derive as large a number A as you can so that $\operatorname{dist}_\infty(f, \Pi_2) \geq A$, where Π_2 is the space of quadratic polynomials, and the distance is measured in the max-norm on the interval $[-1..1]$.

5. Consider a complex-valued $N \times M$ matrix $A = (a_{i,j})$ as a map from $\mathbf{C}^M$ with the 1-norm to $\mathbf{C}^N$ with the max-norm. Show that

$$\|A\| = \max_{i,j} |a_{i,j}|.$$

6. Give a finite-difference scheme for solving the initial value problem for the partial differential equation

$$u_t = u_{xxx} \qquad \text{for } x \in \mathbb{R}, t \geq 0.$$

The scheme must be stable and at least first-order accurate in time and second-order accurate in space. Justify your claims about the stability and accuracy of the scheme.

7. (a) State the Fredholm alternative (for certain linear maps on some Banach space).

(b) Use (a) to **prove** that there is exactly one function f, continuous on the interval [0..3], which satisfies the integral equation

$$f(t) - \frac{1}{2}\int_0^3 \sin(t-s)f(s)\,ds = 1, \quad 0 \leq t \leq 3.$$

8. Consider the linear multistep method

$$2Y_n + Y_{n-1} - 2Y_{n-2} - Y_{n-3} = 3h\left(f(Y_{n-1}) + f(Y_{n-2})\right)$$

with starting values

$$Y_0 = y_0, Y_1 = y_0 + hf(y_0), Y_2 = y_0 + 2hf(y_0).$$

(a) Discuss (1) the consistency, (2) the stability, and (3) the convergence or nonconvergence of the method for the general initial-value problem

$$y' = f(y), \quad y(0) = y_0, \quad 0 \leq t \leq T.$$

(b) Discuss the advantages and/or disadvantages of this method for the particular problem

$$y' = -\frac{3y}{1+y^2}, \quad y(0) = 1, \quad 0 \leq t \leq 10.$$

9. Consider the boundary value problem

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -u^3 + 1 \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega$$

where Ω is the unit square.

(a) Describe a finite-difference scheme to solve this problem. This scheme must produce a unique solution.

(b) Prove that your scheme has a unique solution.

10. Prove the theorem:

Let A be an $n \times n$ matrix and let $\rho := \rho(A)$ be the spectral radius, that is,

$$\rho := \max\{|\lambda| : \lambda \text{ an eigenvalue of } A\}.$$

Then, given any $\epsilon > 0$ there is a vector norm $\|\cdot\|$ and an induced operator norm of A, also denoted $\|\cdot\|$, such that

$$\|A\| \leq \rho + \epsilon.$$