

A Generalization of Stålmарck’s Method*

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Abstract. This paper gives an account of Stålmарck’s method for validity checking of propositional-logic formulas, and explains each of the key components in terms of concepts from the field of abstract interpretation. We then use these insights to present a framework for propositional-logic validity-checking algorithms that is parametrized by an abstract domain and operations on that domain. Stålmарck’s method is one instantiation of the framework; other instantiations lead to new decision procedures for propositional logic.

1 Introduction

A tool for validity checking of propositional-logic formulas (also known as a tautology checker) determines whether a given formula φ over the propositional variables $\{p_i\}$ is true for all assignments of truth values to $\{p_i\}$. Validity is dual to satisfiability: validity of φ can be determined using a SAT solver by checking the satisfiability of $\neg\varphi$ and complementing the answer: $\text{VALID}(\varphi) = \neg\text{SAT}(\neg\varphi)$.

With the advent of SAT-solvers based on conflict-directed clause learning (i.e., CDCL SAT solvers) [11] and their use in a wide range of applications, SAT methods have received increased attention during the last twelve years. Previous to CDCL, a fast validity checker (and hence a fast SAT solver) already existed, due to Stålmарck [13]. Stålmарck’s method was protected by Swedish, European, and U.S. patents [15], which may have discouraged experimentation by researchers. Indeed, one finds relatively few publications that concern Stålmарck’s method—some of the exceptions are by Harrison [9], Cook and Gonthier [2], and Björk [1]. (Kunz and Pradhan [10] discuss a closely related method.)

In this paper, we give a new account of Stålmарck’s method by explaining each of the key components in terms of concepts from the field of abstract interpretation [3]. In particular, we show that Stålmарck’s method is based on a

* Supported, in part, by NSF under grants CCF-{0810053, 0904371}, by ONR under grants N00014-{09-1-0510, 10-M-0251, 11-C-0447}, by ARL under grant W911NF-09-1-0413, by AFRL under grants FA9550-09-1-0279 and FA8650-10-C-7088; and by DARPA under cooperative agreement HR0011-12-2-0012.. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors, and do not necessarily reflect the views of the sponsoring agencies.

** T. Reps has an ownership interest in GrammaTech, Inc., which has licensed elements of the technology discussed in this publication.

certain *abstract domain* and a few *operations* on that domain. For the program-analysis community, the abstract-interpretation account explains the principles behind Stålmarck’s method in terms of familiar concepts. In the long run, our hope is that a better understanding of Stålmarck’s method will lead to

- better program-analysis tools that import principles found in Stålmarck’s method into program analyzers
- improvements to Stålmarck-based validity checkers by (i) incorporating domains other than the ones that have been used (implicitly) in previous implementations of the method, or (ii) improving the method in other ways by incorporating additional techniques from the field of abstract interpretation.

There has been one payoff already: in [17], we describe ways in which ideas from Stålmarck’s method can be adopted for use in program analysis. The techniques described in [17] are quite different from the huge amount of recent work based on reducing a program path π to a formula φ_π via symbolic execution, and then passing φ_π to a decision procedure to determine whether π is feasible. Instead, we adopted—and adapted—the key ideas from Stålmarck’s method to create new algorithms for key program-analysis operations.

In this paper, we use the vantage point of abstract interpretation to describe the elements of the Dilemma Rule—the inference rule that distinguishes Stålmarck’s method from other propositional-reasoning approaches—as follows:

Branch of a Proof: In Stålmarck’s method, each proof-tree branch is associated with a so-called *formula relation* [13]. In abstract-interpretation terms, each branch is associated with an abstract-domain element. As discussed in §4, the set of formula relations associated with a formula φ under consideration is one particular abstract domain.

Splitting: The step of splitting the current goal into sub-goals can be expressed in terms of meet (\sqcap).

Application of simple deductive rules: Stålmarck’s method applies a set of simple deductive rules after each split. In abstract-interpretation terms, the rules perform a *semantic reduction* [4] by means of a technique called *local decreasing iterations* [8].

“Intersecting” results: The step of combining the results obtained from an earlier split are described as an “intersection” in Stålmarck’s papers. In the abstract-interpretation-based framework, the combining step is the *join* (\sqcup) of two abstract-domain values.

This more general view of Stålmarck’s method furnishes insight on when an invocation of the Dilemma Rule fails to make progress in a proof. In particular, both branches of a Dilemma may each succeed (locally) in advancing the proof, but the abstract domain used to represent proof states may not be precise enough to represent the common information when the join of the two branches is performed; consequently, the global state of the proof is not advanced.

We use these insights to present a parametric framework for propositional validity-checking algorithms. The advantages of our approach are

- We prove correctness at the framework level, once and for all, instead of for each instantiation.

- Instantiations that use different abstract domains lead to different decision procedures for propositional logic. Stålmarck’s method is the instantiation of our framework in which the abstract domain tracks equivalence relations between subformulas—or, equivalently, 2-variable Boolean affine relations (2-BAR). By instantiating the framework with other abstract domains, such as k -variable Boolean affine relations (k -BAR) and 2-variable Boolean inequality relations (2-BIR), we obtain more powerful decision procedures.

Contributions. The contributions of the paper can be summarized as follows:

- We explain Stålmarck’s method in terms of abstract interpretation [3]—in particular, we show that it is one instance of a more general algorithm.
- The vantage point of abstract interpretation provides new insights on the existing Stålmarck method.
- Adopting the abstract-interpretation viewpoint leads to a parametric framework for validity checking, parameterized by an abstract domain that supports a small number of operations.

Organization. The remainder of the paper is organized as follows: §2 reviews Stålmarck’s algorithm, and presents our generalized framework at a semi-formal level. §3 defines terminology and notation. §4 describes Stålmarck’s method using abstract-interpretation terminology and presents the general framework. §5 describes instantiations of the framework that result in new decision procedures. §6 presents preliminary experimental results. §7 discusses related work. Proofs are presented in App. A. App. B discusses a few efficiency issues.

2 Overview

In this section, we first review Stålmarck’s method with the help of a few examples. We then present our generalized framework at a semi-formal level. The algorithms that we give are intended to clarify the principles behind Stålmarck’s method, rather than represent the most efficient implementation.

2.1 Stålmarck’s Method

Consider the tautology $\varphi = (a \wedge b) \vee (\neg a \vee \neg b)$. It expresses the pigeonhole principle for two pigeons ($a \wedge b$) and one hole ($\neg a \vee \neg b$). Ex. 1 below shows that the simpler component of the two components of Stålmarck’s method (application of “simple deductive rules”) is sufficient to establish that φ is valid.

Example 1. We use $\mathbf{0}$ and $\mathbf{1}$ to denote the propositional constants false and true, respectively. Propositional variables, negations of propositional variables, and propositional constants are referred to collectively as *literals*. Stålmarck’s method manipulates *formula relations*, which are equivalence relations over literals. A formula relation R will be denoted by \equiv_R , although we generally omit the subscript when R is understood. We use $\mathbf{0} \equiv \mathbf{1}$ to denote the universal (and contradictory) equivalence relation $\{l_i \equiv l_j \mid l_i, l_j \in \text{Literals}\}$.

$$\begin{aligned} v_1 &\Leftrightarrow (v_2 \vee v_3) & (1) \\ v_2 &\Leftrightarrow (a \wedge b) & (2) \\ v_3 &\Leftrightarrow (\neg a \vee \neg b) & (3) \end{aligned}$$

Fig. 1: Integrity constraints corresponding to the formula $\varphi = (a \wedge b) \vee (\neg a \vee \neg b)$. The root variable of φ is v_1 .

$$\begin{array}{c} \frac{p \Leftrightarrow (q \vee r) \quad p \equiv \mathbf{0}}{q \equiv \mathbf{0} \quad r \equiv \mathbf{0}} \text{ OR1} \\ \frac{p \Leftrightarrow (q \wedge r) \quad q \equiv \mathbf{1} \quad r \equiv \mathbf{1}}{p \equiv \mathbf{1}} \text{ AND1} \end{array}$$

Fig. 2: Propagation rules.

Stålmarck's method first assigns to every subformula of φ a unique Boolean variable in a set of propositional variables \mathcal{V} , and generates a list of *integrity constraints* as shown in Fig. 1. An *assignment* is a function in $\mathcal{V} \rightarrow \{0, 1\}$. The integrity constraints limit the set of assignments in which we are interested. Here the integrity constraints encode the structure of the formula.

Stålmarck's method establishes the validity of the formula φ by showing that $\neg\varphi$ leads to a contradiction (which means that $\neg\varphi$ is unsatisfiable). Thus, the second step of Stålmarck's method is to create a formula relation that contains the assumption $v_1 \equiv \mathbf{0}$. Fig. 2 lists some *propagation rules* that enable Stålmarck's method to refine a formula relation by inferring new equivalences. For instance, rule OR1 says that if $p \Leftrightarrow (q \vee r)$ is an integrity constraint and $p \equiv \mathbf{0}$ is in the formula relation, then $q \equiv \mathbf{0}$ and $r \equiv \mathbf{0}$ can be added to the formula relation.

Fig. 3 shows how, starting with the assumption $v_1 \equiv \mathbf{0}$, the propagation rules derive the explicit contradiction $\mathbf{0} \equiv \mathbf{1}$, thus proving that φ is valid. \square

Ex. 1 illustrates several aspects of Stålmarck's method. By starting with the assumption that the formula is equivalent to false, Stålmarck's method derives new equivalences using simple deductive rules. These rules use the currently known equivalences and an integrity constraint to derive new equivalences.

Alg. 1 (Fig. 4) implements the propagation rules of Fig. 2. Given an integrity constraint $J \in \mathcal{I}$ and a set of equivalences $R_1 \subseteq R$, line 1 calls the function `ApplyRule`, which instantiates and applies the derivation rules of Fig. 2 and returns the deduced equivalences in R_2 . The new equivalences in R_2 are incor-

$$\begin{array}{ll} v_1 \equiv \mathbf{0} & \dots \text{ by assumption} \\ v_2 \equiv \mathbf{0}, v_3 \equiv \mathbf{0} & \dots \text{ by rule OR1 using Eqn. (1)} \\ \neg a \equiv \mathbf{0}, \neg b \equiv \mathbf{0} & \dots \text{ by rule OR1 using Eqn. (3)} \\ a \equiv \mathbf{1}, b \equiv \mathbf{1} & \dots \text{ interpretation of logical negation} \\ v_2 \equiv \mathbf{1} & \dots \text{ by rule AND1 using Eqn. (2)} \\ \mathbf{0} \equiv \mathbf{1} & \dots v_2 \equiv \mathbf{0}, v_2 \equiv \mathbf{1} \end{array}$$

Fig. 3: Proof that φ is valid.

<hr/> <p>Algorithm 1: propagate(J, R_1, R, \mathcal{I})</p> <hr/> <pre> 1 $R_2 = \text{ApplyRule}[\mathcal{I}](J, R_1)$ 2 return Close($R \cup R_2$) </pre> <hr/> <p>Algorithm 2: 0-saturation(R, \mathcal{I})</p> <hr/> <pre> 1 repeat 2 $R' \leftarrow R$ 3 foreach $J \in I, R_1 \subseteq R$ do 4 $R \leftarrow \text{propagate}(J, R_1, R, \mathcal{I})$ 5 until ($R = R'$) contradiction(R) 6 return R </pre> <hr/> <p>Algorithm 3: 1-saturation(R, \mathcal{I})</p> <hr/> <pre> 1 repeat 2 $R' \leftarrow R$ 3 foreach v_i, v_j such that $v_i \equiv v_j \notin R$ and $v_i \equiv \neg v_j \notin R$ do 4 $R_1 \leftarrow \text{Close}(R \cup \{v_i \equiv v_j\})$ 5 $R_2 \leftarrow \text{Close}(R \cup \{v_i \equiv \neg v_j\})$ 6 $R'_1 \leftarrow \text{0-saturation}(R_1)$ 7 $R'_2 \leftarrow \text{0-saturation}(R_2)$ 8 $R \leftarrow R'_1 \cap R'_2$ 9 until ($R = R'$) contradiction(R) 10 return R </pre> <hr/>	<hr/> <p>Algorithm 4: k-saturation(R, \mathcal{I})</p> <hr/> <pre> 1 repeat 2 $R' \leftarrow R$ 3 foreach v_i, v_j such that $v_i \equiv v_j \notin R$ and $v_i \equiv \neg v_j \notin R$ do 4 $R_1 \leftarrow \text{Close}(R \cup \{v_i \equiv v_j\})$ 5 $R_2 \leftarrow \text{Close}(R \cup \{v_i \equiv \neg v_j\})$ 6 $R'_1 \leftarrow \text{(k-1)-saturation}(R_1, \mathcal{I})$ 7 $R'_2 \leftarrow \text{(k-1)-saturation}(R_2, \mathcal{I})$ 8 $R \leftarrow R'_1 \cap R'_2$ 9 until ($R = R'$) contradiction(R) 10 return R </pre> <hr/> <p>Algorithm 5: k-Stålmarck(φ)</p> <hr/> <pre> 1 $(v_\varphi, \mathcal{I}) \leftarrow \text{integrity}(\varphi)$ 2 $R \leftarrow \{v_\varphi \equiv \mathbf{0}\}$ 3 $R' \leftarrow \text{k-saturation}(R, \mathcal{I})$ 4 if $R' = \mathbf{0} \equiv \mathbf{1}$ then return <i>valid</i> 5 else return <i>unknown</i> </pre> <hr/>
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Fig. 4: Stålmarck's method. The operation Close performs transitive closure on a formula relation after new tuples are added to the relation.

porated into R and the transitive closure of the resulting equivalence relation is returned. We implicitly assume that if Close derives a contradiction then it returns $\mathbf{0} \equiv \mathbf{1}$. Alg. 2 (Fig. 4) describes *0-saturation*, which calls propagate repeatedly until no new information is deduced, or a contradiction is derived. If a contradiction is derived, then the given formula is proved to be valid.

Unfortunately, 0-saturation is not always sufficient.

Example 2. Consider the tautology $\psi = (a \wedge (b \vee c)) \Leftrightarrow ((a \wedge b) \vee (a \wedge c))$, which expresses the distributivity of \wedge over \vee . The integrity constraints for ψ are:

$$\begin{array}{lll}
u_1 \Leftrightarrow (u_2 \Leftrightarrow u_3) & u_2 \Leftrightarrow (a \wedge u_4) & u_3 \Leftrightarrow (u_5 \vee u_6) \\
u_4 \Leftrightarrow (b \vee c) & u_5 \Leftrightarrow (a \wedge b) & u_6 \Leftrightarrow (a \wedge c)
\end{array}$$

The root variable of ψ is u_1 . Assuming $u_1 \equiv \mathbf{0}$ and then performing 0-saturation does not result in a contradiction; all we can infer is $u_2 \equiv \neg u_3$.

To prove that ψ is a tautology, we need to use the Dilemma Rule, which is a special type of branching and merging rule. It is shown schematically in Fig. 5. After two literals v_i and v_j are chosen, the current formula relation R is split

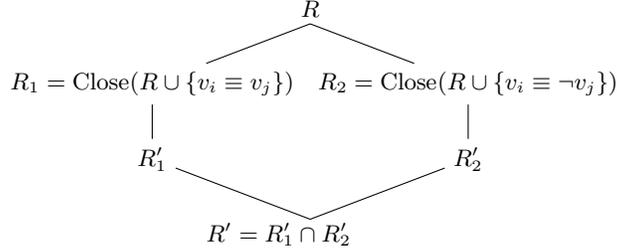


Fig. 5: The Dilemma Rule.

into two formula relations, based on whether we assume $v_i \equiv v_j$ or $v_i \equiv \neg v_j$, and transitive closure is performed on each variant of R . Next, the two relations are 0-saturated, which produces the two formula relations R'_1 and R'_2 . Finally, the two proof branches are merged by intersecting the set of tuples in R'_1 and R'_2 . The correctness of the Dilemma Rule follows from the fact that equivalences derived from *both* of the (individual) assumptions $v_i \equiv v_j$ and $v_i \equiv \neg v_j$ hold irrespective of whether $v_i \equiv v_j$ holds or whether $v_i \equiv \neg v_j$ holds.

The Dilemma Rule is applied repeatedly until no new information is deduced by a process called *1-saturation*, shown in Alg. 3 (Fig. 4). 1-saturation uses two literals v_i and v_j , and splits the formula relation with respect to $v_i \equiv v_j$ and $v_i \equiv \neg v_j$ (lines 4 and 5). 1-saturation finds a contradiction when both 0-saturation branches identify contradictions (in which case $R = R'_1 \cap R'_2$ equals $\mathbf{0} \equiv \mathbf{1}$). The formula ψ in Ex. 2 can be proved valid using 1-saturation, as shown in Fig. 6. The first application of the Dilemma Rule, which splits on the value of b , does not make any progress; i.e., no new information is obtained after the intersection. The next two applications of the Dilemma Rule, which split on the values of a and c , respectively, each deduce a contradiction on one of their branches. Each contradictory branch is eliminated because the (universal) relation $\mathbf{0} \equiv \mathbf{1}$ is the identity element for intersection, and hence the intersection result is the equivalence relation from the non-contradictory branch. We illustrate this fact in Fig. 6 by eliding the merges with contradictory branches. Finally, splitting on the variable b leads to a contradiction on both branches. \square

There are many possible strategies for choosing which equivalence to split on for the Dilemma Rule during 1-saturation. The choices made by a strategy have an impact on the completeness and efficiency of the decision procedure. For expository purposes, Alg. 3 adopts a simple yet complete strategy. We revisit this issue in App. B.

Unfortunately 1-saturation may not be sufficient to prove certain tautologies. The 1-saturation procedure can be generalized to the k -saturation procedure shown in Alg. 4 (Fig. 4). Stålmarck's method (Alg. 5 of Fig. 4) is structured as a semi-decision procedure for validity checking. The actions of the algorithm are parameterized by a certain parameter k that is fixed by the user. For a given tautology, if k is large enough Stålmarck's method can prove validity, but

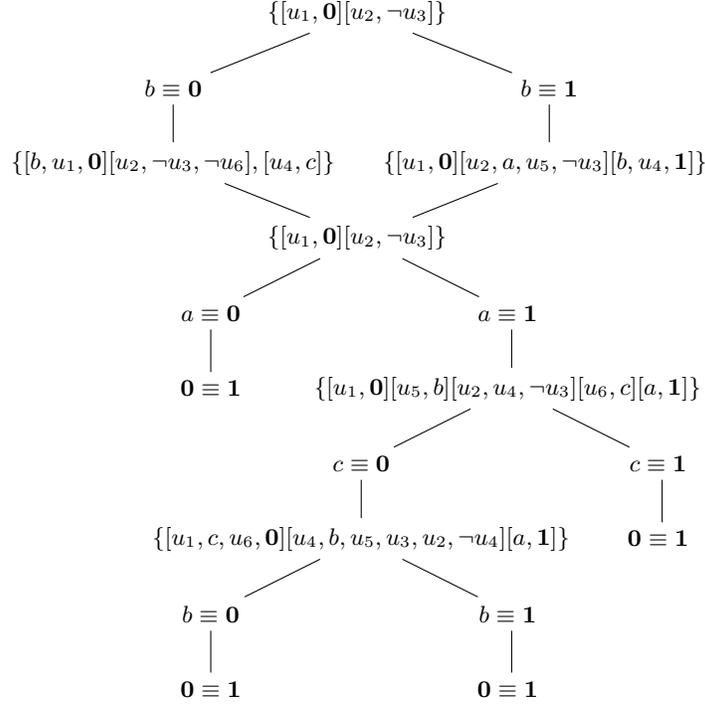


Fig. 6: Sequence of Dilemma Rules in a 1-saturation proof that ψ is valid. (Details of 0-saturation steps omitted.)

if k is too small the answer returned is “unknown”. In the latter case, one can increment k and try again. However, for each k , $(k + 1)$ -saturation is significantly more expensive than k -saturation: the running time of Alg. 5 as a function of k is $O(|\varphi|^k)$ [13].

Each equivalence relation that arises during Stålmarck's method can be viewed as an *abstraction* of a set of variable assignments. More precisely, at any moment during a proof there are some number of open branches. Each branch B_i has its own equivalence relation R_i , which represents a set of variable assignments A_i that might satisfy $\neg\varphi$. In particular, the contradictory equivalence relation $\mathbf{0} \equiv \mathbf{1}$ represents the empty set of assignments. Overall, the proof represents the set of assignments $\bigcup_i A_i$, which is a superset of the set of assignments that might satisfy $\neg\varphi$. Validity of φ is established by showing that the set $\bigcup_i A_i$ equals \emptyset .

2.2 Generalizing Stålmarck's Method

Instead of computing an equivalence relation \equiv on literals, let us compute an inequality relation \leq between literals. Fig. 7 shows a few of the propagation rules

$$\begin{array}{c}
\frac{a \Leftrightarrow (b \Rightarrow c)}{c \leq a \quad \neg a \leq b} \text{IMP1} \qquad \frac{a \Leftrightarrow (b \Leftrightarrow c) \quad a \leq \mathbf{0}}{b \leq \neg c \quad \neg c \leq b} \text{IFF1} \\
\frac{a \Leftrightarrow (b \Rightarrow c) \quad \mathbf{1} \leq b \quad c \leq \mathbf{0}}{a \leq \mathbf{0}} \text{IMP2}
\end{array}$$

Fig. 7: Examples of propagation rules for inequality relations on literals.

that deduce inequalities. Because (i) an equivalence $a \equiv b$ can be represented using two inequality constraints, $a \leq b$ and $b \leq a$, (ii) an inequivalence $a \not\equiv b$ can be treated as an equivalence $a \equiv \neg b$, and (iii) $a \leq b$ cannot be represented with any number of equivalences, inequality relations are a strictly more expressive method than equivalence relations for abstracting a set of variable assignments. Moreover, Ex. 3 shows that, for some tautologies, replacing equivalence relations with inequality relations enables Stålmarck’s method to be able to find a k -saturation proof with a strictly lower value of k .

Example 3. Consider the formula $\chi = (p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$. The corresponding integrity constraints are $w_1 \Leftrightarrow (w_2 \Leftrightarrow w_3)$, $w_2 \Leftrightarrow (p \Rightarrow q)$, and $w_3 \Leftrightarrow (\neg q \Rightarrow \neg p)$. The root variable of χ is w_1 . Using formula relations (i.e., equivalence relations over literals), Stålmarck’s method finds a 1-saturation proof that χ is valid. In contrast, using inequality relations, a Stålmarck-like algorithm finds a 0-saturation proof. The proof starts by assuming that $w_1 \leq \mathbf{0}$. 0-saturation using the propagation rules of Fig. 7 results in the contradiction $\mathbf{1} \leq \mathbf{0}$, as shown in Fig. 8. \square

We say that the instantiation of Stålmarck’s method with inequality relations is *more powerful than* the instantiation with equivalence relations. In general, Stålmarck’s method can be made more powerful by using a more expressive abstraction: when you plug in a more expressive abstraction, a proof may be possible with a lower value of k . This observation raises the following questions:

1. What other abstractions can be used to create more powerful instantiations?
2. Given an abstraction, how do we come up with the propagation rules?
3. How do we split the current abstraction at the start of the Dilemma Rule?
4. How do we perform the merge at the end of the Dilemma Rule?
5. How do we guarantee that the above operations result in a sound and complete decision procedure?

Abstract interpretation provides the appropriate tools to answer these questions.

3 Terminology and Notation

3.1 Propositional Logic

We write propositional formulas over a set of propositional variables \mathcal{V} using the propositional constants $\mathbf{0}$ and $\mathbf{1}$, the unary connective \neg , and the binary con-

$w_1 \leq \mathbf{0}$... by assumption
$w_2 \leq \neg w_3, \neg w_3 \leq w_2$... Rule IFF1 on $w_1 \Leftrightarrow (w_2 \Leftrightarrow w_3)$
$q \leq w_2, \neg w_2 \leq p$... Rule IMP1 on $w_2 \Leftrightarrow (p \Rightarrow q)$
$q \leq \neg w_3$... $q \leq w_2, w_2 \leq \neg w_3$
$w_3 \leq p$... $w_2 \leq \neg w_3$ implies $w_3 \leq \neg w_2, \neg w_2 \leq p$
$\neg p \leq w_3, \neg w_3 \leq \neg q$... Rule IMP1 on $w_3 \Leftrightarrow (\neg q \Rightarrow \neg p)$
$q \leq \mathbf{0}$... $\neg w_3 \leq \neg q$ implies $q \leq w_3, q \leq \neg w_3$
$\mathbf{1} \leq p$... $w_3 \leq p, \neg p \leq w_3$ implies $\neg w_3 \leq p$
$w_2 \leq \mathbf{0}$,	... Rule IMP2 on $w_2 \Leftrightarrow (p \Rightarrow q)$
$w_3 \leq \mathbf{0}$... Rule IMP2 on $w_3 \Leftrightarrow (\neg q \Rightarrow \neg p)$
$\mathbf{1} \leq \mathbf{0}$... $w_2 \leq \neg w_3, \neg w_3 \leq w_2, w_2 \leq \mathbf{0}, w_3 \leq \mathbf{0}$

Fig. 8: 0-saturation proof that χ is valid, using inequality relations on literals.

nectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$, and \oplus (xor). Propositional variables, negations of propositional variables, and propositional constants are referred to collectively as *literals*. $\text{voc}(\varphi)$ denotes the subset of \mathcal{V} that occurs in φ .

The semantics of propositional logic is defined in the standard way:

Definition 1. An **assignment** σ is a (finite) function in $\mathcal{V} \rightarrow \{0, 1\}$. Given a formula φ over the propositional variables x_1, \dots, x_n and an assignment σ that is defined on (at least) x_1, \dots, x_n , the **meaning** of φ with respect to σ , denoted by $\llbracket \varphi \rrbracket(\sigma)$, is the truth value in $\{0, 1\}$ defined inductively as follows:

$$\begin{array}{lll}
\llbracket \mathbf{0} \rrbracket(\sigma) = 0 & \llbracket \neg \varphi \rrbracket(\sigma) = 1 - \llbracket \varphi \rrbracket(\sigma) & \llbracket \varphi_1 \Rightarrow \varphi_2 \rrbracket(\sigma) = (\llbracket \varphi_1 \rrbracket(\sigma) \leq \llbracket \varphi_2 \rrbracket(\sigma)) \\
\llbracket \mathbf{1} \rrbracket(\sigma) = 1 & \llbracket \varphi_1 \wedge \varphi_2 \rrbracket(\sigma) = \min(\llbracket \varphi_1 \rrbracket(\sigma), \llbracket \varphi_2 \rrbracket(\sigma)) & \llbracket \varphi_1 \Leftrightarrow \varphi_2 \rrbracket(\sigma) = (\llbracket \varphi_1 \rrbracket(\sigma) = \llbracket \varphi_2 \rrbracket(\sigma)) \\
\llbracket x_i \rrbracket(\sigma) = \sigma(x_i) & \llbracket \varphi_1 \vee \varphi_2 \rrbracket(\sigma) = \max(\llbracket \varphi_1 \rrbracket(\sigma), \llbracket \varphi_2 \rrbracket(\sigma)) & \llbracket \varphi_1 \oplus \varphi_2 \rrbracket(\sigma) = (\llbracket \varphi_1 \rrbracket(\sigma) \neq \llbracket \varphi_2 \rrbracket(\sigma))
\end{array}$$

Assignment σ **satisfies** φ , denoted by $\sigma \models \varphi$, iff $\llbracket \varphi \rrbracket(\sigma) = 1$. Formula φ is **satisfiable** if there exists σ such that $\sigma \models \varphi$; φ is **valid** if for all σ , $\sigma \models \varphi$.

We overload the notation $\llbracket \cdot \rrbracket$ as follows: $\llbracket \varphi \rrbracket$ means $\{\sigma \mid \sigma : \mathcal{V} \rightarrow \{0, 1\} \wedge \sigma \models \varphi\}$. Thus, we can think of φ as a device for accepting a set of assignments. Given a finite set of formulas $\Phi = \{\varphi_i\}$, $\llbracket \Phi \rrbracket$ means $\bigcap_i \llbracket \varphi_i \rrbracket$. \square

3.2 Abstract Domains

In this paper, the concrete domain \mathcal{C} is $\mathbb{P}(\mathcal{V} \rightarrow \{0, 1\})$. We will work with several abstract domains \mathcal{A} , each of which abstracts \mathcal{C} by a Galois connection $\mathcal{C} \xleftrightarrow[\alpha]{\gamma} \mathcal{A}$. We assume that the reader is familiar with the basic terminology of abstract interpretation [3] ($\perp, \top, \sqcup, \sqcap, \sqsubseteq, \alpha, \gamma$, monotonicity, distributivity, etc.), as well as with the properties of a Galois connection $\mathcal{C} \xleftrightarrow[\alpha]{\gamma} \mathcal{A}$.

Definition 2. An element R of the domain of **equivalence relations** (*Equiv*) over the set $\text{Literals}[\mathcal{V}]$ formed from Boolean variables \mathcal{V} , their negations, and Boolean constants represents a set of assignments in $\mathbb{P}(\mathcal{V} \rightarrow \{0, 1\})$. The special value \perp_{Equiv} represents the empty set of assignments, and will be denoted by “ $\mathbf{0} \equiv \mathbf{1}$ ”. Each other value $R \in \text{Equiv}$ is an equivalence relation on $\text{Literals}[\mathcal{V}]$; the concretization $\gamma(R)$ is the set of all assignments that satisfy all the equivalences in R . The ordering $a_1 \sqsubseteq_{\text{Equiv}} a_2$ means that equivalence relation a_1 is

a coarser partition of $\text{Literals}[\mathcal{V}]$ than a_2 . The value \top_{Equiv} is the identity relation, $\{(v, v) \mid v \in \mathcal{V}\}$, and thus represents the set of all assignments. $R_1 \sqcup R_2$ is the coarsest partition that is finer than both R_1 and R_2 .

An alternative way to define the same domain is to consider it as the domain of **two-variable Boolean affine relations** (2-BAR) over \mathcal{V} . Each element $R \in 2\text{-BAR}$ is a conjunction of Boolean affine constraints, where each constraint has one of the following forms:

$$v_i \oplus v_j = \mathbf{0} \quad v_i \oplus v_j \oplus \mathbf{1} = \mathbf{0} \quad v_i = \mathbf{0} \quad v_i \oplus \mathbf{1} = \mathbf{0},$$

which correspond to the respective equivalences

$$v_i \equiv v_j \quad v_i \equiv \neg v_j \quad v_i \equiv \mathbf{0} \quad v_i \equiv \mathbf{1}.$$

The value $\perp_{2\text{-BAR}}$ is any set of unsatisfiable constraints. The value $\top_{2\text{-BAR}}$ is the empty set of constraints. The concretization function $\gamma_{2\text{-BAR}}$, and abstraction function $\alpha_{2\text{-BAR}}$ are:

$$\begin{aligned} \gamma_{2\text{-BAR}}(R) &= \{c \in (\mathcal{V} \rightarrow \{0, 1\}) \mid R = \bigwedge_i r_i \text{ and for all } i, c \models r_i\} \\ \alpha_{2\text{-BAR}}(C) &= \bigwedge \{r \mid \text{for all } c \in C, c \models r\} \end{aligned}$$

For convenience, we will continue to use equivalence notation (\equiv) in examples that use 2-BAR, rather than giving affine relations (\oplus). \square

Definition 3. An element of the **Cartesian domain** represents a set of assignments in $\mathbb{P}(\mathcal{V} \rightarrow \{0, 1\})$. The special value $\perp_{\text{Cartesian}}$ denotes the empty set of assignments; all other values can be denoted via a 3-valued assignment in $\mathcal{V} \rightarrow \{0, 1, *\}$. The third value “*” denotes an unknown value, and the values 0, 1, * are ordered so that $0 \sqsubset *$ and $1 \sqsubset *$.

The partial ordering \sqsubseteq on 3-valued assignments is the pointwise extension of $0 \sqsubset *$ and $1 \sqsubset *$, and thus $\top_{\text{Cartesian}} = \lambda w. *$ and $\sqcup_{\text{Cartesian}}$ is pointwise join. The concretization function $\gamma_{\text{Cartesian}}$, and abstraction function $\alpha_{\text{Cartesian}}$ are:

$$\begin{aligned} \gamma_{\text{Cartesian}}(A) &= \{c \in (\mathcal{V} \rightarrow \{0, 1\}) \mid c \sqsubseteq A\} \\ \alpha_{\text{Cartesian}}(C) &= \lambda w. \sqcup \{c(w) \mid c \in C\} \end{aligned}$$

We will denote an element of the Cartesian domain as a mapping, e.g., $[p \mapsto 0, q \mapsto 1, r \mapsto *]$, or $[0, 1, *]$ if p, q , and r are understood. \square

Local Decreasing Iterations. Local decreasing iterations [8] is a technique that is ordinarily used for improving precision during the abstract interpretation of a program. During an iterative fixed-point-finding analysis, the technique of local decreasing iterations is applied at particular points in the program, such as, e.g., the interpretation of the true branch of an if-statement whose branch condition is φ . The operation that needs to be performed is the application of the abstract transformer for $\text{assume}(\varphi)$. As the name “local decreasing iterations” indicates, a purely local iterative process repeatedly applies the operator $\text{assume}(\varphi)$ either until some precision criterion or resource bound is attained, or a (local) fixed point is reached. The key theorem is stated as follows:

Theorem 1. ([8, Thm. 2]) An operator τ is a **lower closure operator** if it is monotonic, idempotent ($\tau \circ \tau = \tau$), and reductive ($\tau \sqsubseteq \lambda x. x$). Let τ be a lower closure operator on \mathcal{A} ; let (τ_1, \dots, τ_k) be a k -tuple of reductive operators

Stålmarck’s Method	Abstract-Interpretation Concept
Equivalence relation	Abstract-domain element
Propagation rule	Sound reductive operator
0-saturation	Local decreasing iterations
Split	Meet (\sqcap) in each proof-tree branch: one with a splitting-set element a and one with a ’s companion
Intersection (\cap)	Join (\sqcup)

Table 1: Abstract-interpretation account of Stålmarck’s method.

on \mathcal{A} , each of which over-approximates (\sqsupseteq) τ ; and let $(u_n)_{n \in \mathbb{N}}$ be a sequence of elements in $[1, \dots, k]$. Then the sequence of reductive operators on \mathcal{A} defined by

$$\eta_0 = \tau_{u_0} \quad \eta_{n+1} = \tau_{u_{n+1}} \circ \eta_n$$

is decreasing and each of its elements over-approximates τ . \square

Example 4. The propagation rules of Fig. 2 can be recast in terms of reductive operators that refine an element R of the 2-BAR domain as follows:

Operator	Derived from
$\tau_1(R) = R \cup ((v_1 \equiv \mathbf{0} \in R) ? \{v_2 \equiv \mathbf{0}, v_3 \equiv \mathbf{0}\} : \emptyset)$	$v_1 \Leftrightarrow (v_2 \vee v_3) \in \mathcal{I}$
$\tau_2(R) = R \cup ((\{a \equiv \mathbf{1}, b \equiv \mathbf{1}\} \subseteq R) ? \{v_2 \equiv \mathbf{1}\} : \emptyset)$	$v_2 \Leftrightarrow (a \wedge b) \in \mathcal{I}$
$\tau_3(R) = R \cup ((\{v_3 \equiv \mathbf{0}\} \in R) ? \{a \equiv \mathbf{1}, b \equiv \mathbf{1}\} : \emptyset)$	$v_3 \Leftrightarrow (\neg a \wedge \neg b) \in \mathcal{I}$
$\tau_4(R) = (\{v_2 \equiv \mathbf{0}, v_2 \equiv \mathbf{1}\} \subseteq R) ? \mathbf{0} \equiv \mathbf{1} : R$	

The operators τ_1 , τ_2 , and τ_3 instantiate the rules of Fig. 2 for the three integrity constraints shown in Fig. 1. The derivation described in Fig. 3 can now be stated as $\tau_4(\tau_2(\tau_3(\tau_1(\{v_1 \equiv \mathbf{0}\})))) = (\tau_4 \circ \tau_2 \circ \tau_3 \circ \tau_1)(\{v_1 \equiv \mathbf{0}\})$, which results in the abstract state $\mathbf{0} \equiv \mathbf{1}$.

Note that the operators given above are specialized for the given integrity constraints, while the rules in Fig. 2 are stated in a parametric fashion. We will revisit this issue in App. B. \square

4 The Generalized Framework

In this section, we map the concepts used in Stålmarck’s method to concepts used in abstract interpretation, as summarized in Tab. 1. The payoff is that we obtain a parametric framework for propositional validity-checking algorithms (Alg. 9) that can be instantiated in different ways by supplying different abstract domains. The proofs of all theorems stated in this section are found in App. A.

Definition 4. Given a Galois connection $\mathcal{C} \xleftrightarrow[\alpha]{\gamma} \mathcal{A}$ between abstract domain \mathcal{A} and concrete domain $\mathcal{C} = \mathbb{P}(\mathcal{V} \rightarrow \{0, 1\})$, an **acceptable splitting set** S for \mathcal{A} satisfies

1. $S \subseteq \mathcal{A}$
2. For every $a \in S$, there exists $b \in S$ such that $\gamma(a) \cup \gamma(b) = \gamma(\top)$. Two elements $a, b \in S$ such that $\gamma(a) \cup \gamma(b) = \gamma(\top)$ are called **companions**.

3. For every assignment $C \in \mathcal{V} \rightarrow \{0,1\}$ there exists $M_C \subseteq S$ such that $\gamma(\bigsqcap M_C) = C$. We call M_C the **cover** of C . \square

Example 5. The set of “single-point” partial assignments $\{\top[v \leftarrow 0]\} \cup \{\top[v \leftarrow 1]\}$ is an acceptable splitting set for both the Cartesian domain and the 2-BAR domain. Another acceptable splitting set for the 2-BAR domain is the set consisting of all 2-BAR elements that consist of a single constraint. \square

The assumptions of our framework are rather minimal:

1. There is a Galois connection $\mathcal{C} \xrightleftharpoons[\alpha]{\gamma} \mathcal{A}$ between \mathcal{A} and the concrete domain of assignments $\mathcal{C} = \mathbb{P}(\mathcal{V} \rightarrow \{0,1\})$.
2. \mathcal{A} is at least as expressive as the Cartesian domain (Defn. 3); that is, for all $A_c \in \text{Cartesian}$, there exists $A \in \mathcal{A}$ such that $\gamma_{\text{Cartesian}}(A_c) = \gamma_{\mathcal{A}}(A)$.
3. There is an algorithm to perform the join of arbitrary elements of the domain; that is, for all $A_1, A_2 \in \mathcal{A}$, there is an algorithm that produces $A_1 \sqcup A_2$.
4. There is an algorithm to perform the meet of arbitrary elements of the domain; that is, for all $A_1, A_2 \in \mathcal{A}$, there is an algorithm that produces $A_1 \sqcap A_2$.
5. There is an acceptable splitting set S for \mathcal{A} (Defn. 4).

Assumption 2 ensures that any instantiation that satisfies assumptions 1–4 will satisfy assumption 5: the set of “single-point” partial assignments inherited from the Cartesian domain (Ex. 5) is always an acceptable splitting set.

Note that because the concrete domain \mathcal{C} is over a finite set of Boolean variables, the abstract domain \mathcal{A} has no infinite descending chains. It is not hard to show that 2-BAR meets assumptions (1)–(5). The standard version of Stålmarck’s method (§2.1) is the instantiation of the framework presented in this section with the abstract domain 2-BAR.

At any moment during our generalization of Stålmarck’s method, each open branch B_i represents a set of variable assignments $C_i \in \mathcal{C}$ such that $\bigcup_i C_i \supseteq \llbracket \neg\varphi \rrbracket$. That is, each branch B_i represents an abstract state $A_i \in \mathcal{A}$ such that $\bigcup_i \gamma(A_i) \supseteq \llbracket \neg\varphi \rrbracket$. Let $\bar{A} = \bigsqcup_i A_i$. Then \bar{A} is sound, i.e., $\gamma(\bar{A}) \supseteq \bigcup_i \gamma(A_i) \supseteq \llbracket \neg\varphi \rrbracket$. The net result of the proof rules is to derive a *semantic reduction* \bar{A}' of \bar{A} with respect to the integrity constraints \mathcal{I} ; that is, $\gamma(\bar{A}') \cap \llbracket \mathcal{I} \rrbracket = \gamma(\bar{A}) \cap \llbracket \mathcal{I} \rrbracket$, and $\bar{A}' \sqsubseteq \bar{A}$. If the algorithm derives that $\bar{A}' = \perp_{\mathcal{A}}$, then the formula φ is proved valid.

Generalized Propagation Rules. The propagation rules aim to refine the abstract state by assuming a single integrity constraint $J \in \mathcal{I}$. It is possible to list all the propagation rules in the style of Fig. 2 for the 2-BAR domain; for brevity, Alg. 6 is stated in terms of the semantic properties that an individual propagation rule satisfies, expressed using the operations α , γ , and \sqcap of abstract domain \mathcal{A} . This procedure is sound if the abstract value \bar{A} returned satisfies $\gamma(\bar{A}) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$. Furthermore, to guarantee progress we have to show that Alg. 6 implements a reductive operator, i.e., $\bar{A} \sqsubseteq A$.

Theorem 2. [Soundness of Alg. 6] *Let $\bar{A} := \text{propagate}_{\mathcal{A}}(J, A_1, A, \mathcal{I})$ with $J \in \mathcal{I}$ and $A_1 \supseteq A$. Then $\gamma(\bar{A}) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$ and $\bar{A} \sqsubseteq A$. \square*

<hr/> <p>Algorithm 6: $\text{propagate}_{\mathcal{A}}(J, A_1, A, \mathcal{I})$</p> <hr/> <pre> 1 requires($J \in \mathcal{I} \wedge A_1 \sqsupseteq A$) 2 return $A \sqcap \alpha(\llbracket J \rrbracket \cap \gamma(A_1))$ </pre> <hr/> <p>Algorithm 7: $0\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})$</p> <hr/> <pre> 1 repeat 2 $A' \leftarrow A$ 3 foreach $J \in I, A_1 \sqsupseteq A$ such that $\text{voc}(J) \cup \text{voc}(A_1) < \epsilon$ do 4 $A \leftarrow \text{propagate}_{\mathcal{A}}(J, A_1, A, \mathcal{I})$ 5 until $(A = A') \parallel A = \perp_{\mathcal{A}}$ 6 return A </pre> <hr/>	<hr/> <p>Algorithm 8: $k\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})$</p> <hr/> <pre> 1 repeat 2 $A' \leftarrow A$ 3 foreach a, b that are companions such that $a \not\sqsupseteq A$ and $b \not\sqsupseteq A$ do 4 $A_1 \leftarrow A \sqcap a$ 5 $A_2 \leftarrow A \sqcap b$ 6 $A'_1 \leftarrow (k-1)\text{-saturation}_{\mathcal{A}}(A_1, \mathcal{I})$ 7 $A'_2 \leftarrow (k-1)\text{-saturation}_{\mathcal{A}}(A_2, \mathcal{I})$ 8 $A \leftarrow A'_1 \sqcup A'_2$ 9 until $(A = A') \parallel A = \perp_{\mathcal{A}}$ 10 return A </pre> <hr/>
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Fig. 9: The algorithms that make up the parametric framework for propositional validity checking. Alg. 7 is parameterized on ϵ , which bounds the number of variables considered in each propagation step.

Example 6. Let us apply Alg. 6 with $J = v_1 \Leftrightarrow (v_2 \vee v_3)$, $A_1 = \{v_1 \equiv \mathbf{0}\}$ and $A = \{v_1 \equiv \mathbf{0}, v_4 \equiv \mathbf{0}\}$. To save space, we use 3-valued assignments to represent the concrete states of assignments to v_1, \dots, v_4 .

$$\begin{aligned}
\llbracket J \rrbracket &= \{(1, 0, 1, *), (1, 1, 0, *), (1, 1, 1, *), (0, 0, 0, *)\} \\
\gamma(A_1) &= \{(0, *, *, *)\} \\
C = \llbracket J \rrbracket \cap \gamma(A_1) &= \{(0, 0, 0, *)\} \\
\alpha(C) &= \{v_1 \equiv \mathbf{0}, v_2 \equiv \mathbf{0}, v_3 \equiv \mathbf{0}\} \\
\bar{A} = A \sqcap \alpha(C) &= \{v_1 \equiv \mathbf{0}, v_2 \equiv \mathbf{0}, v_3 \equiv \mathbf{0}, v_4 \equiv \mathbf{0}\}
\end{aligned}$$

Thus, the value \bar{A} computed by Alg. 6 is exactly the abstract value that can be deduced by propagation rule OR1 of Fig. 2. \square

Generalized 0-Saturation. Alg. 7 shows the generalized 0-saturation procedure that repeatedly applies the propagation rules (line 4) using a single integrity constraint (line 3), until no new information is derived or a contradiction is found (line 5); $\text{voc}(\varphi)$ denotes the set of φ 's propositional variables.

To prove soundness we show that the abstract value \bar{A} returned by Alg. 7 satisfies $\gamma(\bar{A}) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$.

Theorem 3. [Soundness of Alg. 7]

For all $A \in \mathcal{A}$, $\gamma(0\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$. \square

Generalized k -Saturation. Alg. 8 describes the generalized k -saturation procedure that repeatedly applies the generalized Dilemma Rule. By requirement 5, there is an acceptable splitting set S for \mathcal{A} . The generalized Dilemma Rule, shown schematically in Fig. 10, splits the current abstract state A into two

Algorithm 9: k -Stålmarck $_{\mathcal{A}}(\varphi)$

```

1  $(v_\varphi, \mathcal{I}) \leftarrow \text{integrity}(\varphi)$ 
2  $A \leftarrow \top_{\mathcal{A}}[v_\varphi \leftarrow 0]$ 
3  $A' \leftarrow k\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})$ 
4 if  $A' = \perp_{\mathcal{A}}$  then return valid
5 else return unknown

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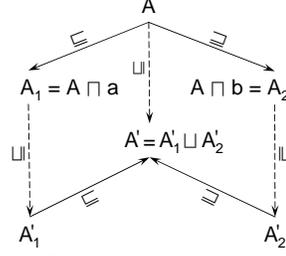


Fig. 10: Generalized Dilemma Rule.

abstract states A_1 and A_2 using companions $a, b \in S$. Using the fact that $\gamma(a) \cup \gamma(b) = \gamma(\top)$ (Defn. 4), we can show that $\gamma(A_1) \cup \gamma(A_2) = \gamma(A)$. This fact is essential for proving the soundness of the generalized Dilemma Rule. To merge the two branches of the generalized Dilemma Rule, we perform a join of the abstract states derived in each branch. The dashed arrows from A to A' , A_1 to A'_1 , and A_2 to A'_2 in Fig. 10 indicate that, in each case, the target value is a semantic reduction of the source value. The next theorem proves that Alg. 8, which utilizes the generalized Dilemma Rule, is sound.

Theorem 4. [Soundness of Alg. 8]

For all $A \in \mathcal{A}$, $\gamma(k\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$. □

Generalized k -Stålmarck. Alg. 9 describes our generalization of Stålmarck’s method, which is parameterized by an abstract domain \mathcal{A} . Line 1 converts the formula φ into the integrity constraints \mathcal{I} , with v_φ representing φ . We have to prove that Alg. 9 returns *valid* when the given formula φ is indeed valid.

Theorem 5. [Soundness of Alg. 9]

If k -Stålmarck $_{\mathcal{A}}(\varphi)$ returns *valid*, then $\llbracket \neg\varphi \rrbracket = \emptyset$. □

Completeness. As we saw in §2, Alg. 9 is not complete for all values of k . However, Alg. 9 is complete if k is large enough. To prove completeness we make use of item 3 of Defn. 4. After performing k -saturation, Alg. 9 has considered all assignments C that have a cover of size k . Let $\text{MinCover}[C] = \min\{|M| \mid M \subseteq S \text{ is a cover of } C\}$, and let $m = \max_{C \in \mathcal{C}} \text{MinCover}[C]$. m -Stålmarck $_{\mathcal{A}}(\varphi)$ will consider all assignments, and thus is complete; that is, if m -Stålmarck $_{\mathcal{A}}(\varphi)$ returns *unknown*, then φ is definitely not valid. App. B discusses the efficiency of our generalization of Stålmarck’s Method.

5 Instantiations

Stålmarck’s method is the instantiation of the framework from §4 with the abstract domain 2-BAR. In this section, we present the details for a few other instantiations of the framework from §4. As observed in §4, any instantiation that

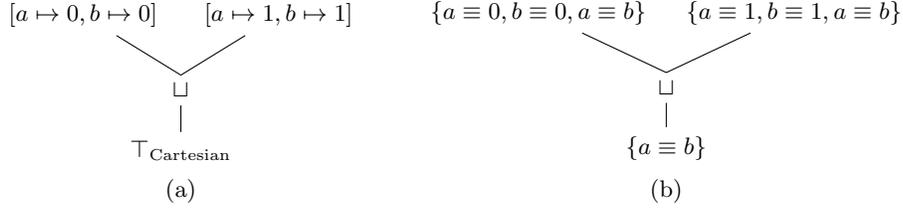


Fig. 11: An example in which the Cartesian domain and 2-BAR start with equivalent information in the respective branches, but (a) the Cartesian domain loses all information at the join, whereas (b) 2-BAR retains the equivalence $a \equiv b$.

satisfies the first four assumptions of the framework has an acceptable splitting set; hence, we only consider the first four assumptions in the discussion below.

Cartesian Domain. The original version of Stålmarck's method [16] did not use equivalence classes of propositional variables (i.e., the abstract domain 2-BAR). Instead, it was based on a weaker abstract domain of partial assignments, or equivalently, the Cartesian domain. It is easy to see that the Cartesian domain meets the requirements of the framework.

1. There is a Galois connection $\mathbb{P}(\mathcal{V} \rightarrow \{0, 1\}) \xleftarrow{\gamma} \text{Cartesian} \xrightarrow{\alpha}$. The α and γ functions are given in Defn. 3.
2. The Cartesian domain obviously meets the expressiveness condition of being as precise as the Cartesian domain.
3. The partial ordering \sqsubseteq on 3-valued assignments is the pointwise extension of $0 \sqsubseteq *$ and $1 \sqsubseteq *$. Thus, $\sqcup_{\text{Cartesian}}$ is pointwise join.
4. $\sqcap_{\text{Cartesian}}$ is pointwise meet.

Example 7. Fig. 11 presents an example in which the Cartesian domain and 2-BAR start with equivalent information in the respective branches, but the Cartesian domain loses all information at a join, whereas 2-BAR retains an equivalence. Consequently, the original version of Stålmarck's method [16] is less powerful than the standard version of Stålmarck's method that came later [15]. \square

Three-Variable Boolean Affine Relations (3-BAR). The abstract domain 3-BAR is defined almost identically to 2-BAR (Defn. 2). In general, a non-bottom element of 3-BAR is a satisfiable conjunction of constraints of the form $\bigoplus_{i=1}^3 (a_i \wedge x_i) \oplus b = \mathbf{0}$, where $a_i, b \in \{\mathbf{0}, \mathbf{1}\}$.

1. The definitions of the γ and α functions of the Galois connection $\mathbb{P}(\mathcal{V} \rightarrow \{0, 1\}) \xleftarrow{\gamma} \text{3-BAR} \xrightarrow{\alpha}$ are identical to those stated in Defn. 2.
2. 3-BAR generalizes 2-BAR, and so is more precise than the Cartesian domain.
3. $A_1 \sqcup A_2$ can be implemented by first extending A_1 and A_2 with all implied constraints, and then intersecting the extended sets.
4. $A_1 \sqcap A_2$ can be implemented by unioning the two sets of constraints.

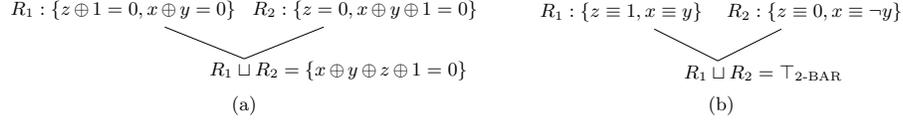


Fig. 12: 3-BAR (a) retains more information at the join than 2-BAR (b).

Example 8. Fig. 12 presents an example in which 2-BAR and 3-BAR start with equivalent information in the respective branches, but 2-BAR loses all information at a join, whereas 3-BAR retains an affine relation. Consequently, the instantiation of our framework with the 3-BAR domain provides a more powerful proof procedure than the standard version of Stålmarck’s method. \square

Two-Variable Boolean Inequality Relations (2-BIR). 2-BIR is yet another constraint domain, and hence defined similarly to 2-BAR and 3-BAR. A non-bottom element of 2-BIR is a satisfiable conjunction of constraints of the form $x \leq y$, $x \leq b$, or $b \leq x$, where $x, y \in \mathcal{V}$ and $b \in \{\mathbf{0}, \mathbf{1}\}$.

1. The definitions of γ and α are again identical to those given in Defn. 2.
2. An equivalence $a \equiv b$ can be represented using two inequality constraints, $a \leq b$ and $b \leq a$, and hence 2-BIR is more precise than 2-BAR, which in turn is more precise than the Cartesian domain.
3. $A_1 \sqcup A_2$ can be implemented by first extending A_1 and A_2 with all implied constraints, and then intersecting the extended sets.
4. $A_1 \sqcap A_2$ can be implemented by unioning the two sets of constraints.

Example 9. Fig. 13 presents an example in which 2-BAR and 2-BIR start with equivalent information in the respective branches, but 2-BAR loses all information at a join, whereas 2-BIR retains a Boolean inequality. Consequently, the instantiation of our framework with the 2-BIR domain provides a more powerful proof procedure than the standard version of Stålmarck’s method. \square

6 Experiments

As discussed in §1, a validity-checking algorithm can be used for checking satisfiability. In this section, we present preliminary experimental results for the following instantiations of our parametric framework:

- 1-Stålmarck[Cartesian]: uses 1-saturation and the Cartesian domain.

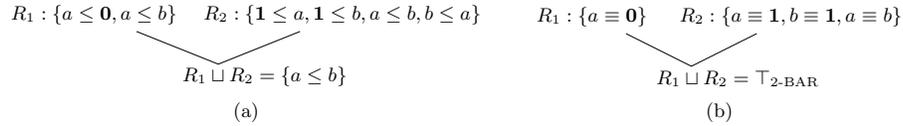


Fig. 13: 2-BIR (a) retains more information at the join than 2-BAR (b).

- 1-Stålmarck[2-BAR]: uses 1-saturation and the 2-BAR domain.
- 1-Stålmarck[2-BIR]: uses 1-saturation and the 2-BIR domain.
- 2-Stålmarck[Cartesian]: uses 2-saturation and the Cartesian domain.

We compared the above algorithms with the mature SAT solver, MiniSat (v2.2.0) solver [7]. For our evaluation, we used the Small, Difficult Satisfiability Benchmark (SDSB) suite, which contains 3,608 satisfiability benchmarks that have up to 800 literals, and have been found to be difficult for solvers [14]. We used a time-out limit of 500 seconds. If an algorithm could not determine whether a benchmark was satisfiable or unsatisfiable, then the solver is recorded as taking the full 500 seconds for that benchmark.

For each of the five algorithms, Fig. 14(a) is a semi-log plot in which each point (n, t) means that there were n benchmarks that were each solved correctly in no more than t seconds. Fig. 14(b) and Fig. 15 give log-log scatter plots of the time taken (in seconds) for each of the benchmarks, for several combinations of the five algorithms. As seen in Fig. 14(a), MiniSat correctly solves 3,484 of 3,608 benchmarks, and is significantly faster than 1-Stålmarck[2-BAR] (Fig. 14(b)).

When comparing among the instantiations of our framework, we expect more benchmarks to be solved correctly as we move to more expressive abstract domains. For instance, 1-Stålmarck[2-BAR] (1,545 benchmarks) solves 36 benchmarks that 1-Stålmarck[Cartesian] (1,509 benchmarks) was unable to solve. On the other hand, 1-Stålmarck[2-BAR] is slower than 1-Stålmarck[Cartesian], as seen in Fig. 15(a), because the join operation of the 2-BAR domain is more expensive than that of the Cartesian domain. The complexity of the join operation plays an even greater role for the 2-BIR domain: although 1-Stålmarck[2-BIR]

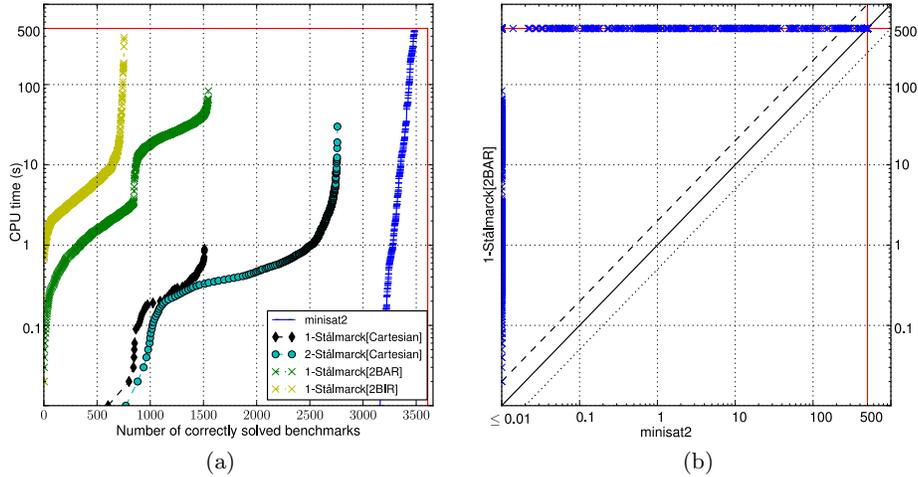


Fig. 14: (a) Semi-log plot showing the number of benchmarks that were each solved correctly in no more than t seconds. (b) Log-log scatter plot of the time taken (in seconds) by MiniSat versus 1-Stålmarck[2-BAR].

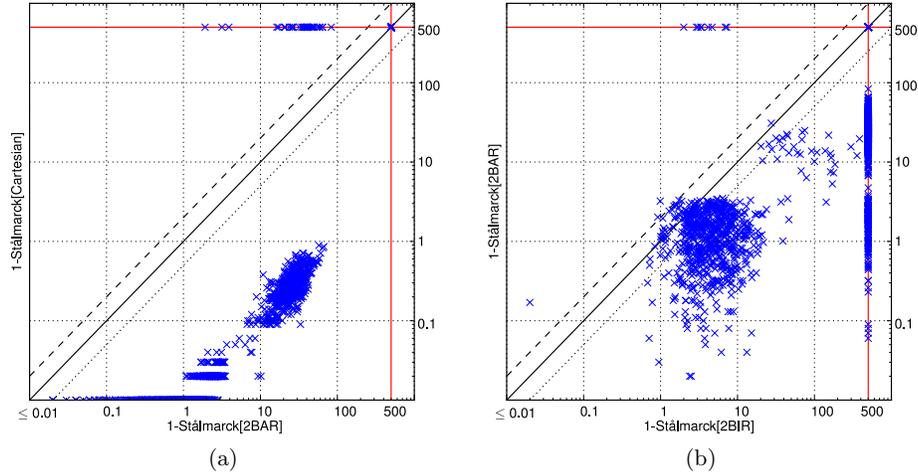


Fig. 15: Log-log scatter plots of the time taken (in seconds) by (a) $1\text{-Stålmärck}[2\text{-BAR}]$ versus $1\text{-Stålmärck}[\text{Cartesian}]$, and (b) $1\text{-Stålmärck}[2\text{-BIR}]$ versus $1\text{-Stålmärck}[2\text{-BAR}]$.

solves 9 benchmarks that $1\text{-Stålmärck}[2\text{-BAR}]$ was unable to solve, overall $1\text{-Stålmärck}[2\text{-BIR}]$ is only able to solve 754 benchmarks in the 500-second time limit. We are currently investigating more efficient implementations of the join algorithms for the various domains.

Using 2-saturation allows Stålmärck’s method instantiated with Cartesian domain to correctly solve 2,758 benchmarks (Fig. 14(a)), including 1,213 benchmarks that $1\text{-Stålmärck}[2\text{-BAR}]$ was unable to solve and 1,774 benchmarks that $1\text{-Stålmärck}[2\text{-BIR}]$ was unable to solve.

7 Related Work

Stålmärck’s method was patented under Swedish, U.S., and European patents [15]. Up until Jan. 4, 2011, when the U.S. patent expired, researchers interested in experimenting with Stålmärck’s method in the U.S. would have needed to obtain a license for the technology. According to Wikipedia [19],

“In 2002, the Court of Appeals for the Federal Circuit dramatically limited the scope of the research exemption [for performing research on patented methods without infringing] in *Madey v. Duke University*, 307 F.3d 1351, 1362 (Fed. Cir. 2002). The court did not reject the defense, but left only a ‘very narrow and strictly limited experimental use defense’ for ‘amusement, to satisfy idle curiosity, or for strictly philosophical inquiry.’ The court also precludes the defense . . . [for research] done ‘in furtherance of the alleged infringer’s legitimate business.’ In the case

of a research university . . . the court held that . . . the defense was inapplicable.”

Sheeran and Stålmarck [13] give a lucid presentation of the algorithm. Björk [1] explored extensions of Stålmarck's method to first-order logic. (Björk credits Stålmarck with making the first extension of the method in that direction, and mentions an unpublished manuscript of Stålmarck's.)

CDCL/DPLL solvers [12] are alternatives to Stålmarck's method for validity checking and SAT. D'Silva et al. [5] give an abstract-interpretation-based account of CDCL/DPLL SAT solvers. Thus, though having similar goals, our work and that of D'Silva et al. are complementary. Our work and that of D'Silva et al. were performed independently and contemporaneously. They have also lifted their technique from a propositional SAT solver to a floating-point decision procedure that makes use of floating-point intervals [6].

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A Proofs

Theorem 2. [Soundness of Alg. 6] *Let $\bar{A} := \text{propagate}_{\mathcal{A}}(J, A_1, A, \mathcal{I})$ with $J \in \mathcal{I}$ and $A_1 \sqsupseteq A$. Then $\gamma(\bar{A}) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$ and $\bar{A} \sqsubseteq A$.*

Proof.

$$\begin{aligned}
J \in \mathcal{I}, A_1 \sqsupseteq A &\Rightarrow \llbracket J \rrbracket \cap \gamma(A_1) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A_1) \\
&\Rightarrow \alpha(\llbracket J \rrbracket \cap \gamma(A_1)) \supseteq \alpha(\llbracket \mathcal{I} \rrbracket \cap \gamma(A_1)) \\
&\Rightarrow A \sqcap \alpha(\llbracket J \rrbracket \cap \gamma(A_1)) \supseteq A \sqcap \alpha(\llbracket \mathcal{I} \rrbracket \cap \gamma(A_1)) \\
&\Rightarrow \gamma(A \sqcap \alpha(\llbracket J \rrbracket \cap \gamma(A_1))) \supseteq \gamma(A \sqcap \alpha(\llbracket \mathcal{I} \rrbracket \cap \gamma(A_1))) \\
&\Rightarrow \gamma(A \sqcap \alpha(\llbracket J \rrbracket \cap \gamma(A_1))) \supseteq \gamma(A_1) \cap \gamma(\alpha(\llbracket \mathcal{I} \rrbracket \cap \gamma(A_1))) \\
&\Rightarrow \gamma(A \sqcap \alpha(\llbracket J \rrbracket \cap \gamma(A_1))) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A_1) \\
&\Rightarrow \gamma(\bar{A}) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A_1)
\end{aligned}$$

This argument proves that the value computed by Alg. 6 is sound.

Furthermore, $A \sqcap \alpha(\llbracket J \rrbracket \cap \gamma(A_1)) \sqsubseteq A$; therefore, $\bar{A} \sqsubseteq A$. This observation proves that Alg. 6 implements a reductive operator. \square

Theorem 3. [Soundness of Alg. 7]

For all $A \in \mathcal{A}$, $\gamma(0\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$.

Proof. In Alg. 7, the propagate procedure is repeatedly called on line 4 in a loop. By Thm. 2, we know that the call to propagate is a sound reductive operator. Thus, by directly applying Thm. 1 we can conclude that the value computed at the end of the loop on lines 3–4 is sound.

Termination of the outermost loop follows from the fact that \mathcal{A} has no infinite descending chains and propagate is reductive. \square

Theorem 4. [Soundness of Alg. 8]

For all $A \in \mathcal{A}$, $\gamma(k\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$.

Proof. We prove this via induction on k . Thm. 3 proves the base case when $k = 0$.

To prove the inductive case, assume that Alg. 8 is sound for $k - 1$, i.e.,

$$\text{for all } A \in \mathcal{A}, \gamma((k-1)\text{-saturation}_{\mathcal{A}}(A, \mathcal{I})) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A) \quad (4)$$

Let A^0 be the value of A passed as input to Alg. 8, and A^i be the value of A computed at the end of the i th iteration of the loop body consisting of lines 3–8; that is, the value of A computed in line 8.

We show via induction that, for each i , $\gamma(A^i) \supseteq \llbracket \mathcal{I} \rrbracket \cup \gamma(A^0)$. We first prove the base case for $i = 1$; that is, $A^1 \supseteq \llbracket \mathcal{I} \rrbracket \cup \gamma(A^0)$.

From line 4 of Alg. 8, we have $A_1 = A^0 \sqcap a$, which implies $\gamma(A_1) \supseteq \gamma(A^0) \cap \gamma(a)$. Similarly, by line 5, we have $\gamma(A_2) \supseteq \gamma(A^0) \cap \gamma(b)$, where b is a companion of a .

Now $\gamma(A_1) \cup \gamma(A_2) \supseteq \gamma(A^0) \cap (\gamma(a) \cup \gamma(b))$. By item 2 of Defn. 4, we know that $\gamma(a) \cup \gamma(b) = \gamma(\top_{\mathcal{A}})$. Thus, we obtain

$$\gamma(A_1) \cup \gamma(A_2) \supseteq \gamma(A^0) \quad (5)$$

By lines 6 and 7, we have $A'_1 = (\text{k-1-saturation}_{\mathcal{A}}(A_1))$ and $A'_2 = (\text{k-1-saturation}_{\mathcal{A}}(A_2))$.

By the induction hypothesis (Eqn. (4)), we have

$$\begin{aligned} \gamma(A'_1) &\supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A_1), \gamma(A'_2) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A_2) \\ \Rightarrow \gamma(A'_1) \cup \gamma(A'_2) &\supseteq \llbracket \mathcal{I} \rrbracket \cap (\gamma(A_1) \cup \gamma(A_2)) \\ \Rightarrow \gamma(A'_1) \cup \gamma(A'_2) &\supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A^0) \text{ (using Eqn. (5))} \\ \Rightarrow \gamma(A'_1 \sqcup A'_2) &\supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A^0) \end{aligned}$$

Thus, at line 8, we have $\gamma(A^1) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A^0)$.

An almost identical proof can be used to prove the inductive case for the induction on i . Thus, for all i , $\gamma(A^i) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A^0)$.

Because the value returned by Alg. 8 is the final value computed at line 8, we have proved the inductive case for the induction on k . Thus, by induction, we have shown that for all $A \in \mathcal{A}$, $\gamma(\text{k-saturation}_{\mathcal{A}}(A, \mathcal{I})) \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$. \square

Theorem 5. [Soundness of Alg. 9]

If $k\text{-Stålmarck}_{\mathcal{A}}(\varphi)$ returns valid, then $\llbracket \neg\varphi \rrbracket = \emptyset$.

Proof. The abstract state A created at line 2 corresponds to the assumption that there exists an assignment that falsifies φ . Thus, $\llbracket \neg\varphi \rrbracket = \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$. By Thm. 4, we have $\gamma(A') \supseteq \llbracket \mathcal{I} \rrbracket \cap \gamma(A)$ at line 3. Thus, if $A' = \perp_{\mathcal{A}}$ we know that $\llbracket \mathcal{I} \rrbracket \cap \gamma(A) = \emptyset$. Therefore, if Alg. 9 returns *valid*, $\llbracket \neg\varphi \rrbracket = \emptyset$. \square

B Efficiency

We now describe how some of the efficient implementation techniques that can be used with 2-BAR, as described in [13], can be generalized to other abstract domains. In each case, we express the essence of each technique in terms of abstract interpretation.

On line 3 of Alg. 7, the quantities J and A_1 are chosen so that $|\text{voc}(J) \cup \text{voc}(A_1)|$ is small. Such a choice enables efficient symbolic implementations of the operations used in Alg. 6, viz., implementing truth-table semantics on the limited vocabulary of size ϵ . Because J in Alg. 6 is a single integrity constraint, there are only a bounded number of Boolean operators involved in each propagation step. By limiting the size of $\text{voc}(J) \cup \text{voc}(A_1)$ (line 3 of Alg. 7), it is possible to generate automatically a bounded number of propagation-rule schemas to implement line 2 of Alg. 6.

Alg. 7 should choose J and A_1 such that $\text{voc}(J) \cap \text{voc}(A_1) \neq \emptyset$ to avoid useless calls to propagate that do not lead to new information being deduced. Furthermore, 0-saturation can be made more efficient by using semi-naive evaluation [18, Alg. 3.4]: 0-saturation should focus on the new information learned about A on each round of the loop on lines 3 and 4 of Alg. 7 to choose which J 's to use in subsequent propagate() steps.

Assuming that all of the required abstract-domain operations can be performed in polynomial time, Alg. 9 retains the property from the standard version of Stålmarck’s method that it runs in polynomial time in the size of the input formula.