High-Order Accurate Schemes for Incompressible Viscous Flow

John C. Strikwerda

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John C. Strikwerda†
Computer Sciences Department
University of Wisconsin-Madison

Abstract

We present new finite difference schemes for the incompressible Navier-Stokes equations. The schemes are based on two spatial differencing methods, one is fourth-order accurate and the other is sixth-order accurate. The temporal differencing is based on backward differencing formulas. The schemes use non-staggered grids and satisfy regularity estimates, guaranteeing smoothness of the solutions. The schemes are computationally efficient. Computational results demonstrating the accuracy are presented.

Keywords: incompressible Navier-Stokes, finite difference schemes, GMRES.

AMS(MOS) classifications: 65M05, 65N05, 76D05

1. Introduction.

High-order accurate finite difference schemes are important in scientific computation because they offer a means to obtain accurate solutions with less work than may be required for methods of lower accuracy. Finite difference methods are attractive because of the relative ease of implementation and flexibility.

In this paper we present new finite difference schemes for the incompressible Navier-Stokes equations. The schemes are based on two spatial differencing methods, one a fourth-order accurate method and one a sixth-order accurate method. There are several temporal differencing methods presented in section 7. These temporal schemes can be used with either of the spatial differencing methods. The temporal differencing is based on backward differencing formulas (BDF) that are used for stiff ordinary differential equations. The schemes are implicit and appear to be unconditionally stable for the Stokes equations. (A rigorous stability analysis is the subject of further research.)

High-order methods have been presented by Rai and Moin [13] and Lele [10] for the fractional step method proposed by Kim and Moin [9]. There is an excellent study of these methods in the paper by Tafti [20]. A disadvantage of these methods is that because they are explicit, there is a severe stability limit on the time step. Moreover, as pointed out by Perot [11], the pressure for fractional-step methods can be no better than first-order accurate in time. Projection methods also have difficulty with higher order accuracy in time, see Shen [16]. This is not so for the methods presented here, where the pressure can be determined to a high order of accuracy. For steady flows the method of Aubert and Deville [1] can be applied to yield fourth-order accuracy, at the expense of increasing the number of unknowns and computational complexity of the system. All of these methods use staggered meshes.

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The schemes presented in this paper are for orthogonal Cartesian grids on non-staggered grids, that is, the velocity components and pressure unknowns are assigned to a common grid. The schemes are for the two-dimensional Navier-Stokes equations, however, the methods for obtaining the equations extend easily to three dimensions and to generalizations of the Navier-Stokes equations. A second-order scheme similar to the ones presented here was presented in [19], although it used a less efficient solution procedure. The methods presented here have been incorporated into a domain decomposition method.

The schemes presented here have been tested on sample problems with low Reynolds numbers. The tests show that the schemes are very accurate and efficient for these low Reynolds number computations. The extension of these methods to high Reynolds number is the subject of further research.

The nondimensional time-dependent incompressible Navier-Stokes system of equations is:

$$\vec{u}_t - \frac{1}{R}\nabla^2\vec{u} + \vec{\nabla}(\vec{u}\vec{u}^T) + \vec{\nabla}p = \vec{f} \tag{1.1}$$

$$\vec{\nabla}\cdot\vec{u} = g. \tag{1.2}$$

The vector function $\vec{u}$ is the velocity and the scalar function p is the pressure. The Reynolds number R measures the strength of the inertial effects relative to the viscous effects. Notice that the pressure appears only in (1.1) and only in terms of its spatial derivatives. We refer to equations (1.1) as the momentum equations and equation (1.2) as the divergence equation.

The functions $\vec{f}$ and g are considered to be given data. In most problems the function g in (1.2) is identically zero, but we include the general case because it fits in naturally with our methods and it is useful in checking the accuracy of the computer implementation of the methods. In particular, the accuracy can be checked by choosing the velocity and pressure to be arbitrary polynomials of the proper degree (see section 8).

In the limit as R tends to zero, with a rescaling of t, p, and $\vec{f}$, the Navier-Stokes system can be replaced by the time-dependent Stokes system, which is:

$$\vec{u}_t - \nabla^2\vec{u} + \vec{\nabla}p = \vec{f} \tag{1.3}$$

$$\vec{\nabla}\cdot\vec{u} = g. \tag{1.4}$$

We consider the Navier-Stokes system holding in a domain Ω and to specify a unique solution boundary conditions must be given. The simplest conditions are to specify the velocity $\vec{u}$ on the boundary, i.e.,

$$\vec{u} = \vec{b} \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

This is called the Dirichlet boundary condition. To limit our discussion we only consider Dirichlet boundary conditions in this paper. The modifications needed for other boundary conditions should not be difficult to implement.

The system (1.1) and (1.2) has a solution only if the integrability condition

$$\int_{\partial\Omega} \vec{n}\cdot\vec{b} = \int_{\Omega} g \tag{1.6}$$

is satisfied. This condition is a constraint relating function g in (1.2) and (1.4) and the boundary data $\vec{b}$ in (1.5).

The schemes we develop are derived using the difference calculus. By considering the total system in the derivation we obtain schemes that are compact, that is the stencil of the scheme is about as small as possible. In particular, the schemes presented here have smaller stencils than those of Lele [10] and Rai and Moin [13]. However, to obtain usable schemes two other aspects must be taken into account. These are the regularity of the scheme and the behavior at boundaries. The regularity of the scheme is important to assure that the solutions are smooth, that is the high frequency modes are prevented from dominating the error. The schemes have parameters that remove these high frequency modes, often referred to as checkerboard pressure oscillations. The difficulty at the boundaries is related to the size of the stencil. Since the stencil increases in size as the order increases, the amount of modification required at the boundary also increases. These topics are addressed in sections 5 and 6.

The schemes we derived can be used for both the steady-state and time-dependent equations. We consider only schemes for which the temporal differencing and spatial differencing are independent of each other. The spatial differencing is discussed in sections 3 and 4 and the temporal differencing is discussed in section 7.

We avoid modifying the Navier-Stokes equations such as is done with the 'Poisson pressure equation' method. One difficulty with such methods is the need to decide on additional boundary conditions, especially on the pressure. This is also true for projection methods, see Gresho [6]. In our approach the linear systems that must be solved to determine the solution at each time step involve both the momentum and divergence equations. These large systems are solved by preconditioned GMRES methods, [15]. One advantage of our approach is that the pressure can be obtained with the same order of accuracy in space as the velocity, and better than first-order in time which is the limit with fractional-step methods [11].

We do not use the finite volume approach, relying on the power of the symbolic difference calculus to obtain high accuracy with compact stencils. Our schemes do not satisfy exact conservation laws for mass or momentum. The accuracy of the solutions implies that the conservation laws should be satisfied to a high degree of accuracy. The schemes are based on the conservation form of the differential equations.

The structure of the paper is as follows. In section 2 we present the notation for the basic difference operators. In sections 3 and 4 we present the spatial differencing methods for orders four and six, respectively. In section 5 we discuss the numerical boundary conditions needed for both schemes. Section 6 discusses the regularity of the two schemes for steady-state computations. The multistep schemes used for the temporal differencing are discussed in section 7, and in section 8, the numerical tests of the methods are discussed. Conclusions are presented in section 9.

2. Notation.

We develop our schemes for regular two-dimensional Cartesian grids, with grid spacing Δx and Δy, respectively. We use the notation δ_{x0} to denote the first-order central difference with respect to x and is defined by

$$\delta_{x0} f_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x}.$$

The forward and backward operators are denoted by δ_{x+} and δ_{x-}, respectively, and are defined by

$$\delta_{x+} f_i = \frac{f_{i+1} - f_i}{\Delta x}$$

and

$$\delta_{x-} f_i = \frac{f_i - f_{i-1}}{\Delta x}.$$

The standard second-order central difference is denoted δ_x^2 and is defined by

$$\delta_x^2 f_i = \delta_{x+}\delta_{x-} f_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}.$$

Difference operators δ_{y0}, δ_{y+}, etc. are defined similarly.

We obtain most of our difference formulas from basic identities relating derivatives to the difference operators δ_x^2 and δ_{x0}. For the first-derivative, we use the identity

$$\frac{\partial}{\partial x} = \left(1 + \left(\frac{\Delta x \delta_x}{2}\right)^2\right)^{-1/2} \frac{\sinh^{-1}\left(\frac{1}{2}\Delta x \delta_x\right)}{\frac{1}{2}\Delta x \delta_x} \delta_{x0}, \tag{2.1}$$

see [18]. The basic identity we use that relates the second derivative to δ_x^2 is, see [18],

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\sinh^{-1}\left(\frac{1}{2}\Delta x \delta_x\right)}{\frac{1}{2}\Delta x}\right)^2 . \tag{2.2}$$

By expanding these expressions as Taylor series in Δx to appropriate powers, we may obtain difference approximations of any order.

To handle the modifications at the boundaries, we use two formulas that relate forward and backward differences. These are

$$\delta_{x-} = \frac{\delta_{x+}}{1 + \Delta x \delta_{x+}} \tag{2.3}$$

and

$$\delta_{x+} = \frac{\delta_{x-}}{1 - \Delta x \delta_{x-}}.$$

These two relations both arise from the identity

$$\delta_x^2 = \delta_{x+}\delta_{x-} = \frac{\delta_{x+} - \delta_{x-}}{\Delta x}.$$

3. The Fourth-Order Scheme.

Our fourth-order scheme for the Navier-Stokes equations is based on the approximations

$$\frac{\partial}{\partial x} = \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0} + O(\Delta x)^4 \tag{3.1}$$

and

$$\frac{\partial}{\partial x} = \left(1 + \frac{\Delta x^2}{6}\delta_x^2\right)^{-1}\delta_{x0} + O(\Delta x)^4 \tag{3.2}$$

for the first derivatives from (2.1) and

$$\frac{\partial^2}{\partial x^2} = \left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2 + O(\Delta x)^4 \tag{3.3}$$

for the second derivatives from (2.2).

Using the approximations (3.1) and (3.3), we have that the first two equations of (1.1) are approximated as

$$u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}(u^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}(uv) + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}p$$
$$= \frac{1}{R}\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2 u + \frac{1}{R}\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)^{-1}\delta_y^2 u + f_1 + O(\Delta)^4$$

and

$$v_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}(uv) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}(v^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}p$$
$$= \frac{1}{R}\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2 v + \frac{1}{R}\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)^{-1}\delta_y^2 v + f_2 + O(\Delta)^4 .$$

We have used the symbol $O(\Delta)^4$ for $O(\Delta x)^4 + O(\Delta y)^4$. The discretization of the derivative in time is discussed in section 7. Operating through with the product

$$\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right) \tag{3.4}$$

we obtain

$$\left(1+\frac{\Delta x^2}{12}\delta_x^2\right)\left(1+\frac{\Delta y^2}{12}\delta_y^2\right)\left(u_t+\left(1-\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}(u^2)+\left(1-\frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}(uv)\right.$$
$$\left.+\left(1-\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}p\right)$$
$$=\frac{1}{R}\left(1+\frac{\Delta y^2}{12}\delta_y^2\right)\delta_x^2u+\frac{1}{R}\left(1+\frac{\Delta x^2}{12}\delta_x^2\right)\delta_y^2u$$
$$+\left(1+\frac{\Delta x^2}{12}\delta_x^2+\frac{\Delta y^2}{12}\delta_y^2\right)f_1+O(\Delta)^4$$

for the first component of the velocity, and similarly for the other component. The stencil for the second-order difference terms from the Laplacian has the shape

$$\begin{array}{ccc}\bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\end{array}$$

where the scheme is centered about the center of the stencil. The coefficients for the difference in the x-direction are

$$\frac{1}{12}\begin{bmatrix}1 & -2 & 1\\ 10 & -20 & 10\\ 1 & -2 & 1\end{bmatrix}.$$

The terms for the first difference in x for the convection terms and pressure gradient become

$$\left(1+\frac{\Delta x^2}{12}\delta_x^2\right)\left(1+\frac{\Delta y^2}{12}\delta_y^2\right)\left(1-\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}+O(\Delta)^4$$
$$=\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}+O(\Delta)^4\,.$$

The stencil for the terms in this last expression (other than the $O(\Delta)^4$ terms) has the shape

$$\begin{array}{ccccc} & \bullet & & \bullet & \\ \bullet & \bullet & \times & \bullet & \bullet\\ & \bullet & & \bullet & \end{array}$$

where the $\times$ marks the center point. The difference approximations in the $y-$direction are similar. Notice that the stencil must be modified for points one grid spacing from the boundary. The modifications are discussed in section 5.

To insure the regularity, we take the pressure gradient expression to be

$$\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}p+\gamma\Delta x^4\delta_x^4\delta_{x+}p.$$

As shown in section 6, the term with γ positive will insure the regularity of the solution. Notice that this term does not degrade the accuracy of the difference formula.

Thus the scheme for the first momentum equation, with the exception of the time differencing, is

$$\begin{aligned}
&\left(1+\frac{\Delta x^2}{12}\delta_x^2+\frac{\Delta y^2}{12}\delta_y^2\right)u_t \\
&+\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}(u^2)+\left(1-\frac{\Delta y^2}{12}\delta_y^2+\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{y0}(uv) \\
&+\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}p+\gamma\Delta x^4\delta_x^4\delta_{x+}p \\
&=\frac{1}{R}\left(1+\frac{\Delta y^2}{12}\delta_y^2\right)\delta_x^2u+\frac{1}{R}\left(1+\frac{\Delta x^2}{12}\delta_x^2\right)\delta_y^2u \\
&+\left(1+\frac{\Delta x^2}{12}\delta_x^2+\frac{\Delta y^2}{12}\delta_y^2\right)f_1
\end{aligned} \tag{3.5}$$

and similarly for the other momentum equation. Notice that the terms on the right-hand side of (3.5) are essentially the standard fourth-order accurate scheme for the Poisson equation derived by Rosser [14].

We next consider the approximation of the divergence equation (1.2) or (1.4). Using the approximation (3.2) on (1.2), we have

$$\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)^{-1}\delta_{x0}u+\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)^{-1}\delta_{y0}v=g+O(\Delta)^4.$$

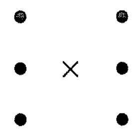
Operating with the product

$$\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)$$

we obtain

$$\begin{aligned}
&\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)\delta_{x0}u+\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{y0}v \\
&=\left(1+\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta y^2}{6}\delta_y^2\right)g+O(\Delta)^4.
\end{aligned} \tag{3.6}$$

The stencil for the terms for the differencing in u has the shape



where the $\times$ marks the center point. The stencil for the differencing of v is similar, but rotated a quarter turn.

To insure the regularity of the scheme we modify (3.6) to give the scheme

$$
\begin{aligned}
&\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)\delta_{x0}u+\gamma\Delta x^4\delta_x^4\delta_{x-}u+\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{y0}v+\gamma\Delta y^4\delta_y^4\delta_{y-}v \\
&=\left(1+\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta y^2}{6}\delta_y^2\right)g.
\end{aligned}
\tag{3.7}
$$

Notice that the divergence operator and the gradient operator are not adjoints of each other.

4. The Sixth-Order Scheme.

The sixth-order accurate scheme for the Navier-Stokes equation is based on expanding (2.1) to terms that are $O(\Delta x)^6$. The approximation for the first derivative is

$$
\frac{\partial}{\partial x}=\left(1-\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}+O(\Delta x)^6. \tag{4.1}
$$

This equation is used to approximate first derivatives in the convection terms and the pressure gradient. It has a stencil involving seven points and for the divergence equation it is desirable to find a formula of sixth-order accuracy with a smaller stencil. An approximation giving a smaller stencil is

$$
\frac{\partial}{\partial x}=\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)^{-1}\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}+O(\Delta x)^6\,. \tag{4.2}
$$

Similarly we obtain from (2.2)

$$
\frac{\partial^2}{\partial x^2}=\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)^{-1}\left(1+\frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2+O(\Delta x)^6 \tag{4.3}
$$

for the second derivatives.

Using the approximations (4.1) and (4.3) we have that the first of the two components of (1.1) may be approximated as

$$
\begin{aligned}
&u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}(u^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2 + \frac{\Delta y^4}{30}\delta_y^4\right)\delta_{y0}(uv) \\
&+ \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}p \\
&= \frac{1}{R}\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)^{-1}\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 u + \frac{1}{R}\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)^{-1}\left(1 + \frac{\Delta y^2}{20}\delta_y^2\right)\delta_y^2 u \\
&+ f_1 + O(\Delta)^6
\end{aligned}
$$

and similarly for the second momentum equation.

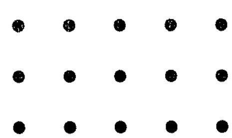
Operating through with the product

$$
\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right) \tag{4.4}
$$

we obtain

$$
\begin{aligned}
&\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left[u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}(u^2)\right. \\
&\left. + \left(1 - \frac{\Delta y^2}{6}\delta_y^2 + \frac{\Delta y^4}{30}\delta_y^4\right)\delta_{y0}(uv) + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}p\right] \\
&= \frac{1}{R}\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 u + \frac{1}{R}\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{\Delta y^2}{20}\delta_y^2\right)\delta_y^2 u \\
&+ \left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right) f_1 + O(\Delta)^6 .
\end{aligned}
$$

The stencil for the second-order difference terms in the x direction has the shape



centered around the center point. Notice that this difference approximation requires modification for two points away from the boundary.

The terms for the first-order difference in x become

$$\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}+O(\Delta)^6$$

$$=\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2+\frac{\Delta x^4}{90}\delta_x^4\right)\delta_{x0}+O(\Delta)^6$$

$$=\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}+O(\Delta)^6$$

The stencil for these terms (other than the $O(\Delta)^6$ terms) has the shape

$$\begin{array}{ccccccc} & \bullet & \bullet & & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \times & \bullet & \bullet & \bullet \\ & \bullet & \bullet & & \bullet & \bullet & \end{array}$$

where the $\times$ marks the center point.

To insure the regularity, we take the pressure gradient expression for the first momentum equation to be

$$\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}p-\gamma\Delta x^6\delta_x^6\delta_{x+}p.$$

As shown in section 6, the term with γ positive will insure the regularity of the solution, and again the additional regularity term does not degrade the order of accuracy.

Thus the scheme, with the exception of the time differencing, is

$$\begin{aligned}
&\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)u_t \\
&\quad+\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}(u^2) \\
&\quad+\left[\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1-\frac{\Delta y^2}{30}\delta_y^2\right)+\frac{\Delta y^4}{90}\delta_y^4\right]\delta_{y0}(uv) \\
&\quad+\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}p-\gamma\Delta x^6\delta_x^6\delta_{x+}p \\
&=\frac{1}{R}\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1+\frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2u\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{\Delta y^2}{20}\delta_y^2\right)\delta_y^2u\right] \\
&\quad+\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)f_1
\end{aligned}\tag{4.5}$$

and similarly for the other momentum equation.

We next consider the approximation of the divergence equation (1.2) or (1.4). Using the approximation (4.2) on (1.2), we have

$$\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)^{-1}\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}u+\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)^{-1}\left(1+\frac{\Delta y^2}{30}\delta_y^2\right)\delta_{y0}v$$
$$=g+O(\Delta)^6\,.$$

Operating with the product

$$\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)$$

we obtain

$$\begin{aligned}&\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}u+\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{30}\delta_y^2\right)\delta_{y0}v\\&=\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)g+O(\Delta)^6\,.\end{aligned}\tag{4.6}$$

To insure the regularity of the scheme we modify (4.6) as

$$\begin{aligned}&\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}u-\gamma\Delta x^6\delta_x^6\delta_{x-}u\\&\quad+\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{30}\delta_y^2\right)\delta_{y0}v-\gamma\Delta y^6\delta_y^6\delta_{y-}v\\&=\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)g.\end{aligned}\tag{4.7}$$

The stencil for the difference operator on u, other than the term multiplied by γ, is

$$\begin{array}{ccccc}\bullet & \bullet & & \bullet & \bullet\\ \bullet & \bullet & \times & \bullet & \bullet\\ \bullet & \bullet & & \bullet & \bullet\end{array}$$

where the $\times$ marks the center point.

One disadvantage of the sixth-order scheme over the fourth-order scheme is that because of the necessary boundary modifications, the stencil for the Laplacian is not symmetric for the sixth-order scheme. Some implications of this are described in the section 7.

5. Boundaries and Extrapolation of Pressure.

For the higher-order methods, the stencils are so wide that some modification of the schemes is required at boundaries. Also, for all of the schemes the pressure values on the boundary must be determined by extrapolation. The regularity terms, those multiplied by γ, are removed whenever they conflict with boundaries.

We index the grid points by nonnegative integers starting from 0. For a rectangle the grid points are indexed by (i, j) for $i = 0, 1, 2, ..., M$ and $j = 0, 1, 2, ..., N$ for some integers M and N. We consider only the boundary points with $i = 0$, the other boundaries are handled similarly.

We use the identity (2.3) to replace backward differences with forward differences. In particular we use

$$\delta_x^2 = \delta_{x+}\delta_{x-} = (1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2 + O(\Delta x^4) \tag{5.1}$$

We also use the relation

$$\begin{aligned}\delta_{x0} &= \frac{1}{2}(\delta_{x+} + \delta_{x-}) = \frac{1}{2}\left(\delta_{x+} + \frac{\delta_{x+}}{1 + \Delta x\delta_{x+}}\right) \\ &= \delta_{x+} - \frac{1}{2}\Delta x\delta_{x+}^2 + O(\Delta x)^2 .\end{aligned} \tag{5.2}$$

For the fourth-order scheme, the differencing for the convection and gradient terms need modification near the boundary. For $i = 1$, the terms

$$\left(1 + \frac{\Delta y^2}{12}\delta_y^2 - \frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}$$

are replaced by

$$\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)\delta_{x0} - \frac{\Delta x^2}{12}\delta_x^2\left(\delta_{x+} - \frac{1}{2}\Delta x\delta_{x+}^2\right)$$

For the sixth-order scheme, both the differences for the Laplacian and the convection terms need modification near the boundary. From the Laplacian, the terms

$$\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 u_{i,j}$$

for $i = 1$ are replaced by

$$\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 + \frac{\Delta x^2}{20}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2\right)\delta_x^2 u .$$

The expression

$$\left(1 + \frac{\Delta x^2}{20}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2\right)\delta_x^2\phi$$

at grid point i is

$$\frac{24\phi_{i-1} - 62\phi_i + 72\phi_{i+1} - 69\phi_{i+2} + 56\phi_{i+3} - 28\phi_{i+4} + 8\phi_{i+5} - \phi_{i+6}}{20\Delta x^2}.$$

Similar modifications are made at the other boundaries.

For the pressure gradient and convection terms, we also use (5.1). At $i = 1$, the expression

$$\left[\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 - \frac{\Delta x^2}{30}\delta_x^2\right) + \frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}p \tag{5.3}$$

from the sixth-order scheme (4.5) is replaced by

$$\begin{aligned}&\left[\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 - \frac{\Delta x^2}{30}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2\right)\right]\delta_{x0}p \\ &\qquad + \frac{\Delta x^4}{90}(1 - \frac{3}{2}\Delta x\delta_{x+})\delta_{x+}^4\delta_{x-}p.\end{aligned} \tag{5.4}$$

In this last expression, the fourth-order divided difference was modified using the relations

$$\delta_x^4 = \delta_x^2\delta_{x+}^2(1 - \Delta x\delta_{x+}) + O(\Delta^2)$$

and the central difference was modified using (5.2).

The expression

$$\left(1 - \frac{\Delta x^2}{30}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2)\delta_{x+}^2\right)\delta_{x0}p_1$$

in (5.4) expands to

$$\frac{-27p_0 - 9p_1 + 37p_2 + 4p_3 - 9p_4 + 5p_5 - p_6}{60\Delta x}$$

and $\frac{\Delta x^4}{90}\delta_x^2\delta_{x+}^3\left(1 - \frac{3}{2}\Delta x\delta_{x+}\right)p_{1,j}$ expands to

$$\frac{-5p_{0,j} + 28p_{1,j} - 65p_{2,j} + 80p_{3,j} - 55p_{4,j} + 20p_{5,j} - 3p_{6,j}}{180\Delta x}.$$

At $i = 2$ the expression (5.3) is replaced by

$$\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 - \frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}p + \frac{\Delta x^4}{90}\delta_x^4(1 - \frac{1}{2}\Delta x\delta_{x+})\delta_{x+}p \tag{5.5}$$

where we used (5.2) on the central first-order difference.

The values of the pressure on the boundaries must be set by extrapolation from values in the interior. For a scheme of order r the extrapolation should have order $r + 1$ to insure that the scheme is exact for polynomials of degree r. In this work the formula used to determine $p_{0,j}$ was

$$\delta_{x+}^{r+1} p_{0,j} = 0 \; . \tag{5.6}$$

For the fourth-order scheme the extrapolation is the fifth-order formula

$$p_{0,j} = 5p_{1,j} - 10p_{2,j} + 10p_{3,j} - 5p_{4,j} + p_{5,j} \; .$$

Similarly, for the sixth-order scheme the extrapolation is the seventh order formula

$$p_{0,j} = 7p_{1,j} - 21p_{2,j} + 35p_{3,j} - 35p_{4,j} + 21p_{5,j} - 7p_{6,j} + p_{7,j} \; .$$

Values of the pressure must also be set in the corners of the Cartesian grids. In this work the formula used to determine $p_{0,0}$ was, for the method with $r = 2t$,

$$\left(\delta_{x+}^{t} \delta_{y+}^{t+1} + \delta_{x+}^{t+1} \delta_{y+}^{t}\right) p_{0,0} = 0 \; ,$$

where the other boundary values of p have been determined by the boundary extrapolation (5.6).

For the fourth-order scheme the corner extrapolation is

$$\begin{aligned} p_{0,0} = (& \qquad\qquad\;\; 5\,p_{1,0} - 4\,p_{2,0} + p_{3,0} \\ & +5\,p_{0,1} - 12\,p_{1,1} + 9\,p_{2,1} - 2\,p_{3,1} \\ & -4\,p_{0,2} + 9\,p_{1,2} - 6\,p_{2,2} + p_{3,2} \\ & + p_{0,3} - 2\,p_{1,3} + p_{2,3})/2 \quad . \end{aligned}$$

Similarly, for the sixth-order scheme the corner extrapolation is

$$\begin{aligned} p_{0,0} = (& \qquad\qquad\;\; 7\,p_{1,0} - 9\,p_{2,0} + 5\,p_{3,0} - p_{4,0} \\ & + 7\,p_{0,1} - 24\,p_{1,1} + 30\,p_{2,1} - 16\,p_{3,1} + 3\,p_{4,1} \\ & - 9\,p_{0,2} + 30\,p_{1,2} - 36\,p_{2,2} + 18\,p_{3,2} - 3\,p_{4,2} \\ & + 5\,p_{0,3} - 16\,p_{1,3} + 18\,p_{2,3} - 8\,p_{3,3} + p_{4,3} \\ & - p_{0,4} + 3\,p_{1,4} - 3\,p_{2,4} + p_{3,4})/2 \quad . \end{aligned}$$

6. The Regularity of the Schemes.

In this section we check the regularity of the schemes. As shown in [3] a scheme must be regular in order to insure that the solution be smooth. We consider only the steady equations since the theory has only been developed for the steady state case. The importance of regularity is shown in the examples in section 8.

To test the regularity of the fourth-order scheme consisting of (3.5), the similar formula for v, and (3.7), we examine the symbol of the principal part of the scheme. We use the Fourier transform to determine the symbol by replacing $u_{\ell,m}$ by $\hat{u}\; e^{2i(x\theta_1 + y\theta_2)}$ and similarly for v and p. Because of the factor of 2 in the exponential, we are only concerned with θ_1 and θ_2 in the range $-\pi/2$ to $\pi/2$.

We may write the symbol as

$$\begin{pmatrix} L(\theta_1,\theta_2) & 0 & iG(\theta_1,\theta_2)/\Delta x \\ 0 & L(\theta_1,\theta_2) & iG(\theta_2,\theta_1)/\Delta y \\ iD(\theta_1,\theta_2)/\Delta x & iD(\theta_2,\theta_1)/\Delta y & 0 \end{pmatrix} \tag{6.1}$$

where

$$L(\theta_1,\theta_2) = 4\left(1-\frac{\sin^2\theta_2}{3}\right)\frac{\sin^2\theta_1}{\Delta x^2} + 4\left(1-\frac{\sin^2\theta_1}{3}\right)\frac{\sin^2\theta_2}{\Delta y^2} \tag{6.2}$$

$$G(\theta_1,\theta_2) = \left(1-\frac{\sin^2\theta_2}{3}+\frac{\sin^2\theta_1}{3}\right)\sin(2\theta_1) + 2^5\gamma\sin^5\theta_1 e^{i\theta_1} \tag{6.3}$$

and

$$D(\theta_1,\theta_2) = \left(1-\frac{2\sin^2\theta_2}{3}\right)\sin(2\theta_1) + 2^5\gamma\sin^5\theta_1 e^{-i\theta_1}\ . \tag{6.4}$$

The determinant of this system is

$$L(\theta_1,\theta_2)\ \left(G(\theta_1,\theta_2)D(\theta_1,\theta_2)/\Delta x^2\ +\ G(\theta_2,\theta_1)D(\theta_2,\theta_1)/\Delta y^2\right)\ . \tag{6.5}$$

The scheme is regular precisely when this determinant vanishes only for $\theta_1 = \theta_2 = 0$, with $|\theta_1|$ and $|\theta_2|$ less than or equal to $\pi/2$. The factor $L(\theta_1,\theta_2)$ does not vanish for nonzero θ_1 and θ_2, so we need consider only the other factor in (6.5). We first evaluate the product $G(\theta_1,\theta_2)D(\theta_1,\theta_2)$. We have

$$\begin{aligned} G(\theta_1,\theta_2)D(\theta_1,\theta_2) = &\Bigg[\left(1-\frac{1}{3}\sin^2\theta_2+\frac{1}{3}\sin^2\theta_1\right)\left(1-\frac{2}{3}\sin^2\theta_2\right) \\ &\quad + i2^5\gamma\sin^4\theta_1\left(1-\frac{1}{2}\sin^2\theta_2+\frac{1}{6}\sin^2\theta_1\right)\Bigg]\sin^2 2\theta_1 \\ &\quad + 2^{10}\gamma^2\sin^{10}\theta_1 - i2^5\gamma\left(\frac{\sin^2\theta_1+\sin^2\theta_2}{3}\right)\sin^6\theta_1\sin 2\theta_1\ . \end{aligned}$$

An examination of this expression easily shows that the real part is a sum of non-negative terms, and for $|\theta_1|$ and $|\theta_2|$ less than or equal to $\pi/2$ it vanishes only for $\theta_1 = 0$. Thus, $G(\theta_1,\theta_2)D(\theta_1,\theta_2)/\Delta x^2 + G(\theta_2,\theta_1)D(\theta_2,\theta_1)/\Delta y^2$ does not vanish except for θ_1 and θ_2 both zero.

The regularity of the sixth-order scheme is analyzed similarly. The symbol has the same form as (6.1) with

$$\begin{aligned} L(\theta_1,\theta_2) = &4\left(1-\frac{8\sin^2\theta_2}{15}\right)\left(1-\frac{2\sin^2\theta_1}{10}\right)\frac{\sin^2\theta_1}{\Delta x^2} \\ &+ 4\left(1-\frac{8\sin^2\theta_1}{15}\right)\left(1-\frac{2\sin^2\theta_2}{10}\right)\frac{\sin^2\theta_2}{\Delta y^2} \end{aligned} \tag{6.6}$$

$$G(\theta_1,\theta_2) = \left(\left(1 - \frac{8\sin^2\theta_2}{15}\right)\left(1 + \frac{2\sin^2\theta_1}{15}\right) + \frac{8\sin^4\theta_1}{45}\right)\sin(2\theta_1) + 2^7\gamma\sin^7\theta_1 e^{i\theta_1} \quad (6.7)$$

and

$$D(\theta_1,\theta_2) = \left(1 - \frac{4\sin^2\theta_2}{5}\right)\left(1 - \frac{2\sin^2\theta_1}{15}\right)\sin(2\theta_1) + 2^7\gamma\sin^7\theta_1 e^{-i\theta_1} \ . \quad (6.8)$$

The symbol L, as defined in (6.6), vanishes only when $\theta_1 = \theta_2 = 0$, with $|\theta_1|$ and $|\theta_2|$ less than or equal to $\pi/2$.

For the sixth-order scheme, the product $G(\theta_1,\theta_2)D(\theta_1,\theta_2)$ is

$$\begin{aligned}
&G(\theta_1,\theta_2)D(\theta_1,\theta_2) = \\
&\left[\left(1 - \frac{8}{15}\sin^2\theta_2\right)\left(1 - \frac{4}{5}\sin^2\theta_2\right)\left(1 - \frac{4}{225}\sin^4\theta_1\right)\right. \\
&+ \frac{8}{45}\sin^4\theta_1\left(1 - \frac{4}{5}\sin^2\theta_2\right)\left(1 - \frac{2}{15}\sin^2\theta_1\right) \\
&\left.+ 2^7\gamma\sin^6\theta_1\left(1 - \frac{2}{3}\sin^2\theta_2 + \frac{4}{225}\sin^2\theta_1\sin^2\theta_2 + \frac{4}{45}\sin^4\theta_1\right)\right]\sin^2 2\theta_1 \\
&+ 2^{14}\gamma^2\sin^{14}\theta_1 \\
&- \frac{2^9 i\gamma}{15}\left(\sin^2\theta_1(1 + \frac{2}{3}\sin^2\theta_1) + \sin^2\theta_2(1 - \frac{2}{3}\sin^2\theta_1)\right)\sin^8\theta_1\sin 2\theta_1 \ .
\end{aligned}$$

As with the fourth-order scheme, with γ positive, this expression vanishes only for θ_1 equal to 0. Therefore the expression (6.5), for the sixth-order scheme, does not vanish for non-zero values of θ_1 and θ_2, and thus the scheme is regular.

7. The Temporal Differencing.

In this section we discuss the temporal differencing of the time-dependent Navier-Stokes equations. We consider only schemes based on multistep methods from ordinary differential equations. To simplify the discussion we consider the Navier-Stokes and Stokes equations in the form

$$\begin{aligned}
\vec{u}_t &= \mathcal{L}\vec{u} - \vec{\nabla}p + \vec{f} \\
\nabla\cdot\vec{u} &= g
\end{aligned} \quad (7.1)$$

where $\vec{u}$ denotes the velocity and p denotes the pressure. The operator $\mathcal{L}$ represents the terms involving spatial derivatives of the velocity. Our approach is motivated by the similarity of the system (7.1) to differential-algebraic systems for ordinary differential equations. The incompressible Navier-Stokes equations are similar to differential-algebraic systems of index 2, see [2].

For the system (7.1) the temporal differencing using a general multistep method is defined by

$$\begin{aligned} \frac{1}{\Delta t}\sum_{k=0}^{K}\alpha_k \vec{u}^{n-k} &= \sum_{k=0}^{K}\beta_k \mathcal{L}\vec{u}^{n-k} - \sum_{k=0}^{K}\beta_k \vec{\nabla} p^{n-k} + \sum_{k=0}^{K}\beta_k \vec{f}^{n-k} \\ \nabla\cdot\vec{u}^n &= g^n \,. \end{aligned} \tag{7.2}$$

The two arrays of coefficients α_k and β_k. are normalized by

$$\sum_{k=0}^{K}\alpha_k = 0 \quad \text{and} \quad \sum_{k=0}^{K}\beta_k = 1.$$

Any of the second-order, fourth-order, or sixth-order spatial differencing can be used with any of these time-dependent schemes. We consider the temporal differencing (7.2) to be applied before the spatial differencing, such as applying the operations (3.4) or (4.4).

The stability of the scheme depends on the two polynomials

$$\mathcal{A}(z) = \sum_{k=0}^{K}\alpha_k z^{K-k} \quad \text{and} \quad \mathcal{B}(z) = \sum_{k=0}^{K}\beta_k z^{K-k}.$$

A necessary condition for the stability of the overall scheme is that the polynomial $\mathcal{A}$ satisfy the standard root condition for stability in the sense of ordinary differential equations. That is, the roots of $\mathcal{A}(z) = 0$ must be inside the unit circle or simple on the unit circle [5], [8].

As the following theorem shows, for the standard multistep methods (7.2) stability also requires that the polynomial $\mathcal{B}(z)$ satisfy the root condition.

Theorem 7.1. *A necessary condition for stability of the multistep method* (7.2) *is that the roots of* $\mathcal{B}(z) = 0$ *be inside the unit circle or simple on the unit circle.*

Proof:

Let z be a root of $\mathcal{B}(z) = 0$ and let q be any non-constant function of the spatial variables. A solution of (7.2) is constructed by setting $\vec{u}^\nu = 0$ and $p^\nu = z^\nu q$. If the magnitude of z is larger than 1, then this solution will be unbounded in norm. If z is a multiple root with magnitude 1, then take $p^\nu = \nu z^\nu q$. Thus it is necessary for $\mathcal{B}(z)$ to satisfy the root condition. ∎

As is seen in the proof, the roots of the polynomial $\mathcal{B}(z)$ govern the growth of the pressure errors. The restriction on the roots of $\mathcal{B}(z)$ is a result of the pressure appearing only in the spatial differencing portion of the equations. Similar situations occur with semi-explicit differential-algebraic equations of index 2. However, defining $\bar{p}$ by

$$\bar{p}^n = \sum_{k=0}^{K}\beta_k p^{n-k}$$

we can replace (7.2) with

$$\begin{aligned} \frac{1}{\Delta t}\sum_{k=0}^{K}\alpha_k \vec{u}^{n-k} &= \sum_{k=0}^{K}\beta_k \mathcal{L}\vec{u}^{n-k} - \vec{\nabla}\bar{p}^n + \sum_{k=0}^{K}\beta_k \vec{f}^{n-k} \\ \nabla\cdot\vec{u}^n &= g^n\ . \end{aligned} \tag{7.3}$$

The function $\bar{p}^n$ is at least a first-order accurate approximation to $p(t_n - \mu\Delta t)$ where

$$\mu = \sum_{k=0}^{K} k\beta_k.$$

Note that a disadvantage of this modified scheme is that the pressure is not be obtained to the same accuracy as the velocity without some post-processing.

Theorem 7.1 is a severe limit on multistep schemes, however, the modified multistep scheme (7.3) allows for many schemes to be used. An adequate theory for the stability of schemes for the Navier-Stokes and Stokes equations has not been developed. Here we rely on the experience and theory of differential-algebraic equations, see [2], to guide our choice.

Primarily, we have used schemes based on backward time-differences. These schemes, called backward differencing formula (BDF) schemes, are widely used for solving stiff ordinary differential equations and in differential-algebraic equations, see [2]. For these schemes β_0 is 1, the other β_k are 0, and the α_k are chosen from the formula

$$\frac{\partial}{\partial t} = -\frac{\ln(1-\Delta t\ \delta_{t-})}{\Delta t} = \left(1 + \frac{1}{2}\Delta t\ \delta_{t-} + \frac{1}{3}(\Delta t\ \delta_{t-})^2 + \frac{1}{4}(\Delta t\ \delta_{t-})^3 + \cdots\right)\delta_{t-}$$

truncated after K terms for a scheme with order of accuracy K. The coefficients α_k for a scheme of order K are given by

$$\alpha_0 = \sum_{j=1}^{K}\frac{1}{j} \quad \text{and} \quad \alpha_k = (-1)^k \sum_{j=k}^{K}\binom{j}{k}\frac{1}{j} \quad \text{for} \quad k = 1,\cdots,K.$$

Our finite difference schemes are obtained by replacing the temporal derivative in equations (3.5) and (4.5) with the BDF operator of order K. In this paper we consider only K equal to 2, 3, and 4.

These schemes are stable as multistep schemes for ordinary differential equations for $K \le 6$, unstable for $K = 7$ and possibly all larger values of K, see [7]. In the numerical tests, the BDF schemes out performed the non-BDF schemes.

The backward differencing schemes we have used are:

1. Second-order backward in time ($K = 2$)
   $\alpha_0 = 3/2,\ \alpha_1 = -2,\ \alpha_2 = 1/2;\ \beta_0 = 1$
2. Third-order backward in time ($K = 3$)

$\alpha_0 = 11/6,\ \alpha_1 = -3,\ \alpha_2 = 3/2,\ \alpha_3 = -1/3;\ \beta_0 = 1$

3. Fourth-order backward in time ($K = 4$)

   $\alpha_0 = 25/12,\ \alpha_1 = -4,\ \alpha_2 = 3,\ \alpha_3 = -4/3,\ \alpha_4 = 1/4;\ \beta_0 = 1$

In addition, three schemes were used in the first time steps, when the above schemes could not be used since they require several past time steps.

4. Crank-Nicolson (second-order accurate)

   $\alpha_0 = 1,\ \alpha_1 = -1;\ \beta_0 = \beta_1 = 1/2$

5. Fourth-order in time using three time levels.

   $\alpha_0 = 1,\ \alpha_1 = 0,\ \alpha_2 = -1;\ \beta_0 = 1/6,\ \beta_1 = 2/3,\ \beta_2 = 1/6$

6. Fourth-order in time using four time levels.

   $\alpha_0 = 17/24,\ \alpha_1 = \alpha_2 = -3/8,\ \alpha_3 = 1/24;\ \beta_0 = 1/4,\ \beta_1 = 3/4$

These schemes are obtained by factoring the backward operators as done in sections 3 and 4 for spatial difference operators. (Scheme 5 is equivalent to (3.2) applied in time rather than space.)

As multistep schemes, as in (7.2), schemes 5 and 6 are unstable by Theorem 7.1, but appear stable when used as in (7.3). Runs using the second-order scheme 1 used scheme 4 to compute the first time step. Runs using the third-order accurate scheme 2used scheme 4for the first time step and scheme 1for the second time step. Subsequent steps then used scheme 2.

Runs using the fourth-order scheme 3 used scheme 4 to compute the first time step, scheme 5 to compute the second time step, and scheme 6 to compute the third time step. Subsequent steps then used scheme 3. The use of a second-order accurate scheme to initialize a fourth-order accurate appears not to reduce the overall order of accuracy. There was no reason to use a first-order accurate scheme. The BDF schemes are dissipative of order 2 when applied to parabolic equations. For dissipative schemes with order of accuracy r, initializing schemes may be accurate of order $r - 2$ and still have the overall order of accuracy be r.

For the Stokes equations, because they are linear, the determination of the solution at the next time step requires the solution of a linear system. The system can be written as

$$\begin{aligned} \alpha_0 \frac{\vec{u}^n}{\Delta t} - \beta_0 \mathcal{L} \vec{u}^n + \vec{\nabla} \vec{p}^n &= - \sum_{k=1}^{K} \alpha_k \frac{\vec{u}^{n-k}}{\Delta t} + \sum_{k=1}^{K} \beta_k \mathcal{L} \vec{u}^{n-k} + \sum_{k=0}^{K} \beta_k \vec{f}^{n-k} \\ \nabla \cdot \vec{u}^n &= g^n . \end{aligned} \tag{7.4}$$

This system determines the solution, $\vec{u}^n$ and $\vec{p}^n$, for the new time level. Note that the spatial operators such as (3.4) or (4.4) must be applied to (7.4).

We have used a preconditioned GMRES method [15] for the solution of this linear system. The usual method was GMRES(7) with a restart. The preconditioner was an inversion of the operator

$$\alpha_0 \frac{1}{\Delta t} - \beta_0 \nabla^2 \tag{7.5}$$

on the first two equations in the system. Other methods of solving the linear system could also be used. An advantage of GMRES is that it does not require the system to be symmetric.

As mentioned, the preconditioner for the system (7.4) was the inversion of the operator (7.5). For the second-order and fourth-order accurate schemes this was done using the preconditioned conjugate gradient method, with SSOR as the preconditioner. For the sixth-order scheme, because the operator is not symmetric, the GMRES method was used. Although accurate solutions were obtained, this method was not particularly efficient, More experience is needed with these methods to improve the overall efficiency.

Because the pressure can only be determined to within an additive constant, the system (7.4) is singular. Moreover, the existence of the solution is dependent on satisfying the integrability condition. An important issue is the choice of norms to determine convergence of the system (7.4). By requiring only that the quantity $\vec{\nabla} \cdot \vec{u}^n - g^n$ be constant as described in [19], we effectively have a nonsingular system. The average value of $\vec{\nabla} \cdot \vec{u}^n - g^n$ over the grid is a measure of the consistency of the data. In all the results shown in section 8 this value is less than the errors themselves by several orders of magnitude.

For the nonlinear Navier-Stokes equations, we modify the equations to obtain a linear system for the solution that does not degrade the accuracy. We linearize the quadratic expressions u^2, uv, and v^2 at time level n using the following idea. Consider a two functions $A(t)$ and $B(t)$ depending on the independent variable t. Using the relations

$$\delta_{t-}^r A(t) = 0 \quad \text{and} \quad \delta_{t-}^r B(t) = 0$$

we obtain approximations $\overline{A}$ and $\overline{B}$ to order r for $A(t)$ and $B(t)$. The formula for $\overline{A}^n$ is

$$\overline{A}^n = \sum_{k=1}^{r} (-1)^{k-1} \binom{r}{k} A^{n-k}$$

and similarly for $\overline{B}^n$.

The product AB involving past values of the variable t is approximated using the relation

$$(A - \overline{A})(B - \overline{B}) = O(\Delta t)^{2r} \ .$$

In particular, at $t_n = n\Delta t$ with $A^n = A(t_n)$, we can write

$$A^n B^n = A^n \overline{B}^n + \overline{A}^n B^n - \overline{A}^n \overline{B}^n + O(\Delta t)^{2r} \ .$$

where $\overline{A}^n$ and $\overline{B}^n$ depend on values of t less than t_n.

We use these formulas for $t = n\Delta t$ and with A and B being velocity components. We take r equal to the number of time levels available in the scheme. The expression $\partial(uv)/\partial x$ at time step n is approximated by

$$\partial(u^n \overline{v}^n + \overline{u}^n v^n - \overline{u}^n \overline{v}^n)/\partial x$$

and the spatial derivative is approximated using either the fourth-order or sixth-order methods given previously. In this way the equation being solved for the solution at each time step is a linear system and the accuracy of the solution is not affected. For schemes 4, 5, and 6, the above approximations were used only at the time step being solved for.

8. Numerical Experiments.

Several tests are described in this section that illustrate the accuracy of the methods. The first series of tests check the formal order of accuracy and the second tests examined the accuracy for an analytically known solution. for which there is no analytic solution. The finite difference methods were implemented using the C programming language with double precision variables.

The finite difference schemes were tested extensively to assure that when the velocity and pressure were polynomials of appropriate degree, the solutions satisfied the schemes to within machine precision. This served as both a test of the methods and a means to detect programming errors in the implementation of the methods. For example, the solution

$$\begin{aligned} u &= x^6 - y^6 & f_1 &= -30(x^4 - y^4) + 4x^3y^2 \\ v &= x^3y^3 & f_2 &= -6(xy^3 + x^3y) + 2x^4y \\ p &= x^4y^2 & g &= 6x^5 + 3x^3y^2 \end{aligned}$$

was used with the steady-state Stokes equations (1.3) and (1.4) for the sixth-order method. Other sixth-degree polynomials were also used. By considering these solutions after translations and rotations in the plane, a large class of solutions could be obtained. Similar tests were made for the fourth-order scheme.

For positive Reynolds numbers, because of the quadratic convection terms, the fourth-order scheme is exact for all polynomials of degree two for the velocity and of degree four for pressure. Similarly, the sixth-order method is exact for third-degree polynomials for the velocity and sixth-degree polynomials for the pressure. In all these tests with polynomial solutions, the solutions were computed to within machine precision. Similar tests were used to check the temporal differencing.

For a test of the method on a less trivial application, we take the solution used by Pearson [12], Chorin [4] and others to test their methods. The solution is given by

$$\begin{aligned} u &= -e^{-2t/R} \cos x \sin y \qquad v = e^{-2t/R} \sin x \cos y \\ p &= -\frac{1}{4} e^{-4t/R} (\cos 2x + \sin 2y) \end{aligned}$$

on the square $0 < x < \pi$ and $0 < y < \pi$ for $0 \leq t \leq 0.5$. We took the Reynolds number to be 2.

Computational results are displayed in Tables 1 and 2. The order of accuracy of the method is given as an ordered pair (p, q) where p is the temporal order of accuracy and q is the spatial order of accuracy. The runs marked with an asterisk are those for which the parameter γ was nonzero. These are discussed later in this section.

Table 1 displays the error for runs in which the initial steps were given by the exact solution. For the runs given in Table 2, the solutions on the initial steps were computed from schemes with fewer time levels. The pressure errors were computed using the 'standard deviation' as a norm. That is, the norm is the mean square of the error minus the average error, see [19].

There are two principles illustrated by the data in the tables. The first is that, for a given choice of method, the smaller values of the time step and grid spacing gives better

case	order	Δt	n	error u	error v	error p
1	(2,4)	0.01	20	1.984(-6)	1.770(-6)	4.058(-5)
2	(2,4)	0.02	60	5.565(-6)	5.527(-6)	8.608(-5)
3	(2,4)	0.01	60	1.390(-6)	1.378(-6)	2.140(-5)
4	(3,4)	0.05	20	8.834(-6)	9.268(-6)	1.315(-4)
5	(3,4)	0.05	40	1.295(-6)	1.297(-6)	2.103(-5)
6	(4,6)	0.05	20	7.327(-7)	7.253(-7)	3.986(-6)
7	(4,6)	0.02	20	7.440(-7)	7.446(-7)	5.065(-6)
8	(4,6)	0.02	40	1.710(-8)	1.446(-8)	2.932(-6)
9	*(4,6)	0.02	40	1.242(-8)	1.238(-8)	3.923(-7)

Table 1

accuracy. The second principle is that higher order methods give better order accuracy than lower order methods. Moreover, for fewer grid points and fewer time steps, a higher order method can give better accuracy than lower order accurate methods.

A difficulty with interpreting the numerical results is that the errors due to the time discretization and the spatial discretization are combined. Thus in few cases do we observe a nice reduction in the error that corresponds to the order of accuracy in time or space. Notice however, that because of the unconditional stability of the implicit method, the time steps are much larger than that required by the fractional-step method.

The errors in the runs in Table 1 with order $(2,4)$ are principally due to the time discretization. This is due in part to the smoothness of the solution in space. The error for case 3 is not that much smaller than that for case 1. Also, the ratio of the error between cases 2 and 3 is approximately 4, as would be expected if the error where predominately due to the time discretization. Notice that the reduction in the pressure error is consistent with the second-order accuracy in time.

For the $(3,4)$ runs in Table 1, the error is affected primarily by the spatial step size. There is a reduction in the error of about 7 as the mesh parameter n is increased from 20 to 40. Also, the run with $n = 40$ and $\Delta t = 0.05$, case 5, produces smaller errors than does the run in case 1 for less effort. (The work for a given run is proportional to $n^2(\Delta t)^{-1}$.)

The first two $(4,6)$ runs in Table 1, cases 6 and 7, illustrate that for this choice of scheme the spatial error for the grid with $n = 20$ dominates the temporal discretization error. The smaller error for the pressure for case 6 may be due to the initial data being exact and there being only 7 time steps of actual computation in case 6, and 23 such steps in case 7, and is also influenced by the oscillatory modes, which are allowed by taking γ equal to 0. Note however, that the computation in case 6 is much more efficient than is case 5 with better accuracy.

Cases 8 and 9 illustrate the effect of the regularizing parameter γ on the accuracy. Because the exact solution is so smooth the oscillations allowed by having γ equal to 0 are small initially, and because there are relatively few time steps in these tests, the error due to pressure oscillations remain small and are dominated by the usual discretization error. However, for case 8 the dominant error for the pressure is due to this oscillatory mode. The errors are reduced in case 9 by taking $\gamma = 0.001$. For runs involving more time steps the importance of having a nonzero value for γ is even more important.

Also, comparing the errors for cases 7 and 9 in Table 1, we see that there is a reduction in the velocity errors that is approximately 60, in excellent agreement with the sixth-order accuracy in space. If the error were due solely to the spatial differencing we could expect a reduction of 2^6. The reduction in the pressure is only a factor of 13. The pressure errors are still somewhat oscillatory for this case, a different value of γ might give even smaller errors.

case	order	Δt	n	error u	error v	error p
1	(2,4)	0.01	20	1.982(-6)	1.769(-6)	4.058(-5)
2	(2,4)	0.02	60	5.581(-6)	5.542(-6)	8.610(-5)
3	(3,4)	0.02	20	1.451(-6)	1.551(-6)	3.382(-6)
4	(3,4)	0.02	40	4.977(-8)	3.859(-8)	1.816(-6)
5	(3,6)	0.05	20	9.309(-7)	8.705(-7)	1.803(-5)
6	(4,4)	0.02	20	1.448(-6)	1.533(-6)	3.391(-5)
7	(4,4)	0.02	40	7.469(-8)	5.753(-8)	1.745(-6)
8	*(4,4)	0.02	40	7.439(-8)	5.682(-8)	5.624(-7)
9	(4,4)	0.01	60	8.665(-9)	7.228(-9)	1.335(-6)
10	(4,6)	0.05	20	1.337(-6)	1.271(-6)	4.856(-6)
11	(4,6)	0.01	40	3.489(-8)	3.064(-8)	1.043(-5)
12	*(4,6)	0.01	40	1.317(-8)	1.278(-8)	6.162(-6)
13	*(4,6)	0.01	40	1.358(-8)	1.286(-8)	6.873(-7)

Table 2

The runs in Table 2 with order $(2,4)$, as in Table 1, have errors that are principally due to the time discretization. The errors are also approximately the same for similar (2,4) runs in Tables 1 and 2, showing that the initialization does not significantly affect the accuracy. (Compare cases 1 and 2 in Table 1 with cases 1 and 2 in Table 2.)

Case 3, a (3,4) scheme, obtains results comparable to, but better than, case 1 with less computational effort. There is a dramatic decrease in the velocity errors as the grid is refined between cases 3 and 4.

Case 5 in Table 2, with accuracy (3,6), has errors smaller than cases 1 and 2 with far fewer grid points and time steps. This shows the advantages of the more accurate methods.

Cases 6, 7 and 8 demonstrate the gains in accuracy for the higher order accurate methods. Increasing the spatial resolution by 2 between cases 6 and 7 results in a reduction in the errors by a factor of about 20 for the velocity and the pressure, a bit better than the factor of 16 expected if the errors were due solely to the spatial discretization. They also show that, for fourth-order accurate methods, the accuracy of the solution is not adversely affected by the use of second-order methods as part of the initialization. This is a consequence of the dissipativity of the BDF methods for parabolic equations. Recall from section 7 that that the solution on the first time-step was computed using the Crank-Nicolson scheme, which is second-order accurate in time.

For case 8 a value of γ equal to 0.001 reduced the pressure error by factor of about 3, again showing the importance of regularity. This same value of γ caused a slight increase in the errors for case 6.

Case 9 gives the best results for the accuracy of the velocity of any of the cases, showing the high accuracy of the methods. The pressure errors could be improved by choosing γ to be positive, see the discussion of cases 7 and 8 and cases 11, 12, and 13.

Case 10 when compared to cases 3 and 6 shows that the (4,6) method can produce comparable accuracy for less effort. As mentioned in section 4, a disadvantage of using the sixth-order method is that the iterative solution method that was employed was less efficient, in terms of iterations required for convergence, than was the iterative method for the fourth-order scheme. Hopefully, this inefficiency can be removed by experimenting with preconditioners and with methods.

Cases 11, 12, and 13 illustrate the effect of the parameter γ on the computation. For computations which required few time steps, notice that the stopping time was 0.5, taking γ to be 0 was adequate and gave very good accuracy. In fact, taking γ positive increased the pressure errors for the $(2,4)$ cases. However, in the cases with γ equal to 0, the pressure error was noticeably oscillatory for many cases, and for runs involving more time steps, such as case 11, this error was significant and seriously degraded the total accuracy. Notice that the pressure error for case 11 is greater than that for case 10.

For case 12 the value of γ was 0.01 and for case 13 it was 0.02. Notice the dramatic improvement in the error for the pressure as γ increases in these three cases. Further work is needed to determine how to set the parameter γ for different computations. For computations involving many more time steps than the tests here, and for runs with initial data containing more high Fourier modes than this solution, a non-zero value of γ would be critical to obtaining good results.

As discussed in section 7, the method GMRES(7) was used to solve for the solution variables at each time step. An average of about 10 applications of GMRES(7) were required per time step.

9. Conclusions.

In this paper we presented several new finite difference schemes for the incompressible Navier-Stokes equations. We have shown that these schemes have a high-order of accuracy.

The schemes are based on two spatial differencing methods, one a fourth-order accurate method and one a sixth-order accurate method. The temporal differencing methods are BDF methods of order 2, 3, and 4. These temporal schemes can be used with either of the spatial differencing methods. The schemes, as presented, are for orthogonal Cartesian grids. The schemes can be used for both the steady-state and time-dependent equations.

For the time-dependent methods, the storage requirements of the second-order scheme appear to be greater than needed by fractional-step methods. However, this is compensated for by the larger time-steps and smaller stencils for the higher-order spatial discretization. For the higher-order accuracy in time, the storage requirements are reasonable, increasing by one level of storage with each order of accuracy.

The methods presented in this paper have been demonstrated to be accurate and effective methods for solving the time-dependent incompressible Navier-Stokes and Stokes equations. Further research is needed to improve the efficiency of the solution of the linear systems. For higher Reynolds number solutions, better iterative methods will be needed to solve the linear systems at each time step.

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