Convergence Estimates for Finite Difference Approximations of the Stokes Equations

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Abstract. For three finite difference approximations of the Stokes equations, the Schur complement $Q_h$ of the linear system generated by each of these approximations is shown to have its condition number $\kappa(Q_h)$ independent of mesh size. This result is used to prove convergence estimates of the solutions generated by $Q_h$ for these approximations. One of the convergence estimates is for a staggered mesh scheme and the estimate for this scheme shows that both the pressure and the velocity are second-order accurate.

Key words. pressure equation method, Stokes equations, incompressible Navier-Stokes equations, inf-sup conditions, finite difference schemes, iterative methods

AMS(MOS) subject classifications. 65F10, 65N06, 65N22

1. Introduction. The pressure equation (PE) method, a new fast iterative method to solve finite difference approximations of the Stokes and the incompressible Navier-Stokes equations has been introduced by Shin and Strikwerda [10]. The PE method and many other iterative methods to solve the Stokes equations are heavily dependent on the properties the Schur complements of the linear systems resulting from discretizations of these equations, which we investigate in this paper.

We first review the PE method. The steady-state Stokes equations in $\mathbb{R}^d$ are

$$\begin{align*}
\nabla^2 \tilde{u} - \nabla p &= f \\
\nabla \cdot \tilde{u} &= g
\end{align*}$$

in $\Omega \subset \mathbb{R}^d$. \hspace{1cm} (1.1)

The velocity $\tilde{u}$ is a vector of dimension $d$ and the pressure $p$ is a scalar. Consider the the Dirichlet boundary condition

$$\tilde{u} = \tilde{v} \quad \text{on} \quad \partial \Omega.$$ \hspace{1cm} (1.2)

Let $A_h, G_h$ and $D_h$ be the operators generated by discretizations of the differential operators

$$\begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{pmatrix}, \quad -\nabla \cdot, \quad \text{and} \quad (\nabla \cdot),$$

respectively. The discretization of (1.1) and (1.2) may then be written as

$$\begin{align*}
A_h u_h + G_h p_h &= f_h \\
D_h u_h &= g_h
\end{align*}$$

\hspace{1cm} (1.3)

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and \[ u_h = b_h \quad \partial \Omega, \]
respectively.

Note that
\[ A_h u_h = f_h - G_h p_h, \quad u_h = b_h \quad \text{on} \quad \partial \Omega \quad (1.4) \]
by the first row in (1.3) and (1.2). Hence
\[ u_h = v_h - w_h \quad (1.5) \]
where
\[ A_h v_h = f_h, \quad v_h = b_h \quad \text{on} \quad \partial \Omega \]
and
\[ A_h w_h = G_h p_h, \quad w_h = 0 \quad \text{on} \quad \partial \Omega. \]
If \( A_0^{-1} \) and \( A_b^{-1} \) are the inverse operators of \( A_h \) using the zero boundary condition and the boundary condition for \( u_h \), respectively, then
\[ v_h = A_b^{-1} f_h \quad \text{and} \quad w_h = A_0^{-1} G_h p_h. \quad (1.6) \]

Using the second row in (1.3), (1.5), and (1.6), one obtains
\[ D_h (v_h - w_h) = g_h \]
and
\[ D_h A_0^{-1} G_h p_h = D_h A_b^{-1} f_h - g_h. \]

Thus
\[ Q_h p_h = D_h A_b^{-1} f_h - g_h, \quad (1.7) \]
if we let
\[ Q_h := D_h A_0^{-1} G_h. \]

The operator \( Q_h \) is called the Schur complement of the linear system (1.3).

In this paper, we show that, for three finite difference schemes, the operator \( Q_h \) is self-adjoint, positive definite with eigenvalues bounded independently of mesh size. In each of these cases, one can use the conjugate gradient (CG) method to solve (1.7), and the number of the CG iterations required to solve (1.7) should be independent of the grid parameters. The iterative method based on solving (1.7) by the CG method is called the PE method.

A popular iterative method to solve the Stokes and Navier-Stokes equations is the pressure poisson equation (PPE) method, see [4]. Applying the divergence operator \( \nabla \cdot \) to the first row in (1.1), we have
\[ \nabla^2 (\nabla \cdot u) - \nabla \cdot \nabla p = \nabla \cdot \vec{f} \]
and, by the second row in (1.1),
\[ \nabla^2 p = \nabla^2 g - \nabla \cdot \vec{f}. \]

The PPE method is based on solving the above Poisson equation for pressure. A boundary condition for pressure is needed to solve this equation and it is not clear which boundary condition is appropriate, see [4]. The PE method is similar to the pressure poisson equation (PPE) method in the sense that they use equations for pressure only, but the PE method doesn't require a boundary condition for pressure.
2. Definitions and inf-sup conditions. Let \( \Omega \) be a domain in \( \mathbb{R}^d \) and let \( \Gamma \) be its boundary. For simplicity, we focus on the case when \( d = 2 \), but the results in this paper will hold for any \( d \geq 2 \). We denote by \( L^2(\Omega) \) the space of real functions defined on \( \Omega \) which are integrable in the \( L^2 \) sense with the following usual inner product and norm

\[
(u, v)_\Omega := \int_\Omega uv \, dA, \quad \|u\|_\Omega^2 := (u, u)_\Omega.
\]

Let

\[
H_0^1(\Omega) := \{ u \in L^2(\Omega) \mid u_x, u_y \in L^2(\Omega) \text{ and } u|_\Gamma = 0 \}
\]

have the following inner product and norm

\[
(u, v)_{1,\Omega} := \int_\Omega \nabla u \cdot \nabla v \, dA, \quad \|u\|_{1,\Omega}^2 := (u, u)_{1,\Omega}
\]

and

\[
L^2_0(\Omega) := \{ p \in L^2(\Omega) \mid (p, 1)_\Omega = 0 \}.
\]

We use the notation \( \vec{u} = (u_i) \) for a vector. We shall often be concerned with two-dimensional vector functions with components in \( L^2(\Omega) \) or \( H_0^1(\Omega) \). The notation \( L^2(\Omega)^2, \, H_0^1(\Omega)^2 \) will be used for the product spaces. Define, for \( \vec{u} \) and \( \vec{v} \in L^2(\Omega)^2 \),

\[
(\vec{u}, \vec{v})_\Omega := \sum_{i=1}^2 (u_i, v_i)_\Omega, \quad \|\vec{u}\|_\Omega^2 := (\vec{u}, \vec{u})_\Omega
\]

and, for \( \vec{u} \) and \( \vec{v} \in H_0^1(\Omega)^2 \),

\[
(\vec{u}, \vec{v})_{1,\Omega} := \sum_{i=1}^2 (u_i, v_i)_{1,\Omega}, \quad \|\vec{u}\|_{1,\Omega}^2 := (\vec{u}, \vec{u})_{1,\Omega}.
\]

We also make some definitions analogous to the above on discrete subsets of the unit square \( S \) in \( \mathbb{R}^2 \). Let

\[
S := \{ (x, y) \in \mathbb{R}^2 \mid 0 < x, y < 1 \}
\]

and \( T \) its boundary. Let

\[
h := \frac{1}{N}, \quad \text{for some } N \in \mathbb{N},
\]

\[
\mathbb{R}_h^2 := \{ (lh, mh) \in \mathbb{R}^2 \mid l, m \in \mathbb{N} \},
\]

\[
S_h := \bar{S} \cap \mathbb{R}_h^2
\]

where \( \bar{S} \) is the closure of \( S \).

For an arbitrary discrete set \( \Omega_h \) of the form

\[
\Omega_h := \{ (lh, mh) \in S_h \mid l_0 \leq l \leq l_1 \text{ and } m_0 \leq m \leq m_1 \},
\]

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we define
\[ \Omega_h^I := \{ (lh, mh) \in S_h \mid l_0 + 1 \leq l \leq l_1 - 1, \ m_0 + 1 \leq m \leq m_1 - 1 \}, \]
\[ e(\Omega_h) := \{ (lh, mh) \in S_h \mid l_0 + 1 \leq l \leq l_1, \ m_0 \leq m \leq m_1 \}, \]
\[ w(\Omega_h) := \{ (lh, mh) \in S_h \mid l_0 \leq l \leq l_1 - 1, \ m_0 \leq m \leq m_1 \}, \]
\[ s(\Omega_h) := \{ (lh, mh) \in S_h \mid l_0 \leq l \leq l_1, \ m_0 \leq m \leq m_1 - 1 \}, \]
\[ n(\Omega_h) := \{ (lh, mh) \in S_h \mid l_0 \leq l \leq l_1, \ m_0 + 1 \leq m \leq m_1 \} \]
as the interior, east, west, south and the north sides of \( \Omega_h \) and define
\[ se(\Omega_h) := s(\Omega_h) \cap e(\Omega_h), \]
\[ sw(\Omega_h) := s(\Omega_h) \cap w(\Omega_h) \]
\[ ne(\Omega_h) := n(\Omega_h) \cap e(\Omega_h), \]
\[ nw(\Omega_h) := n(\Omega_h) \cap w(\Omega_h). \]
For the boundary \( \Gamma_h \) of \( \Omega_h \), we define
\[ e(\Gamma_h), \quad w(\Gamma_h), \quad s(\Gamma_h), \quad n(\Gamma_h) \]
as the east, west, south and north parts of \( \Gamma_h \) including the end points.

In this paper, we want to study both standard and staggered grids. The staggered mesh schemes use different grids that are staggered for the pressure and the velocity. A staggered grid is shown in Figure 1. The points marked by \( P, I, \) and \( II \) are where the pressure and the first and the second components of the velocity are defined, respectively.

![Figure 1](image-url)

Let
\[ S_P := \{ (l - \frac{1}{2})h, (m - \frac{1}{2})h \} \in S \mid l, m = 1, \ldots, N \}, \]
\[ S_I := \{ lh, (m - \frac{1}{2})h \} \in S \mid l = 0, \ldots, N, \ m = 0, \ldots, N + 1 \}, \]
\[ S_{II} := \{ (l - \frac{1}{2})h, \ mh \} \in S \mid l = 0, \ldots, N + 1, \ m = 0, \ldots, N \}, \]
then these are the sets for $P$, $I$, and $II$. Figure 1 shows $S_P$, $S_I$, and $S_{II}$ when $N = 3$. Staggered mesh schemes have been used by Amsden and Harlow [1], Brandt and Dinar [2], Harlow and Welch [6], Patankar and Spalding [8], and Raithby and Schneider [9] and others.

Let

$$S_w := \{ (x, y) \in \mathbb{R}^2 | 0 < x < 1 - h, 0 < y < 1 \},$$

$$S_e := \{ (x, y) \in \mathbb{R}^2 | h < x < 1, 0 < y < 1 \},$$

$$S_s := \{ (x, y) \in \mathbb{R}^2 | 0 < x < 1, 0 < y < 1 - h \},$$

$$S_n := \{ (x, y) \in \mathbb{R}^2 | 0 < x < 1, h < y < 1 \},$$

$$S_1 := \{ (x, y) \in \mathbb{R}^2 | 0 < x < 1, \frac{-h}{2} < y < 1 + \frac{h}{2} \},$$

$$S_2 := \{ (x, y) \in \mathbb{R}^2 | -\frac{h}{2} < x < 1 + \frac{h}{2}, 0 < y < 1 \},$$

$$S_{sw} := S_s \cap S_w, \quad \text{and} \quad S_{ne} := S_n \cap S_e$$

be the continuous analogues of $w(S_h)$, $e(S_h)$, and so forth, respectively.

Let $L^2(\Omega_h)$ be the space of all discrete functions defined on $\Omega_h$ with the following inner product and norm

$$(U, V)_{\Omega_h} := h^2 \sum_{(x, y) \in \Omega_h} U(x, y)V(x, y), \quad \|U\|^2_{\Omega_h} := (U, U)_{\Omega_h}$$

and let

$$L^2_0(\Omega_h) := \{ P \in L^2(\Omega_h) | (P, 1)_{\Omega_h} = 0 \},$$

then $L^2(\Omega_h)$ and $L^2_0(\Omega_h)$ are the discrete analogues of $L^2(\Omega)$ and $L^2_0(\Omega)$.

For notational convenience, we introduce

$$U_{l,m} := U(lh, mh),$$

and define the forward, backward and central differencings on the $x$ axis and $y$ axis, respectively, as

$$(\delta_{2+} U)_{l,m} := \frac{U_{l+1,m} - U_{l,m}}{h}, \quad (\delta_{2-} U)_{l,m} := \frac{U_{l,m+1} - U_{l,m}}{h},$$

$$(\delta_x U)_{l,m} := \frac{U_{l,m} - U_{l-1,m}}{h}, \quad (\delta_y U)_{l,m} := \frac{U_{l,m} - U_{l,m-1}}{h},$$

$$(\delta_{2x} U)_{l,m} := \frac{U_{l+\frac{1}{2},m} - U_{l-\frac{1}{2},m}}{h}, \quad (\delta_{2y} U)_{l,m} := \frac{U_{l,m+\frac{1}{2}} - U_{l,m-\frac{1}{2}}}{h}.$$
and let $\nabla_h^2$ be the five-point discrete Laplacian, then

$$\nabla_h^2 = \vec{\nabla} \cdot \vec{\nabla}_+ = \vec{\nabla}_- \cdot \vec{\nabla}.$$

The inner product and the norm of

$$H_0^1(\Omega_h) := \{ U \in L^2(\Omega_h) \mid U|_{\Gamma_h} = 0 \}$$

are defined as

$$(U, V)_1,\Omega_h := (\vec{\nabla}_+ U, \vec{\nabla}_+ V)_{sw(\Omega_h)} = (\vec{\nabla}_- U, \vec{\nabla}_- V)_{ne(\Omega_h)},$$

$$\|U\|_1^2,\Omega_h := (U, U)_1,\Omega_h$$

which are the sums over all points in $\Omega_h$ where difference quotients are defined. The inner product and the norm of the product spaces $L^2(\Omega_h)^2$ and $H_0^1(\Omega_h)^2$ are defined naturally from $L^2(\Omega_h)$ and $H_0^1(\Omega_h)$.

The following inf-sup conditions are essential to study $Q_h$, the matrix in the pressure equation (1.7). Refer to Shin and Strikwerda [11] for the proofs.

**Theorem 2.1.** There exists a positive constant $C$, which is independent of $h$, such that

1. $$\sup_{\vec{U} \in H_0^1(S_t) \times H_0^1(S_H)} \frac{(\vec{\nabla}_0 \cdot \vec{U}, P)^2_{S_p}}{\|U_1\|_{1,S_t}^2 + \|U_2\|_{1,S_H}^2} \geq C\|P\|_{S_p}^2, \quad \forall P \in L^2(S_p),$$

2. $$\sup_{\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))} \frac{(\vec{\nabla}_- \cdot \vec{U}, P)^2_{S_h}}{\|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2} \geq C\|P\|_{S_h}^2, \quad \forall P \in L^2(S_h),$$

3. $$\sup_{\vec{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))} \frac{(\vec{\nabla}_+ \cdot \vec{U}, P)^2_{S_h}}{\|U_1\|_{1,e(S_h)}^2 + \|U_2\|_{1,n(S_h)}^2} \geq C\|P\|_{S_h}^2, \quad \forall P \in L^2(S_h).$$

3. Approximations by Finite Differences. Three finite difference approximations $Q_h$ are introduced in this section. Let $P \in L^2(S_h)$, then $\delta_x P$ and $\delta_y P$ are defined in $w(S_h)$ and $s(S_h)$, respectively. Note that

$$w(\Omega_h) = w(\Omega_h)^0$$

for any rectangular subset $\Omega_h$ of $S_h$. Hence if $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ is the solution of

$$\nabla_h^2 U_1 = \delta_x P \quad \text{in} \ w(S_h)^0 \quad \text{and} \quad \nabla_h^2 U_2 = \delta_y P \quad \text{in} \ s(S_h)^0, \quad (3.1)$$

then

$$Q_P := \vec{\nabla} \cdot \vec{U} \quad \text{in} \ S_h$$

is well-defined. The above finite difference problem is similar to the following partial differential problem: For $p \in L^2(S_w \cup S_s)$, define

$$Q_p := \vec{\nabla} \cdot \vec{u} \quad \text{in} \ S_{sw}$$

(3.3)
where $\tilde{u} \in H^1_0(S_w) \times H^1_0(S_s)$ is the solution of
\[
\nabla^2 u_1 = p_x \quad \text{in } S_w \quad \text{and} \quad \nabla^2 u_2 = p_y \quad \text{in } S_s.
\] (3.4)

Similarly, for $P \in L^2(S_h)$, let $\tilde{U} \in H^1_0(e(S_h)) \times H^1_0(n(S_h))$ be the solution of
\[
\nabla_h^2 U_1 = \delta_{x+} P \quad \text{in } e(S_h)^0 \quad \text{and} \quad \nabla_h^2 U_2 = \delta_{y+} P \quad \text{in } n(S_h)^0,
\] (3.5)

then
\[
Q_+ P := \nabla_+ \cdot \tilde{U} \quad \text{in } S_h^0
\]
is well-defined. The above finite difference problem is similar to the following partial differential problem: For $p \in L^2(S_e \cup S_n)$, define
\[
Q p := \nabla \cdot \tilde{u} \quad \text{in } S_{ne}
\]
where $\tilde{u} \in H^1_0(S_e) \times H^1_0(S_n)$ is the solution of
\[
\nabla^2 u_1 = p_x \quad \text{in } S_e \quad \text{and} \quad \nabla^2 u_2 = p_y \quad \text{in } S_n.
\] (3.6)

Another approximation comes from the staggered mesh schemes. For $P \in L^2(S_P)$, let $\tilde{U} \in H^1_0(S_I) \times H^1_0(S_H)$ be the solution of
\[
\nabla_h^2 U_1 = \delta_{x+} P \quad \text{in } S_I^0 \quad \text{and} \quad \nabla_h^2 U_2 = \delta_{y+} P \quad \text{in } S_H^0,
\] (3.7)

then
\[
Q_+ P := \nabla_0 \cdot \tilde{U} \quad \text{in } S_P
\] (3.8)
is well-defined. The above finite difference problem is similar to the following partial differential problem: For $p \in L^2(S_1 \cup S_2)$, define
\[
Q p := \nabla \cdot \tilde{u} \quad \text{in } S
\] (3.9)
where $\tilde{u} \in H^1_0(S_1) \times H^1_0(S_2)$ is the solution of
\[
\nabla^2 u_1 = p_x \quad \text{in } S_1 \quad \text{and} \quad \nabla^2 u_2 = p_y \quad \text{in } S_2.
\] (3.10)

4. Preliminaries. In this section, we get some basic results for the next sections and also show that $Q_+$ and $Q_-$ are self-adjoint. The next lemma resembles integration by parts.

**Lemma 4.1.** If $U, V \in L^2(\Omega_h)$ with $UV|_{e(\Gamma_h) \cup w(\Gamma_h)} = 0$, then
\[
(\delta_{x+} U, V)_{w(\Omega_h)} = -(U, \delta_{x-} V)_{e(\Omega_h)}.
\]
Proof. Let
\[ \Omega_h := \{ (lh, mh) \in S_h \mid l_0 \leq l \leq l_1 \text{ and } m_0 \leq m \leq m_1 \}, \]
then we note that \( U_{l,m} V_{l,m} = 0 \) if \( l = l_0 \) or \( l_1 \). Hence
\[
\begin{align*}
& (\delta_{x+} U, V)_{w(\Omega_h)} = h^2 \sum_{m=m_0}^{m_1} \sum_{l=l_0}^{l_1-1} \left( \frac{U_{l+1,m} - U_{l,m}}{h} \right) V_{l,m} \\
& = h \left( \sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m} V_{l-1,m} - \sum_{m=m_0}^{m_1} \sum_{l=l_0}^{l_1-1} U_{l,m} V_{l,m} \right) \\
& = h \left( \sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m} V_{l-1,m} - \sum_{m=m_0}^{m_1} \sum_{l=l_0}^{l_1-1} U_{l,m} V_{l,m} \right) \\
& = -h^2 \sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m} \left( \frac{V_{l,m} - V_{l-1,m}}{h} \right) = -(U, \delta_{x-} V)_{e(\Omega_h)} . \quad \Box
\end{align*}
\]
A similar result for staggered grids is stated in the next lemma. Other similar results which arise from different discrete domains and different differencings will be used without proof.

Lemma 4.2. Let \( P \in L^2(S_P) \).
1. For \( U \in H^1_0(S_I) \), \((U, \delta_{x+} P)_{S^*_I} = (-\delta_{x+} U, P)_{S_P} \).
2. For \( U \in H^1_0(S_{II}) \), \((U, \delta_{x+} P)_{S^*_II} = (-\delta_{x+} U, P)_{S_P} \).

By Lemma 4.1, we get the next lemma.

Lemma 4.3. For any \( U \in H^1_0(\Omega_h) \),
1. \( \| \delta_{x-} U \|^2_{e(\Omega_h)} = \| \delta_{x+} U \|^2_{w(\Omega_h)} = (U, -\delta_{x-} \delta_{x+} U)_{\Omega_h} \)
2. \( \| \delta_{y+} U \|^2_{n(\Omega_h)} = \| \delta_{y+} U \|^2_{e(\Omega_h)} = (U, -\delta_{y-} \delta_{y+} U)_{\Omega_h} \)
3. \( \| U \|^2_{1, \Omega_h} = (U, -\nabla_h^2 U)_{\Omega_h} \).

Proof. Let
\[ V = \begin{cases} 
\delta_{x+} U, & \text{in } w(\Omega_h) ; \\
\text{any finite number,} & \text{on } e(\Gamma_h),
\end{cases} \]
then, by Lemma 4.1,
\[
\| \delta_{x-} U \|^2_{e(\Omega_h)} = \| \delta_{x+} U \|^2_{w(\Omega_h)} = (\delta_{x+} U, \delta_{x+} U)_{\Omega_h} = (\delta_{x+} U, V)_{w(\Omega_h)} \\
= (U, -\delta_{x-} V)_{e(\Omega_h)} = (U, -\delta_{x-} V)_{\Omega_h} = (U, -\delta_{x-} \delta_{x+} U)_{\Omega_h} .
\]
The proof for (2) is similar. The statement (3) follows from (1) and (2). \quad \Box
Relation (3) in Lemma 4.3 extends to $H_0^1(\Omega_h)^2$ and implies that $-\nabla_h^2$ is positive definite, and hence the Schwarz inequality for $-\nabla_h^2$,

$$((\vec{V}, -\nabla_h^2 \vec{U})_{\Omega_h}^2 \leq (\vec{V}, -\nabla_h^2 \vec{V})_{\Omega_h}^2 (\vec{U}, -\nabla_h^2 \vec{U})_{\Omega_h}^2, \quad (4.1)$$

holds for any $\vec{U}$ and $\vec{V}$ in $H_0^1(\Omega_h)^2$.

By Lemma 4.1 and Lemma 4.2, we can show that the approximations $Q_{\pm}$ and $Q_0$ are self-adjoint.

**Theorem 4.4.**

(1) $(Q_{\pm} P_1, P_2)_{S_h^2} = (P_1, Q_{\pm} P_2)_{S_h^2}, \quad \forall P_1, P_2 \in L^2(S_h^0)$

(2) $(Q_0 P_1, P_2)_{S_P} = (P_1, Q_0 P_2)_{S_P}, \quad \forall P_1, P_2 \in L^2(S_P)$

**Proof.** Let $\vec{U}, \vec{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ be the solutions of

$$\nabla_h^2 U_1 = \delta_{2-} P_1 \quad \text{in} \ w(S_h)^0 \quad \text{and} \quad \nabla_h^2 U_2 = \delta_{y-} P_1 \quad \text{in} \ s(S_h)^0$$

and

$$\nabla_h^2 V_1 = \delta_{2-} P_2 \quad \text{in} \ w(S_h)^0 \quad \text{and} \quad \nabla_h^2 V_2 = \delta_{y-} P_2 \quad \text{in} \ s(S_h)^0,$$

respectively, then

$$Q_+ P_1 = \vec{\nabla}_- \cdot \vec{U} \quad \text{and} \quad Q_- P_2 = \vec{\nabla}_- \cdot \vec{V} \quad \text{in} \ S_h^0.$$

Hence, by Lemma 4.1 and Lemma 4.3,

$$(Q_+ P_1, P_2)_{S_h^0} = (\vec{\nabla}_- \cdot \vec{U}, P_2)_{S_h^0} = (\delta_{2-} U_1, P_2)_{c(w(S_h))} + (\delta_{y-} U_2, P_2)_{c(s(S_h))}$$

$$= (U_1, -\delta_{2+} P_2)_{w(w(S_h))} + (U_2, -\delta_{y+} P_2)_{s(s(S_h))}$$

$$= (U_1, -\nabla_h^2 V_1)_{w(S_h)^0} + (U_2, -\nabla_h^2 V_2)_{s(s(S_h))^0}$$

$$= (\delta_{2+} U_1, \delta_{2+} V_1)_{w(S_h))} + (\delta_{y+} U_1, \delta_{y+} V_1)_{s(S_h))}$$

$$= (\delta_{2+} U_2, \delta_{y+} V_2)_{w(S_h))} + (\delta_{y+} U_2, \delta_{y+} V_2)_{s(S_h))}$$

$$= (Q_+ P_2, P_1)_{S_h^0} = (P_1, Q_- P_2)_{S_h^0}.$$ 

The proof for $Q_+$ is similar. The proof for $Q_0$ is similar, but we include the proof for completeness. Let $\vec{U}, \vec{V} \in H_0^1(S_I) \times H_0^1(S_{II})$ be the solutions of

$$\nabla_h^2 U_1 = \delta_{20} P_1 \quad \text{in} \ S_I^0 \quad \text{and} \quad \nabla_h^2 U_2 = \delta_{y0} P_1 \quad \text{in} \ S_{II}^0$$

and

$$\nabla_h^2 V_1 = \delta_{20} P_2 \quad \text{in} \ S_I^0 \quad \text{and} \quad \nabla_h^2 V_2 = \delta_{y0} P_2 \quad \text{in} \ S_{II}^0$$

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respectively, then
\[ Q_\delta P_1 = \vec{\delta}_\delta \cdot \vec{U} \quad \text{and} \quad Q_\delta P_2 = \vec{\delta}_\delta \cdot \vec{V} \quad \text{in} \quad SP. \]

Hence, by Lemma 4.2,
\[
(Q_\delta P_1, P_2)_{SP} = (\vec{\delta}_\delta \cdot \vec{U}, P_2)_{SP} = (\delta_{\omega} U_1, P_2)_{SP} + (\delta_{\mu} U_2, P_2)_{n(S_{II})}
= (U_1, -\delta_{\omega} P_2)_{S_{II}} + (U_2, -\delta_{\mu} P_2)_{S_{II}} = (U_1, -\nabla^2_h V_1)_{S_{II}} + (U_2, -\nabla^2_h V_2)_{S_{II}}
= (\delta_{\omega} U_1, \delta_{\omega} V_1)_{w(S_I)} + (\delta_{\mu} U_1, \delta_{\mu} V_1)_{s(S_I)} + (\delta_{\omega} U_2, \delta_{\mu} V_2)_{w(S_{II})} + (\delta_{\mu} U_2, \delta_{\omega} V_2)_{s(S_{II})}
= (\delta_{\omega} U_1, \delta_{\omega} V_1)_{w(S_I)} + (\delta_{\mu} U_1, \delta_{\mu} V_1)_{s(S_I)} + (\delta_{\omega} U_2, \delta_{\mu} V_2)_{w(S_{II})} + (\delta_{\mu} U_2, \delta_{\omega} V_2)_{s(S_{II})}
= (Q_\delta P_1, P_2)_{SP} = (P_1, Q_\delta P_2)_{SP}. \]

\[ \square \]

5. The Condition Number of $Q_h$. We first prove that $Q_{\pm}$ and $Q_\delta$ are bounded above by 1.

**Theorem 5.1.**

(1) \( (Q_{\pm} P, P)_{S^h_k} \leq ||P||^2_{S^h_k}, \forall P \in L^2(S^h_k) \)

(2) \( (Q_\delta P, P)_{SP} \leq ||P||^2_{SP}, \forall P \in L^2(SP) \),

(3) \( ||Q_{\pm}||_{S^h_k} > ||Q_\delta||_{SP} \leq 1 \).

**Proof.** Let \( \vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h)) \) be the solution of (3.1), then

\[
||\vec{\delta} \cdot \vec{U}||^2_{S^h_k} \leq ||U_1||^2_{w(S_h)} + ||U_2||^2_{s(S_h)} = (U_1, -\nabla^2_h U_1)_{w(S_h)} + (U_2, -\nabla^2_h U_2)_{s(S_h)}
= (U_1, -\delta_{\omega} P)_{w(S_h)} + (U_2, -\delta_{\mu} P)_{s(S_h)} = (\delta_{\omega} U_1, P)_{w(S_h)} + (\delta_{\mu} U_2, P)_{s(S_h)}
= (\vec{\delta} \cdot \vec{U}, P)_{S^h_k} \leq ||\vec{\delta} \cdot \vec{U}||_{S^h_k} ||P||_{S^h_k}
\]

which implies

\[ ||\vec{\delta} \cdot \vec{U}||_{S^h_k} \leq ||P||_{S^h_k}. \]

By (3.2),

\[ Q_{\pm} P = \vec{\delta} \cdot \vec{U} \]

and hence

\[ ||Q_{\pm} P||_{S^h_k} = ||\vec{\delta} \cdot \vec{U}||_{S^h_k} \leq ||P||_{S^h_k}. \]

Thus

\[ (Q_{\pm} P, P)_{S^h_k} \leq ||Q_{\pm} P||_{S^h_k} ||P||_{S^h_k} \leq ||P||^2_{S^h_k}. \]

The proof for $Q_{\delta}$ is similar. The proof for (2) is also similar, but we show it for completeness. Let \( \vec{U} \in H_0^1(S_I) \times H_0^1(S_{II}) \) be the solution of (3.7), then, by Lemma 4.2,

\[
||\vec{\delta}_\omega \cdot \vec{U}||_{SP}^2 \leq ||U_1||^2_{1, S_I} + ||U_2||^2_{1, S_{II}} = (U_1, -\nabla^2_h U_1)_{S_I} + (U_2, -\nabla^2_h U_2)_{S_{II}}
= (U_1, -\delta_{\omega} P)_{S_I} + (U_2, -\delta_{\omega} P)_{S_{II}} = (\vec{\delta}_\omega \cdot \vec{U}, P)_{SP} \leq ||\vec{\delta}_\omega \cdot \vec{U}||_{SP} ||P||_{SP}.
\]

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By this relation and (3.8), we obtain
\[ \|Q_+ P\|_{S^p} \leq \|P\|_{S^p} \] (5.3)
and hence
\[ (Q_+ P, P)_{S^p} \leq \|Q_+ P\|_{S^p} \|P\|_{S^p} \leq \|P\|_{S^p}^2. \]

The statement (3) follows from (5.2) and (5.3). ∎

To prove that \( Q_\pm \) and \( Q_s \) are bounded below, we need the next lemma.

**Lemma 5.2.**

1. \((Q_- P, P)_{S^p} = \sup_{\vec{v} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))} \frac{(\vec{\nabla} \cdot \vec{v}, P)^2}{\|V_1\|_{1, w(S_h)}^2 + \|V_2\|_{1, s(S_h)}^2}, \quad \forall P \in L^2(S_h) \)

2. \((Q_+ P, P)_{S^p} = \sup_{\vec{v} \in H_2^1(\omega(S_h)) \times H_2^1(s(S_h))} \frac{(\vec{\nabla} \cdot \vec{v}, P)^2}{\|V_1\|_{1, w(S_h)}^2 + \|V_2\|_{1, s(S_h)}^2}, \quad \forall P \in L^2(S_h) \)

3. \((Q_s P, P)_{S^p} = \sup_{\vec{v} \in H_1^1(S_h)} \frac{(\vec{\nabla} \cdot \vec{v}, P)^2}{\|V_1\|_{1, S_h}^2 + \|V_2\|_{1, S_h}^2}, \quad \forall P \in L^2(S_h). \)

**Proof.** Let \( \vec{U} \) be the solution of (3.1), then
\((Q_- P, P)_{S^p} = (\vec{\nabla} \cdot \vec{U}, P)_{S^p} = (U_1, -\nabla_h^2 U_1)_{w(S_h)} + (U_2, -\nabla_h^2 U_2)_{s(S_h)} \)
by (5.1). Using (4.1), we have
\[(V_1, -\nabla_h^2 U_1)^2_{w(S_h)} \leq (V_1, -\nabla_h^2 U_1)_{w(S_h)} (V_1, -\nabla_h^2 U_1)_{w(S_h)} \]
\[(V_2, -\nabla_h^2 U_2)^2_{s(S_h)} \leq (V_2, -\nabla_h^2 U_2)_{s(S_h)} (V_2, -\nabla_h^2 U_2)_{s(S_h)} \]
for any nonzero \( \vec{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h)). \) Hence
\[2(V_1, -\nabla_h^2 U_1)_{w(S_h)} (V_2, -\nabla_h^2 U_2)_{s(S_h)} \leq 2(V_1, -\nabla_h^2 U_1)_{w(S_h)} \]
\[\sqrt{(V_1, -\nabla_h^2 U_1)_{w(S_h)}} \sqrt{(V_2, -\nabla_h^2 U_2)_{s(S_h)}} \leq \sqrt{(V_1, -\nabla_h^2 U_1)_{w(S_h)}} \sqrt{(V_2, -\nabla_h^2 U_2)_{s(S_h)}} \]
\[\leq (V_1, -\nabla_h^2 U_1)_{w(S_h)} (V_2, -\nabla_h^2 U_2)_{s(S_h)} + (U_1, -\nabla_h^2 U_1)_{w(S_h)} (V_2, -\nabla_h^2 U_2)_{s(S_h)} \]
and
\[(V_1, -\nabla_h^2 U_1)_{w(S_h)} + (V_2, -\nabla_h^2 U_2)_{s(S_h)} \leq \]
\[(V_1, -\nabla_h^2 U_1)_{w(S_h)} + (V_2, -\nabla_h^2 U_2)_{s(S_h)} \leq (V_1, -\nabla_h^2 U_1)_{w(S_h)} + (V_2, -\nabla_h^2 U_2)_{s(S_h)} \]
We have
\[\|V_1\|_{1, w(S_h)} + \|V_2\|_{1, s(S_h)} \]
and, by following the steps in (5.1),
\[(V_1, -\nabla_h^2 U_1)^2_{w(S_h)} + (V_2, -\nabla_h^2 U_2)^2_{s(S_h)} = (\vec{\nabla} \cdot \vec{V}, P)_{S^p} \]
Thus (1) is proved. The proofs for (2) and (3) are similar. ∎

Theorem 2.1 and Lemma 5.2 imply that \( Q_\pm \) and \( Q_s \) are bounded away from zero uniformly with respect to the mesh size \( h \).
Theorem 5.3. There exists a positive constant \( C_L \) which is independent of \( h \) such that
\[
(1) \quad \forall P \in L_0^2(S_h^0), \quad (Q_+ P, P)_{S_h^0} \geq C_L \|P\|_{S_h^0}^2
\]
\[
(2) \quad \forall P \in L_0^2(S_P), \quad (Q_+ P, P)_{S_P} \geq C_L \|P\|_{S_P}^2.
\]

Theorem 5.1 and Theorem 5.3 imply that the condition numbers of \( Q_+ \) and \( Q_- \) are independent of \( h \).

6. Preliminaries for Convergence Estimation. This section is to prepare to get the convergence rates of the solutions computed by \( Q_+ \) and \( Q_- \). We define
\[
|f|_{\Omega_h} := \sup_{(x,y) \in \Omega_h} |f(x,y)|, \quad |f|_{r_0, \Omega_h} := \sum_{r \leq r_0} \sup_{(x,y) \in \Omega_h} |\partial^r f(x,y)|
\]
where \( r_0 \) is a positive integer and \( \partial^r f \) denotes all possible \( r \)'th partial derivatives of \( f \). For a vector \( \tilde{f} = (f_i) \), let
\[
|\tilde{f}|_{\Omega_h} := \sum_i |f_i|_{\Omega_h}, \quad |\tilde{f}|_{r_0, \Omega_h} := \sum_i |f_i|_{r_0, \Omega_h}.
\]
The relation between \( \| . \| \) and \( | . | \) is stated in the next lemma.

Lemma 6.1. Let \( \Omega_h \) be a subset of \( S_h \) and \( U \in L^2(\Omega_h) \), then
\[
\|U\|_{\Omega_h^*} \leq |U|_{\Omega_h^*}.
\]

Proof. Let the number of points in \( \Omega_h^* \) be \( M \), then \( M \leq (N - 1)^2 \). Hence
\[
\|U\|_{\Omega_h^*}^2 \leq h^2 \sum_{(x,y) \in \Omega_h^*} |U(x,y)|^2 \leq h^2 M |U|_{\Omega_h^*}^2 = \frac{(N - 1)^2}{N^2} |U|_{\Omega_h^*}^2 \leq |U|_{\Omega_h^*}^2. \quad \square
\]

Refer to [3] for the next lemma which is a result of the maximum principle.

Lemma 6.2. Let \( \Omega_h \) be a subset of one of the sets \( S_h, S_I \) and \( S_H \) and let \( \Gamma_h \) be its boundary, then there exists a positive constant \( C_M = C_M(\Omega_h) \) such that
\[
|U|_{\Omega_h^*} \leq C_M |\nabla^2 U|_{\Omega_h^*} + |U|_{\Gamma_h} \quad \text{for } U \in L^2(\Omega_h).
\]
Moreover, there exists a positive constant \( C_S = C_S(S) \) such that
\[
|u|_S \leq C_S |\nabla^2 u|_S + |u|_T
\]
for any \( u \) which is twice differentiable in \( S \).

From the above lemma, we can estimate the norm of the difference quotients of the solution of the Poisson’s equation.
Lemma 6.3. Let \( \Omega_h \subseteq S_h \) and \( U \in H^1_0(\Omega_h) \) be such that
\[
\nabla_h^2 U = F \quad \text{in} \quad \Omega_h^0,
\]
then there exists a positive constant \( C_Q \) such that
\[
\| U \|_{1, \Omega_h} \leq C_Q | F |_{\Omega_h^0}.
\]

Proof. By Lemma 4.3, Lemma 6.1 and Lemma 6.2,
\[
\| U \|_{1, \Omega_h}^2 = (U, -\nabla_h^2 U)_{\Omega_h^0} = (U, -F)_{\Omega_h^0} \leq \frac{1}{2} (\| U \|_{\Omega_h^0}^2 + \| F \|_{\Omega_h^0}^2 )
\]
\[
\leq \frac{1}{2} (\| U \|_{\Omega_h^0}^2 + \| F \|_{\Omega_h^0}^2 ) \leq \frac{1}{2} (C_M | F |_{\Omega_h^0}^2 + | F |_{\Omega_h^0}^2 ).
\]
The claim follows by setting \( C_Q = \sqrt{(C_M + 1)/2} \). \( \square \)

We prove the discrete Poincaré inequality.

Lemma 6.4. Let \( U \in H^1_0(S_h) \), then
(1) \( \| U \|_{S_h^0} \leq \| \delta_{Xh} U \|_{S_h^0} \) and \( \| U \|_{S_h^0} \leq \| \delta_{Xh} U \|_{S_h^0} \)
(2) \( 2\| U \|_{S_h^0}^2 \leq \| U \|_{1, S_h}^2 \).

Proof. One can easily show that
\[
(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2 \tag{6.1}
\]
for any positive integer \( n \). Since \( U_{l,0} = 0 \),
\[
\sum_{m' = 1}^m (\delta_{Xh} U)_{l,m'} = \frac{U_{l,m}}{h} \quad \text{for} \quad l, m = 1, \ldots, N - 1. \tag{6.2}
\]

Using (6.1) and (6.2), we get
\[
U_{l,m}^2 \leq h^2 (\sum_{m' = 1}^m (\delta_{Xh} U)_{l,m'})^2 \leq mh^2 \sum_{m' = 1}^m (\delta_{Xh} U)_{l,m'}^2 \leq h \sum_{m' = 1}^{N-1} (\delta_{Xh} U)_{l,m'}^2
\]
and
\[
\sum_{m = 1}^{N-1} U_{l,m}^2 \leq (N-1)h \sum_{m' = 1}^{N-1} (\delta_{Xh} U)_{l,m'}^2 \leq \sum_{m = 1}^{N-1} (\delta_{Xh} U)_{l,m'}^2.
\]

Hence
\[
\| U \|_{S_h^0}^2 = h^2 \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} U_{l,m}^2 \leq h^2 \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} (\delta_{Xh} U)_{l,m}^2 = \| \delta_{Xh} U \|_{1, S_h}^2.
\]
The other inequalities in (1) are similar and (2) follows from (1). \( \square \)

The general boundary value problems for second-order elliptic equations on a polygon are discussed by Grisvard [5]. The next theorem follows from the work for the zero Dirichlet boundary condition in [5].

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Theorem 6.5. Let $k$ be a positive even integer and let $\Omega$ be a polygon with $\Gamma$ its boundary. If $f \in H^k(\Omega)$, then the Poisson’s equation

$$\nabla^2 u = f \quad \text{in } \Omega \quad \text{with } u|_{\Gamma} = 0$$

has a unique solution $\bar{u} \in H_0^{k+2}(\Omega)^2$.

The proof for the following imbedding theorem is in [3].

Theorem 6.6. For any positive integer $k$ and a polygon $\Omega$,

$$H_0^k(\Omega) \subset C^{k-2}(\bar{\Omega}).$$

7. Convergence Estimation. The next lemma shows how smooth the solution of (3.4) is.

Lemma 7.1. Let $\bar{u}$ be the solution of (3.4) with $p \in C^5(\overline{S_w} \cup \overline{S_s})$, then $\bar{u} \in C^4(\overline{S_w}) \times C^4(\overline{S_s})$.

Proof. Since $p \in C^5(\overline{S_w} \cup \overline{S_s})$, we have

$$\bar{\nabla} \ p \in C^4(\overline{S_w}) \times C^4(\overline{S_s}) \subset H^4(S_w) \times H^4(S_s).$$

Applying Theorem 6.5 and Theorem 6.6 to the Poisson’s equation (3.4), we have

$$\bar{u} \in H_0^6(S_w) \times H_0^6(S_s) \subset C^4(\overline{S_w}) \times C^4(\overline{S_s}). \quad \square$$

We show that $Q_-$ gives a first-order accurate solution.

Theorem 7.2. There exists a positive constant $C$ such that the following is true: Let $p \in C^5(\overline{S_w} \cup \overline{S_s})$ be the solution of

$$Qp = f \quad \text{in } \overline{S_{sw}} \quad (7.1)$$

and let $P \in L^2(S_h^0)$ be the solution of

$$Q_- P = f \quad \text{in } S_h^0 \quad (7.2)$$

Let $P$ be chosen up to a constant so that $p - P \in L^2(S_h^0)$. Let $\bar{u} \in C^4(\overline{S_w}) \times C^4(\overline{S_s})$ and $\bar{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ be the solutions of (3.4) and (3.1), respectively, then

$$\|u_1 - U_1\|_{w(S_h)} + \|u_2 - U_2\|_{s(S_h)} + \|p - P\|_{S_h^0} \leq Ch\left(\|u_1\|_{4, S_w} + \|u_2\|_{4, S_s} + \|p\|_{2, S_w \cup S_s}\right).$$
Proof. Let \( \tilde{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h)) \) be the solution of
\[
\nabla_h^2 V_1 = \delta_{x+p} \text{ in } w(S_h) \quad \text{ and } \quad \nabla_h^2 V_2 = \delta_{y+p} \text{ in } s(S_h),
\]
then
\[
Q_\cdot p = \nabla \cdot \tilde{V} \quad \text{ in } S_h^0
\] (7.3)
and
\[
\|u_1 - U_1\|_{w(S_h)} + \|u_2 - U_2\|_{s(S_h)} \leq \|u_1 - V_1\|_{w(S_h)} + \|u_2 - V_2\|_{s(S_h)} + \|V_1 - U_1\|_{w(S_h)} + \|V_2 - U_2\|_{s(S_h)}. \quad (7.4)
\]
Using Taylor expansions, for \((x, y) \in w(S_h)\), we have
\[
\nabla_h^2 u_1(x, y) = \nabla^2 u_1(x, y) + u_1^*(x, y) = p_x(x, y) + u_1^*(x, y)
\] (7.5)
\[
\nabla_h^2 V_1(x, y) = \delta_{x+p}(x, y) = p_x(x, y) + p_1^*(x, y)
\]
and, for \((x, y) \in s(S_h)\),
\[
\nabla_h^2 u_2(x, y) = \nabla^2 u_2(x, y) + u_2^*(x, y) = p_y(x, y) + u_2^*(x, y)
\]
\[
\nabla_h^2 V_2(x, y) = \delta_{y+p}(x, y) = p_y(x, y) + p_2^*(x, y)
\] (7.6)
where
\[
u_1^*(x, y) = \frac{h^2}{12} u_1(x^1, y), \quad p_1^*(x, y) = \frac{h}{2} p_{xx}(x^2, y)
\]
\[
u_2^*(x, y) = \frac{h^2}{12} u_2(y^3, x), \quad p_2^*(x, y) = \frac{h}{2} p_{yy}(x^2, y^2)
\] (7.7)
for some points \((x^i, y)\) in \(S_w\) and \((x, y^i)\) in \(S_s\) around \((x, y)\).

By (7.5) and (7.6),
\[
\nabla_h^2 (u_1 - V_1) = E_1 := u_1^* - p_1^* \quad \text{ in } w(S_h)
\]
\[
\nabla_h^2 (u_2 - V_2) = E_2 := u_2^* - p_2^* \quad \text{ in } s(S_h)
\] (7.8)
and, by Lemma 6.1, Lemma 6.2, (7.7), and (7.8),
\[
\|u_1 - V_1\|_{w(S_h)} + \|u_2 - V_2\|_{s(S_h)} \leq C_M \left( |E_1|_{w(S_h)} + |E_2|_{s(S_h)} \right)
\]
\[
\leq M_u h^2 \left( |u_1|_{S_w} + |u_2|_{S_s} \right) + M_p h |p|_{2, S_w} + M_p h |p|_{2, S_s},
\] (7.9)
for some positive constants \(M_u\) and \(M_p\).
Using Lemma 4.1, Lemma 4.3, Theorem 5.1, Lemma 6.4, and the fact that \( \tilde{V}, \tilde{U} \in H^1_0(w(S_h)) \times H^1_0(s(S_h)) \), we have

\[
2(\| V_1 - U_1 \|^2_{w(S_h)^0} + \| V_2 - U_2 \|^2_{s(S_h)^0}) \leq \| V_1 - U_1 \|^2_{1,w(S_h)} + \| V_2 - U_2 \|^2_{1,s(S_h)}
= \| (V_1 - U_1, -\nabla h^2 (V_1 - U_1)) \|^2_{w(S_h)^0} + \| (V_2 - U_2, -\nabla h^2 (V_2 - U_2)) \|^2_{s(S_h)^0}
= \| (V_1 - U_1, -\delta_{\gamma^+}(p - P)) \|^2_{w(S_h)} + \| (V_2 - U_2, -\delta_{\gamma^+}(p - P)) \|^2_{s(S_h)}
= \| (\delta_{\gamma^-}(V_1 - U_1), -p + P) \|^2_{e(w(S_h))} + \| (\delta_{\gamma^-}(V_2 - U_2), -p + P) \|^2_{s(s(S_h))}
= (\nabla_{\gamma^+} \cdot \tilde{V} - \tilde{U}, p - P)_{S_h^0} = (Q_-(p - P), p - P)_{S_h^0} \leq \| p - P \|_{S_h^0}^2.
\]

Hence

\[
(\| V_1 - U_1 \|^2_{w(S_h)^0} + \| V_2 - U_2 \|^2_{s(S_h)^0}) \leq 2(\| V_1 - U_1 \|^2_{1,w(S_h)} + \| V_2 - U_2 \|^2_{1,s(S_h)}) \leq \| p - P \|_{S_h^0}^2
\]
and

\[
\| V_1 - U_1 \|^2_{w(S_h)^0} + \| V_2 - U_2 \|^2_{s(S_h)^0} \leq \| p - P \|_{S_h^0}^2. \tag{7.10}
\]

Combining (7.4), (7.9), and (7.10), we have

\[
\| u_1 - U_1 \|^2_{w(S_h)^0} + \| u_2 - U_2 \|^2_{s(S_h)^0} \leq M_u h^2 (\| u_1 \|_{1,w(S_h)} + \| u_2 \|_{1,s(S_h)}) + M_p h^2 p \| 2, S_{w \cup S_s} + \| p - P \|_{S_h^0}. \tag{7.11}
\]

Now let's estimate \( \| p - P \|_{S_h^0} \). By (3.3), Theorem 5.3, (7.1), (7.2), and (7.3), we have

\[
C_L \| p - P \|_{S_h^0} \leq \| Q_-(p - P) \|_{S_h^0} = \| Q_-(p - Q_p) \|_{S_h^0}
= \| \nabla_{\gamma^+} \cdot \tilde{V} - \nabla \cdot \tilde{u} \|_{S_h^0} \leq \| \nabla_{\gamma^+} \cdot (\tilde{V} - \tilde{u}) \|_{S_h^0} + \| \nabla_{\gamma^+} \cdot \tilde{u} \|_{S_h^0} \tag{7.12}
\]
and, by Lemma 6.3, (7.7), and (7.8),

\[
\| \nabla_{\gamma^+} \cdot (\tilde{V} - \tilde{u}) \|_{S_h^0} \leq \| u_1 - V_1 \|_{1,w(S_h)} + \| u_2 - V_2 \|_{1,s(S_h)} \tag{7.13}
\]
\[
\leq C_Q (\| E_1 \|_{w(S_h)^0} + \| E_2 \|_{s(S_h)^0}) \leq Q_u h^2 (\| u_1 \|_{1,w(S_h)} + \| u_2 \|_{1,s(S_h)}) + Q_p h^2 p \| 2, S_{w \cup S_s}
\]
for some positive constants \( Q_u \) and \( Q_p \).

Using Taylor expansions, we get

\[
\nabla_{\gamma^+} \cdot \tilde{u}(x, y) = \nabla \cdot \tilde{u}(x, y) - \frac{h}{2} (\nabla \cdot \tilde{u}(x, y) + \tilde{u}(x, \bar{y})) \text{ in } S_h^0
\]
where \((\bar{x}, y)\) and \((x, \bar{y})\) are points in \( S \) around \((x, y)\). Thus

\[
\| \nabla_{\gamma^+} \cdot \tilde{u} - \nabla \cdot \tilde{u} \|_{S_h^0} \leq \frac{h}{2} \| \tilde{u} \|_{1,S} \tag{7.14}
\]

By (7.12), (7.13), and (7.14), we have

\[
\| p - P \|_{S_h^0} \leq C_p h (\| u_1 \|_{1,w(S_h)} + \| u_2 \|_{1,s(S_h)} + \| p \| 2, S_{w \cup S_s}) \tag{7.15}
\]
for some positive constant \( C_p \). The claim follows from (7.11) and (7.15). \( \square \)

The proof of the next lemma is similar to that of Lemma 7.1.
Lemma 7.3. Let \( \tilde{u} \) be the solution of (3.6) with \( p \in C^5(\overline{S} \cup \overline{S}_n) \), then \( \tilde{u} \in C^4(\overline{S}) \times C^4(\overline{S}_n) \).

It is proved similarly that \( Q_+ \) gives a first-order accurate solution.

Theorem 7.4. There exists a positive constant \( C \) such that the following is true: Let \( p \in C^5(\overline{S} \cup \overline{S}_n) \) be the solution of

\[
Q_p = f \quad \text{in} \quad \overline{S}_{ne}
\]

and let \( P \in L^2(S_h^o) \) be the solution of

\[
Q_+ P = f \quad \text{in} \quad S_h^o
\]

Let \( P \) be chosen up to a constant so that \( p - P \in L_0^2(S_h^o) \). Let \( \tilde{u} \in C^4(\overline{S}) \times C^4(\overline{S}_n) \) and \( \tilde{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h)) \) be the solutions of (3.6) and (3.5), respectively, then

\[
\|u_1 - U_1\|_{e(S_h)} + \|u_2 - U_2\|_{n(S_h)} + \|p - P\|_{S_h^o} \leq Ch(\|u_1|_{4, S} + \|u_2|_{4, S} + \|p|_{2, S_e \cup S_n}).
\]

The proof of the next lemma is also similar to that of Lemma 7.1.

Lemma 7.5. Let \( \tilde{u} \) be the solution of (3.10) with \( p \in C^5(\overline{S}_1 \cup \overline{S}_2) \), then \( \tilde{u} \in C^4(\overline{S}_1) \times C^4(\overline{S}_2) \).

We show the second-order accuracy of the solution computed by staggered mesh schemes.

Theorem 7.6. There exists a positive constant \( C \) such that the following is true: Let \( p \in C^5(\overline{S}_1 \cup \overline{S}_2) \) be the solution of

\[
Q_p = f \quad \text{in} \quad S
\]

and let \( P \in L^2(S_P) \) be the solution of

\[
Q_p P = f \quad \text{in} \quad S_P
\]

Let \( P \) be chosen up to a constant so that \( p - P \in L_0^2(S_P) \). Let \( \tilde{u} \in C^4(\overline{S}_1) \times C^4(\overline{S}_2) \) and \( \tilde{U} \in H_0^1(S_1) \times H_0^1(S_2) \) be the solutions of (3.10) and (3.7), respectively, then

\[
\|u_1 - U_1\|_{S_1} + \|u_2 - U_2\|_{S_2} + \|p - P\|_{S_P} \leq Ch^2(\|u_1|_{4, S_1} + \|u_2|_{4, S_2} + \|p|_{3, S}).
\]
Proof. Let \( \bar{V} \in H^1_0(S_I) \times H^1_0(S_H) \) be the solution of
\[
\nabla_h^2 V_1 = \delta_{\omega} p \quad \text{in} \quad S_I^o \quad \text{and} \quad \nabla_h^2 V_2 = \delta_\rho p \quad \text{in} \quad S_H^o,
\]
then
\[
Q_1 p = \nabla_\omega \cdot \bar{V} \quad \text{in} \quad S_P
\]
and
\[
\|u_1 - U_1\|_{S_I^o} + \|u_2 - U_2\|_{S_H^o} \leq \|u_1 - V_1\|_{S_I^o} + \|u_2 - V_2\|_{S_H^o} + \|V_1 - U_1\|_{S_I^o} + \|V_2 - U_2\|_{S_H^o}. \tag{7.19}
\]

Using Taylor expansions, for \((x, y) \in S_I^o\), we have
\[
\nabla_h^2 u_1(x, y) = \nabla^2 u_1(x, y) + u_1^*(x, y) = p_x(x, y) + u_1^*(x, y) \tag{7.20}
\]
and, for \((x, y) \in S_H^o\),
\[
\nabla_h^2 u_2(x, y) = \nabla^2 u_2(x, y) + u_2^*(x, y) = p_y(x, y) + u_2^*(x, y) \tag{7.21}
\]
where
\[
u_1^*(x, y) = \frac{h^2}{12} (u_1)_x x_x x(x^1, y), \quad p_1^*(x, y) = \frac{h}{2} p_x x(x^2, y), \tag{7.22}
u_2^*(x, y) = \frac{h^2}{12} (u_2)_y y_y y(x, y^3), \quad p_2^*(x, y) = \frac{h}{2} p_y y(x, y^4)
\]
for some points \((x^i, y)\) and \((x, y^i)\) in \(S\) around \((x, y)\).

By (7.20) and (7.21),
\[
\nabla_h^2 (u_1 - V_1)(x, y) = E_1(x, y) := u_1^* - p_1^* \quad \text{in} \quad S_I^o \tag{7.23}
\]
and, by Lemma 6.1, Lemma 6.2, (7.22), and (7.23),
\[
\|u_1 - V_1\|_{S_I^o} + \|u_2 - V_2\|_{S_H^o} \leq C_M \left( \|E_1\|_{S_I^o} + \|E_2\|_{S_H^o} \right) \leq C_m h^2 \left( \|u_1\|_{1, S_1} + \|u_2\|_{1, S_2} + |p|_{3, S} \right). \tag{7.24}
\]

for some positive constant \(C_m\).

Using Lemma 4.2, Lemma 4.3, Theorem 5.1, Lemma 6.4, and the fact that \(\bar{V}, \bar{U} \in H^1_0(S_I) \times H^1_0(S_H)\),
\[
2\left( \|V_1 - U_1\|^2_{S_I^o} + \|V_2 - U_2\|^2_{S_H^o} \right) \leq \|V_1 - U_1\|_{1, S_I^o} + \|V_2 - U_2\|_{1, S_H^o}
\]
\[
= (V_1 - U_1, -\nabla_h^2 (V_1 - U_1))_{S_I^o} + (V_2 - U_2, -\nabla_h^2 (V_2 - U_2))_{S_H^o}
\]
\[
= (V_1 - U_1, -\delta_{\omega}(p - P))_{S_I^o} + (V_2 - U_2, -\delta_\rho(p - P))_{S_H^o}
\]
\[
= (\nabla_\omega \cdot (\bar{V} - \bar{U}), p - P)_{S_P} = (Q_1(p - P), p - P)_{S_P} \leq \|p - P\|_{S_P}.
\]

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Hence
\[
(\|V_1 - U_1\|_{S^2_H}^2 + \|V_2 - U_2\|_{S^3_H}^2)^2 \leq 2\left(\|V_1 - U_1\|_{S^2_H}^2 + \|V_2 - U_2\|_{S^3_H}^2\right) \leq \|p - P\|_{S^p}^2
\]
and
\[
\|V_1 - U_1\|_{S^2_H} + \|V_2 - U_2\|_{S^3_H} \leq \|p - P\|_{S^p}.
\]
Combining (7.19), (7.24), and (7.25), we have
\[
\|u_1 - U_1\|_{S^2_H} + \|u_2 - U_2\|_{S^3_H} \leq C_m h^2 (|u_1|_{4, S_1} + |u_2|_{4, S_2} + |p|_{3, S}) + \|p - P\|_{S^p}.
\]
Now let’s estimate \(\|p - P\|_{S^2_H}\). By (3.9), (7.16), (7.17), (7.18), and Theorem 5.3,
\[
C_L \|p - P\|_{S^p} \leq \|Q, p - P\|_{S^p} = \|Q, p - Q, p\|_{S^p}
= \|\nabla \cdot (\nabla - \tilde{\nabla})\|_{S^p} \leq \|\nabla \cdot (\nabla - \tilde{\nabla})\|_{S^p} + \|\nabla \cdot u - \tilde{\nabla} \cdot u\|_{S^p}
\]
and, by Lemma 6.4, (7.22), and (7.23),
\[
\|\nabla \cdot (\nabla - \tilde{\nabla})\|_{S^p} \leq \|u_1 - V_1\|_{1, S_1} + \|u_2 - V_2\|_{1, S_2}
\leq C_Q (|E_1|_{S^2_H} + |E_2|_{S^3_H}) \leq C_q h^2 (|u_1|_{4, S_1} + |u_2|_{4, S_2} + |p|_{3, S})
\]
for some positive constant \(C_q\).
Using Taylor expansions around \((x, y) \in S_P\), we have
\[
\nabla \cdot u(x, y) = \tilde{\nabla} \cdot u(x, y) + \frac{1}{6} \left(\frac{h}{2}\right)^3 \left((u_1)_{x}(x', y) - (u_1)_{x}(x, y') + (u_2)_{yy}(x, y) - (u_2)_{yy}(x, y')\right)
\]
where \((x', y')\) are points in \(S_1\) and \((x, y')\) are points in \(S_2\) around \((x, y)\). Hence
\[
\|\nabla \cdot u - \tilde{\nabla} \cdot u\|_{S^p} \leq C_o h^2 (|u_1|_{3, S_1} + |u_2|_{3, S_2})
\]
for some positive constant \(C_o\). By equations (7.27), (7.28), and (7.29),
\[
\|p - P\|_{S^p} \leq C_p h^2 (|u_1|_{4, S_1} + |u_2|_{4, S_2} + |p|_{3, S})
\]
for some positive constant \(C_p\). By (7.26) and (7.30), the claim follows. □

One may ask whether
\[
Q_o P := \frac{Q_- P + Q_+ P}{2}
\]
would give a second-order accurate solution. Note that the domains of the velocity parts in the solutions of (3.1) and (3.5) are different. If one changes (3.1) and (3.5) so that the domains of the velocity parts are same, then the domains of the pressure parts become different. This difference in domains makes problems for getting a second-order accurate solution.
8. Conclusions. The condition numbers of the matrices generated by three finite difference approximations of the Stokes problem in the pressure equation (PE) method are shown to be independent of mesh size. Moreover, the convergence estimations of the solutions generated by these matrices are shown to be first or second-order accurate. These results were basically by the inf-sup conditions that are proved by Shin and Strikwerda [11]. Current research is on getting the inf-sup conditions for other finite difference approximations.

The PE method has been extended to the Navier-Stokes equations for low Reynolds numbers by Shin and Strikwerda [10]. Many algorithms that use different linearizing techniques could be applied for the extension of the PE method to the Navier-Stokes equations. Research is continuing on getting a better algorithm that works for higher Reynolds numbers.

Work is also being done on applying the PE method to time-dependent problems on more general domains. The method works better with polar domains since the Poisson equation can be solved directly with the help of the line SOR method. A lot of domains in applications are decomposed into some rectangular and some polar domains. Using the Schwarz alternating procedure [7] with a parallel algorithm, the PE method should work efficiently.

References


