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PARALLEL OPTIMIZATION

INTERIOR PROXIMAL POINT ALGORITHM FOR LINEAR PROGRAMS

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Computer Sciences Technical Report #949

July 1990

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Abstract

An interior proximal point algorithm for finding a solution of a linear program is presented. The distinguishing feature of this algorithm is the addition of a quadratic proximal term to the linear objective function. This perturbation has allowed us to obtain solutions with better feasibility. Implementation of this algorithm shows that the algorithm is competitive with MINOS 5.3 and other interior point algorithms. We also establish global convergence and local linear convergence of the algorithm.

1 Introduction

The method we are proposing is a modification of the logarithmic barrier algorithm discussed in [Gill et al, 1986]. At each iteration, the logarithmic penalty function is perturbed by a quadratic term, giving an algorithm that is similar to the proximal point method except that instead of solving the proximal point subproblem exactly, a single Newton step is taken. Adding this quadratic perturbation to the logarithmic penalty function gives a better

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conditioned subproblem and this has enabled us to obtain solutions with improved feasibility.

We consider linear program in the following standard form

$$\min_{x} cx \ subject\ to \ \ x \in S := \{x | Ax = b, x \geq 0\} \tag{1}$$

Here c is an n dimensional vector, A is an m by n matrix and b is an m dimensional vector

Applying the proximal point algorithm to solve (1), one starts with an initial feasible point $\hat{x}^0$ and generates sequence $\{\hat{x}^k\}$ such that

$$\begin{aligned} \hat{x}^{k+1} &:= arg \min_{x \in S} g^k(x) && (2) \\ where\ g^k(x) &:= cx + \frac{\alpha^k}{2} \|x - \hat{x}^k\|^2 \\ \alpha^k &> 0 && (3) \end{aligned}$$

The problem (2) is of course at least as hard to solve as the original problem, but Rockafellar [Rockafellar, 1976] showed that (2) need not to be solved exactly. If the following two conditions are satisfied

$$\begin{aligned} \|\hat{x}^{k+1} - arg \min_{x \in S} g^k(x)\| &\leq \epsilon_k && (4) \\ \sum_{k=0}^{\infty} \epsilon_k &< \infty && (5) \end{aligned}$$

then the sequence $\{\hat{x}^k\}$ converges to x^*, a solution of the original linear problem (1).

To use an interior point approach, we move the nonnegativity constraints from problem (2) to the objective function and obtain $\tilde{x}^{k+1}$ by minimizing the logarithmic barrier function

$$\begin{aligned} \tilde{x}^{k+1} &:= arg \min_{x \in S_1} F^k(x) && (6) \\ where\ F^k(x) &:= cx + \frac{\alpha^k}{2} \|x - \tilde{x}^k\|^2 - \gamma^k \sum_{j=1}^{n} \log x_j \\ S_1 &:= \{x | Ax = b\} \\ and\ \ \alpha^k, \gamma^k &> 0 \end{aligned}$$

For the algorithm proposed here, we do not attempt to get $\tilde{x}^{k+1}$ as the exact solution of problem (6), instead we get an approximation for it from yet another problem which is quadratic. Given a strictly feasible point x^k , positive scalars α^k and γ^k, we solve

$$\min_x \nabla F^k(x^k)(x - x^k) + \frac{1}{2}(x - x^k)\nabla^2 F^k(x^k)(x - x^k) \tag{7}$$

$$subject\, to \;\; Ax = b$$

where

$$\begin{aligned} \nabla F^k(x) &:= c + \alpha^k(x - x^k) - \gamma^k X^{-1} e \\ \nabla^2 F^k(x) &:= \alpha^k I + \gamma^k X^{-2} \end{aligned}$$

where $X := diag(x)$.

If we denote the solution of the above quadratic problem by $\overline{x}^k$, the next iterate is defined to be

$$x^{k+1} := x^k + \lambda^k(\overline{x}^k - x^k) \tag{8}$$

where λ^k is chosen such that the strict feasibility of x^{k+1} is ensured. We call this algorithm the **Interior Proximal Point (IPP)** and describe it in Section 2. The rest of the paper is organized as follows. In Section 3 we give the convergence of the algorithm, in Section 4 we present computational results from our implementation of the algorithm. In Section 4 we describe a refinement technique for improving the accuracy of the optimal solutions obtained by the algorithm. We give summary of the paper in Section 6.

2 Statement of the algorithm

Algorithm IPP:

- Initialization
  We assume that at the start of the algorithm $x^0 > 0$ such that $Ax^0 = b$ is known and $\alpha^0 > \alpha_{min} > 0$ and $\gamma^0 > \gamma_{min} > 0$ are given.
  Set $k = 0$.

- Iterations
  Solve the following quadratic minimization problem

$$\min_x \nabla F^k(x^k)(x - x^k) + \frac{1}{2}(x - x^k)\nabla^2 F^k(x^k)(x - x^k) \tag{9}$$

$$\textit{subject to} \;\; Ax = b$$

  Denote the solution of this minimization problem by $\overline{x}^k$.

  1. Determine λ_{max} as follows

$$\lambda_{max} := \begin{cases} 1 & \text{if} \;\; \overline{x}^k \geq 0 \\ \min_{j \in J} \left(\frac{x_j^k}{x_j^k - \overline{x}_j^k} \right) & \text{otherwise} \end{cases} \tag{10}$$

     where $J := \{j | x_j^k - \overline{x}_j^k > 0\}$
  2. Update

$$x^{k+1} := x^k + \lambda^k(\overline{x}^k - x^k) \tag{11}$$

     where $\lambda^k = \eta\lambda_{max}$, $0 < \eta < 1$

- Termination
  If some convergence criterion is satisfied, then Stop
  Else

  1. if $\alpha^k > \alpha_{min}$, reduce the value of α^k
  2. If $\gamma^k > \gamma_{min}$, reduce the value of γ^k
  3. Set k := k + 1
  4. Go to Iterations

3 Convergence of IPP

We begin by stating the following lemmas that will be used in proving the convergence of our algorithm.

Lemma 1 *Let $\overline{x}$ be a solution of linear program (1) and for some $\gamma > 0$ let $x(\gamma)$ be the solution of*

$$\min_x \; cx - \gamma \sum_{j=1}^{n} \log x_j \;\; \textit{subject to } Ax = b \tag{12}$$

then

$$\langle c, x(\gamma) - \overline{x} \rangle \leq n\gamma$$

Proof

Since $x(\gamma)$ is the solution of problem (12), there exists $u(\gamma) \in \mathbb{R}^m$ such that $(x(\gamma), u(\gamma))$ satisfies the following KKT conditions

$$\begin{aligned} c - \gamma X(\gamma)^{-1} e - A^t u(\gamma) &= 0 \\ Ax(\gamma) &= b \end{aligned}$$

where $X(\gamma)$ is a diagonal matrix with $x(\gamma)$ as its diagonal. Consider the dual of linear program (1)

$$\begin{aligned} \max \;\; & bu \\ \textit{subject to } \; c - A^t u - v &= 0 \\ v &\geq 0 \end{aligned}$$

Let

$$\begin{aligned} x^* &= x(\gamma) \\ u^* &= u(\gamma) \\ v^* &= \gamma X(\gamma)^{-1} e \end{aligned}$$

then (x^*, u^*, v^*) is dual feasible, hence by duality

$$\begin{aligned} c\overline{x} &\geq bu^* \\ &= cx^* - \langle u^*, Ax^* - b \rangle - v^* x^* \\ &= cx^* - n\gamma \end{aligned}$$

this completes proof of the lemma. □

Lemma 2 *Let $y \in \mathbb{R}^n$ be such that $\|y\| = \beta$ for some $\beta \in [0, 1)$. Then*

$$\sum_{j=1}^{n} \sum_{i=2}^{\infty} \frac{(-1)^i}{i} y_j^i \leq \frac{\beta^2}{2(1-\beta)}$$

Proof

$$\begin{aligned}
\sum_{j=1}^{n} \sum_{i=2}^{\infty} \frac{(-1)^i}{i} y_j^i &\leq \frac{1}{2} \sum_{j=1}^{n} \sum_{i=2}^{\infty} |y_j|^i \\
&= \frac{1}{2} \sum_{j=1}^{n} |y_j|^2 \sum_{i=0}^{\infty} |y_j|^i \\
&\leq \frac{1}{2} \sum_{j=1}^{n} |y_j|^2 \frac{1}{1-\beta} \quad (\textit{since } |y_j| \leq \beta < 1 \textit{ for all } j) \\
&= \frac{\beta^2}{2(1-\beta)}
\end{aligned}$$

□

Lemma 3 *Let $x \in \mathbb{R}^n$ be such that*

$$\begin{aligned}
\langle e, x \rangle &= n \\
x &> 0
\end{aligned}$$

Then

$$\sum_{j=1}^{n} \log x_j \leq 0$$

Proof

$$\begin{aligned}
\sum_{j=1}^{n} \log x_j &= \log \prod_{j=1}^{n} x_j \\
&\leq n \log \left(\frac{\sum_{j=1}^{n} x_j}{n} \right) \\
& \quad (\textit{By Geometric Mean} \leq \textit{Arithmetic Mean}) \\
&= 0
\end{aligned}$$

□

To prove the convergence of the algorithm, we assume that the linear problem is of the following form

$$\min cx \tag{13}$$

subject to

$$\begin{aligned} Ax &= 0 \\ \langle e, x\rangle &= n \\ x &\geq 0 \end{aligned}$$

There is no loss of generality in this, since every linear problem in standard form (1) can be transformed into problem (13) (see for example [Dennis Jr. et al, 1987]). We also assume that for the above linear program e is feasible and that $c \neq 0$.

Regarding this linear problem, we have the following lemmas.

Lemma 4 *Let x^k be an iterate of the Algorithm IPP and let $\overline{x}^k$ be the possibly nonpositive solution of quadratic problem (9). Define an ellipsoid*

$$E(\beta) = \{x | \sum_{j=1}^{n} \left(\frac{x_j - x_j^k}{x_j^k}\right)^2 \leq \beta^2\}$$

for some $\beta > 0$. Let

$$\begin{aligned} f^k(x) &:= cx - \gamma^k \sum_{j=1}^{n} \log x_j \\ \gamma^{k+1} &:= \rho^k \gamma^k \;\; for\ some\ \rho^k \in (0,1] \end{aligned}$$

and suppose that $\overline{x}^k \notin E(0.25)$, then there exists a point $\tilde{x}^{k+1}$ on the line joining x^k and $\overline{x}^k$, is on the boundary of $E(0.25)$, and $f^{k+1}(\tilde{x}^{k+1}) < f^k(x^k) - \frac{1}{48}\gamma^k - \alpha^k \left\|\tilde{x}^{k+1} - x^k\right\|^2$

Proof

Define $\tilde{x}^{k+1}$ as follows

$$\tilde{x}^{k+1} = x^k + (\beta/\delta)(\overline{x}^k - x^k) \tag{14}$$

where

$$\beta^2 < \delta^2 = \sum_{j=1}^{n} \left(\frac{\overline{x}_j^k - x_j^k}{x_j^k} \right)^2$$

then $\tilde{x}^{k+1}$ is on the line joining x^k and $\overline{x}^k$, and $\tilde{x}^{k+1}$ lies on the boundary of $E(\beta)$.

Since $\overline{x}^k$ is the solution of quadratic problem (9), there exists $\overline{u}^k \in \mathbb{R}^m$ such that

$$c - \gamma^k (X^k)^{-1} e + (\alpha^k I + \gamma^k (X^k)^{-2})(\overline{x}^k - x^k) - A^t \overline{u}^k = 0$$

or equivalently

$$c - \gamma^k (X^k)^{-1} e + (\alpha^k I + \gamma^k (X^k)^{-2})(\tilde{x}^{k+1} - x^k) + (\alpha^k I + \gamma^k (X^k)^{-2})(\overline{x}^k - \tilde{x}^{k+1}) - A^t \overline{u}^k = 0$$

Multiplying both sides of the last equation by $\tilde{x}^{k+1} - x^k$ and letting

$$y_j = \frac{\tilde{x}_j^{k+1} - x_j^k}{x_j^k}$$

gives

$$\begin{aligned} 0 \ = \ & \left\langle c, \tilde{x}^{k+1} - x^k \right\rangle - \gamma^k \sum_{j=1}^{n} y_j + \alpha^k \left\| \tilde{x}^{k+1} - x^k \right\|^2 + \\ & \gamma^k \sum_{j=1}^{n} y_j^2 + (\tilde{x}^{k+1} - x^k)(\alpha^k I + \gamma^k (X^k)^{-2})(\overline{x}^k - \tilde{x}^{k+1}) \end{aligned}$$

Since $\tilde{x}^{k+1}$ is on the boundary of the ellipsoid $E(\beta)$, it follows that $\|y\| = \beta, \ \beta = 0.25$. Rearranging terms in the last equation and substituting from (14), we get

$$\begin{aligned} \left\langle c, \tilde{x}^{k+1} - x^k \right\rangle \ = \ & \gamma^k \sum_{j=1}^{n} y_j - \alpha^k \left\| \tilde{x}^{k+1} - x^k \right\|^2 \\ & - \gamma^k \sum_{j=1}^{n} y_j^2 - \frac{\delta - \beta}{\beta} (\tilde{x}^{k+1} - x^k)(\alpha I + \gamma^k (X^k)^{-2})(\tilde{x}^{k+1} - x^k) \\ < \ & \gamma^k \sum_{j=1}^{n} y_j - \gamma^k \sum_{j=1}^{n} y_j^2 - \alpha^k \left\| \tilde{x}^{k+1} - x^k \right\|^2 \end{aligned}$$

Subtracting

$$\gamma^{k+1}\sum_{j=1}^{n}\log\tilde{x}_j^{k+1}-\gamma^k\sum_{j=1}^{n}\log x_j^k$$

from both sides of the previous relation gives

$$\begin{aligned}
&(c\tilde{x}^{k+1}-\gamma^{k+1}\sum_{j=1}^{n}\log\tilde{x}_j^{k+1})-(cx^k-\gamma^k\sum_{j=1}^{n}\log x_j^k)\\
<\ & \gamma^k\left(\sum_{j=1}^{n}y_j-\sum_{j=1}^{n}y_j^2-\sum_{j=1}^{n}\log(\frac{\tilde{x}_j^{k+1}}{x_j^k})+(1-\rho^k)\sum_{j=1}^{n}\log\tilde{x}_j^{k+1}\right)-\alpha^k\left\|\tilde{x}^{k+1}-x^k\right\|^2\\
\leq\ & \gamma^k\left(\sum_{j=1}^{n}y_j-\sum_{j=1}^{n}y_j^2-\sum_{j=1}^{n}\log(1+\frac{\tilde{x}_j^{k+1}-x_j^k}{x_j^k})\right)-\alpha^k\left\|\tilde{x}^{k+1}-x^k\right\|^2\\
&(by\ Lemma\ 3)\ \ (\rho^k\in(0,1])\\
=\ & -\gamma^k\left(\sum_{j=1}^{n}y_j^2-\sum_{j=1}^{n}\sum_{i=2}^{\infty}\frac{(-1)^i}{i}y_j^i\right)-\alpha^k\left\|\tilde{x}^{k+1}-x^k\right\|^2\\
\leq\ & -\gamma^k\left[\beta^2\left(1-\frac{1}{2(1-\beta)}\right)\right]-\alpha^k\left\|\tilde{x}^{k+1}-x^k\right\|^2\ \ (by\ Lemma\ 2)\ \ (\beta=0.25)\\
=\ & -\frac{1}{48}\gamma^k-\alpha^k\left\|\tilde{x}^{k+1}-x^k\right\|^2
\end{aligned}$$

hence $f^{k+1}(\tilde{x}^{k+1})<f^k(x^k)-\frac{1}{48}\gamma^k-\alpha^k\left\|\tilde{x}^{k+1}-x^k\right\|^2$. □

Remark 5 *Instead of $\beta=0.25$, we can pick $\beta=0.3596$ and maximize the expression in the square bracket above, but any $\beta\in(0,0.5)$ will also work for our convergence proof.*

Corollary 6 *Let $\overline{x}^k$ be the solution of quadratic problem (9) and suppose that $\overline{x}^k\in E(0.25)$, then*

$$f^{k+1}(\overline{x}^k)\leq f^k(x^k)-\alpha^k\left\|\overline{x}^k-x^k\right\|^2$$

Proof
We consider the following 2 cases

1. $\overline{x}^k = x^k$
   We have

$$\begin{aligned} f^{k+1}(\overline{x}^k) - f^k(x^k) &= c\overline{x}^k - \gamma^{k+1}\sum_{j=1}^{n}\log \overline{x}_j^k - cx^k + \gamma^k \sum_{j=1}^{n}\log x_j^k \\ &= (\gamma^k - \gamma^{k+1})\sum_{j=1}^{n}\log x_j^k \\ &\leq 0 \;\; (by\ definition\ of\ \gamma^{k+1}\ and\ Lemma\ \ 3) \end{aligned}$$

2. $\overline{x}^k \neq x^k$
   If this is true, then there must exists $\beta' \in (0, 0.25)$ such that $\overline{x}^k$ lies on the boundary of $E(\beta')$ and

$$f^{k+1}(\overline{x}^k) - f^k(x^k) \leq -\gamma^k(\beta')^2\left(1 - \frac{1}{2(1-\beta')}\right) - \alpha^k \left\|\overline{x}^k - x^k\right\|^2$$

   The proof is similar to that of Lemma 4. □

If a line search procedure is employed, we can establish the nonincreasing property of the iterates by using the results of the above lemma and corollary.

Lemma 7 *Suppose that x^{k+1} in the Algorithm IPP is computed using a line search procedure as follows*

$$x^{k+1} = x^k + \lambda^k(\overline{x}^k - x^k)$$

where

$$\lambda^k := arg \min_{\lambda\in[0,\lambda_{max}]} f^{k+1}(x^k + \lambda(\overline{x}^k - x^k)) + \alpha^k\lambda^2 \left\|\overline{x}^k - x^k\right\|^2$$

where λ_{max} is defined as in (10), then

$$f^{k+1}(x^{k+1}) - f^k(x^k) \leq -\alpha^k \left\|x^{k+1} - x^k\right\|^2$$

Proof
Let $\overline{x}^k$ be the solution of the quadratic problem (9) and consider the following 2 cases

1. $\overline{x}^k \notin E(0.25)$
   By Lemma 4, there exists $\tilde{x}^{k+1}$ on the boundary of $E(0.25)$ such that

$$f^{k+1}(\tilde{x}^{k+1}) < f^k(x^k) - \alpha^k \left\|\tilde{x}^{k+1} - x^k\right\|^2$$

   Let $\tilde{\lambda}^k > 0$ be such that $\tilde{x}^{k+1} = x^k + \tilde{\lambda}^k(\overline{x}^k - x^k)$, we have

$$\begin{aligned} f^{k+1}(x^{k+1}) &+ \alpha^k \left\|x^{k+1} - x^k\right\|^2 \\ &= f^{k+1}(x^k + \lambda^k(\overline{x}^k - x^k)) + \alpha^k(\lambda^k)^2 \left\|\overline{x}^k - x^k\right\|^2 \\ &\leq f^{k+1}(x^k + \tilde{\lambda}^k(\overline{x}^k - x^k)) + \alpha^k(\tilde{\lambda}^k)^2 \left\|\overline{x}^k - x^k\right\|^2 \\ &\quad (by\ definition\ of\ \lambda^k) \\ &= f^{k+1}(\tilde{x}^{k+1}) + \alpha^k \left\|\tilde{x}^{k+1} - x^k\right\|^2 \\ &< f^k(x^k) \end{aligned}$$

2. $\overline{x}^k \in E(0.25)$
   By similar derivation as in case 1 above, we have

$$f^{k+1}(x^{k+1}) + \alpha^k \left\|x^{k+1} - x^k\right\|^2 \leq f^k(x^k)$$

This completes the proof. □

The following theorem establishes the convergence of the algorithm.

Theorem 8 *Let $\overline{x}$ be the solution of linear program (13), and consider the problem*

$$\min \overline{f}(x) \tag{15}$$

$$\begin{aligned} where\ \overline{f}(x) &:= cx - \overline{\gamma}\sum_{j=1}^{n} \log x_j \\ subject\ to\ Ax &= 0 \\ \langle e, x\rangle &= n \\ x &\geq 0 \end{aligned}$$

Suppose that the algorithm IPP is terminated only when $\gamma^k \leq \gamma_{min} = \overline{\gamma}$ and $\left\|x^{k+1} - x^k\right\| = 0$, then the sequence $\{x^k\}$ generated by the algorithm applied to the linear program (13) either terminates at or converges to x^* which solves problem (15) and $cx^* \leq c\overline{x} + n\overline{\gamma}$.

Proof
The algorithm terminates only when $\gamma^k \leq \overline{\gamma}$ and $x^{k+1} = x^k$, and hence for $x^* = x^k$, $cx^* \leq c\overline{x} + n\overline{\gamma}$ by Lemma 1. Suppose the algorithm does not terminate. Then by Lemma 7 we have

$$\overline{f}(x^k) \geq \overline{f}(x^{k+1}) + \alpha^k \left\|x^{k+1} - x^k\right\|^2 \tag{16}$$

The nonincreasing sequence $\{\overline{f}(x^k)\}$ is bounded below by $\overline{f}(x^*)$ and hence converges. Since the sequences $\{\lambda^k\}$ and $\{x^k\}$ lie in a compact set, they must have accumulation points λ^* and x^* respectively, and since $\alpha^k \geq \alpha_{min} > 0$, from (16) we have

$$\begin{aligned}
0 &= \lim_{j\to\infty} \left\|x^{i_j+1} - x^{i_j}\right\|^2 \\
&= \lim_{j\to\infty} (\lambda^{i_j})^2 \left\|\overline{x}^{i_j} - x^{i_j}\right\|^2 \\
&= \lim_{j\to\infty} (\lambda^{i_j})^2 \lim_{j\to\infty} \left\|\overline{x}^{i_j} - x^{i_j}\right\|^2
\end{aligned}$$

We either have
Case I : $\lim_{j\to\infty} \|\overline{x}^{i_j} - x^{i_j}\|^2 = 0$
or
Case II : $\lim_{j\to\infty} (\lambda^{i_j})^2 = 0$
In the Case II, by definition of λ^{i_j} we have

$$0 = \left\langle \nabla\overline{f}(x^{i_j} + \lambda^{i_j}(\overline{x}^{i_j} - x^{i_j}), \overline{x}^{i_j} - x^{i_j}\right\rangle + 2\alpha^{i_j}\lambda^{i_j}\left\|\overline{x}^{i_j} - x^{i_j}\right\|^2$$

Define diagonal matrices D and D^* such that

$$\begin{aligned}
D_{kk} &= \left(x_k^{i_j}\right)^{-1} \\
D^*_{kk} &= \left(\lim_{j\to\infty} x_k^{i_j}\right)^{-1}
\end{aligned}$$

for all $k = 1, 2, 3..., n$. By letting $j \to \infty$ we get

$$\begin{aligned}
0 &= \lim_{j\to\infty} \left\langle \nabla\overline{f}(x^{i_j}), \overline{x}^{i_j} - x^{i_j}\right\rangle \\
&= \lim_{j\to\infty} -\left\langle \overline{x}^{i_j} - x^{i_j}, (\alpha^{i_j} I + \gamma^{i_j} D^2)(\overline{x}^{i_j} - x^{i_j})\right\rangle
\end{aligned}$$

Since the matrix $\alpha^{i_j} I + \gamma^{i_j} D^{-2}$ is positive definite, we have

$$\lim_{j\to\infty} (\overline{x}^{i_j} - x^{i_j}) = 0$$

We have that Case II implies Case I. From the KKT conditions for quadratic problem (9), we must have

$$\begin{array}{rcl} c - \gamma^{i_j} De + (\alpha^{i_j} + \gamma^{i_j} D^2)(\overline{x}^{i_j} - x^{i_j}) - \overline{A}^t u^{i_j} & = & 0 \\ \overline{A}\overline{x}^{i_j} = \overline{b} & & \end{array}$$

where $\overline{A}^t = [A^t \ \ e]$ and $\overline{b}^t = (0 \ \ n)$. As $j \to \infty$ we have

$$\begin{array}{rcl} c - \overline{\gamma} D^* e - \overline{A}^t u^* & = & 0 \\ \overline{A} x^* & = & \overline{b} \end{array}$$

Hence (x^*, u^*) satisfies the KKT conditions for problem (15) and by Lemma 1 we have that $cx^* \leq c\overline{x} + n\overline{\gamma}$. Since all accumulation points of the bounded sequence $\{x^k\}$ are equal to the unique solution of (15), it follows that $\{x^k\}$ converges to x^*, and this completes the proof of the theorem. □

The next lemma shows that if we have an iterate x^k such that $\overline{f}(x^k)$ is sufficiently close to the minimum value of problem (15) then the solution of quadratic problem (9) is strictly feasible.

Lemma 9 *Suppose for some k, x^k satisfies the following*

$$\overline{f}(x^k) - \overline{f}(x^*) \leq \frac{1}{48}\overline{\gamma} \tag{17}$$

where x^* is the solution of (15), then $\overline{x}^k$ which is the solution of the quadratic problem (9) is strictly feasible, that is $\overline{x}^k > 0$.

Proof

Suppose $\overline{x}^k$ is not strictly feasible, then $\overline{x}^k \notin E(0.25)$. By Lemma 4 there exists $\tilde{x}^{k+1}$on the boundary of $E(0.25)$ such that

$$\begin{array}{rcl} \overline{f}(\tilde{x}^{k+1}) - \overline{f}(x^k) & < & -\frac{1}{48}\overline{\gamma} \\ & \leq & \overline{f}(x^*) - \overline{f}(x^k) \end{array}$$

which is a contradiction. □

For local convergence of our algorithm, we use the results given in the paper by Garcia Palomares and Mangasarian [Garcia Palomares & Mangasarian, 1976]. In this paper they describe an algorithm that solves a nonlinear program by successively solving quadratic problems which contain an estimate of the Hessian of the original nonlinear problem. The nonlinear problem that they consider is

$$\min f(x) \;\; \textit{subject to}\; g(x) \leq 0 \tag{18}$$

where f and g are differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$ respectively. And the quadratic problem that is solved at each iteration is

$$\begin{array}{ll} \min & \nabla f(x^k)(x - x^k) + \frac{1}{2}(x - x^k)G(z^k)(x - x^k) \\ \textit{subject to} & g(x^k) - \left\langle \nabla g(x^k), x - x^k \right\rangle \leq 0 \end{array} \tag{19}$$

where $z^k = (x^k, u^k)$ and $G(z^k)$ is an approximation of $\nabla^2 L(z^k)$, the n *by* n Hessian with respect to x of the Lagrangian $L(z) = f(x) + \langle u, g(x) \rangle$. Let $\overline{z} = (\overline{x}, \overline{u})$ be a point that satisfies the second order sufficiency conditions for problem (18) and suppose that $G(z^k)$ satisfies the conditions

$$\left\| G(z^k) - \nabla^2 L(z^k) \right\| \;\; \leq \;\; \frac{1}{10\beta}$$

$$\beta = \frac{3}{2} \left\| \nabla_z h(\overline{z})^{-1} \right\|$$

where the function $h(z) : \mathbb{R}^{n+m}$ to $\mathbb{R}^{n+m}$ defined as

$$h(z) := \begin{pmatrix} \nabla f(x) + \nabla g(x) u \\ u_1 g_1(x) \\ \vdots \\ u_m g_m(x) \end{pmatrix}$$

If for the entire algorithm, a step size equal to one is taken (i.e. (x^{k+1}, u^{k+1}) is equal to the solution of the quadratic problem (19), if the starting point is sufficiently close to the solution point, if the second order sufficient conditions are satisfied at this solution point, if the active constraints are linearly independent and if the multipliers associated with them are positive, then the algorithm converges locally with a linear rate.

By using their result, we have the following local linear convergence result for our algorithm.

Theorem 10 *Suppose the rows of the matrix A and the vector e that form the equality constraints of problem (13) are linearly independent, and suppose that when x^k is sufficiently close to the solution point of problem (15) (that is, if (17) is satisfied) no line search is done in the algorithm, that is $x^{k+1} = \overline{x}^k$. If $\alpha^k \to \alpha_{min}$ and $\alpha_{min} \leq \frac{1}{10\beta}$ where β is the constant defined below, then $\{x^k\}$ converges to x^* with a linear rate and x^* solves problem (15).*

Proof

Since we only have equality constraints, define function $d(x, u)$ as follows

$$d(x,u) := \begin{pmatrix} \nabla \overline{f}(x) - \overline{A}^t u \\ \overline{A}x - \overline{b} \end{pmatrix}$$

where $\overline{A}^t = [A^t \;\; e]$, $\overline{b}^t = (0^t \;\; n)$ and $\overline{f}(x)$ is defined in problem (15). Let $\overline{z} = (\overline{u}, \overline{x})$ be a point that satisfies the second order sufficiency conditions for problem (15). The linear independence of the rows of $\overline{A}$ ensures that $\nabla_z d(\overline{x}, \overline{u})$ is nonsingular (see [McCormick, 1982]), hence we can define β to be

$$\beta := \frac{3}{2} \left\| \nabla_z d(\overline{x}, \overline{u})^{-1} \right\|$$

In our algorithm, we have

$$\left\| G(z^k) - \nabla^2 L(z^k) \right\| = \alpha^k$$

Since α^k is decreased at each iteration, for sufficiently large k, we must have

$$\alpha^k \leq \alpha_{min} \leq \frac{1}{10\beta}$$

and by Theorem 3.1 [Garcia Palomares & Mangasarian, 1976], the linear convergence property of Algorithm IPP follows. □

Finally, we state the following lemma and proposition to show that the solution of problem (15) is bounded away from zero.

Lemma 11 *Let $\gamma > 0$ and let $i \in \{1, 2, ..., n\}$. Then there exists $\delta_i > 0$ such that*

$$\left. \begin{array}{c} 0 < x_i \leq \delta_i \\ Ax = 0 \\ \langle e, x \rangle = n \\ x > 0 \end{array} \right\rangle \Longrightarrow cx - \gamma \sum_{j=1}^{n} \log x_j > ce$$

Proof
Suppose not, then there exist sequences $\{\delta_i^k\} \downarrow 0$, such that $\{x^k\} > 0, \left\langle e, x^k \right\rangle = n, Ax^k = 0,\ 0 < x_i^k \le \delta_i^k$ and

$$cx^k - \gamma \sum_{j=1}^{n} \log x_j^k \le ce \tag{20}$$

Let $k \to \infty$ and let $\overline{x}$ be an accumulation point of $\{x^k\}$ such that $\overline{x}_I = 0$ (note that $I \neq \emptyset$ because $\overline{x}_i = 0$), we get the contradiction

$$\infty = \lim_{k\to\infty} -\gamma \sum_{j\in I} \log \overline{x}_j \le ce - c\overline{x} + \gamma \sum_{j \notin I} \log \overline{x}_j < \infty$$

□

Proposition 12 *Let $\gamma > 0$ and assume that the point $e \in S$, where*

$$S := \{x | Ax = 0, \langle e, x\rangle = n\}$$

The problem

$$\min\ cx - \gamma \sum_{j=1}^{n} \log x_j \tag{21}$$

$$\begin{aligned} \textit{subject to}\ \ Ax &= 0 \\ \langle e, x\rangle &= n \\ x &\ge 0 \end{aligned}$$

has a unique solution $x(\gamma) \ge \delta e$ for some $\delta = \delta(\gamma) > 0$.

Proof
The point $e > 0$ is feasible, by Lemma 11, taking $\delta := \min_{1 \le i \le n} \delta_i > 0$, we have

$$x > \delta e \Longleftarrow cx - \gamma \sum_{j=1}^{n} \log x_j \le ce$$

Hence problem (21) is equivalent to the following problem

$$\min\ cx - \gamma \sum_{j=1}^{n} \log x_j \tag{22}$$

$$\begin{aligned} \textit{subject to } Ax &= 0 \\ \langle e, x\rangle &= n \\ x &\geq \delta e \end{aligned}$$

Since this problem has a continuous objective function on a compact set, it must have a solution satisfying its constraint $x \geq \delta e$ which also solves (21). Uniqueness follows from the strict convexity of the objective function of (22) in its feasible region.

Corollary 13 *The same δ works for all γ bigger than the γ given in the above proposition.*

Proof
Implication of Lemma 11 holds for all γ greater than the one given, because $-\sum_{j=1}^{n} \log x_j^k \geq 0$. □

4 Numerical results

The proximal minimization problem that we have at each iteration is

$$\min_x g^k(x) \textit{ subject to } x \in S := \{x | Ax = b, x \geq 0\}$$

where

$$g^k(x) := cx + \frac{\alpha^k}{2}\|x - x^k\|^2 \quad \textit{and } \alpha^k > 0$$

In our implementation, the above problem is reformulated as the following equivalent problem

$$\min_x h^k(x) \textit{ subject to } x \in S := \{x | Ax = b, x \geq 0\}$$

where

$$h^k(x) := \frac{1}{2}\|x - x^k + \epsilon^k c\|^2 \quad \textit{and } \epsilon^k > 0$$

The logarithmic penalty problems associated with $h^k(x)$ is

$$f^k(x) := h^k(x) - \gamma^k \sum_{j=1}^{n} \log x_j$$

The gradient and Hessian of $f^k(x)$ are

$$\begin{aligned}\nabla f^k(x) &= x - x^k + \epsilon^k c - \gamma^k X^{-1} e \\ \nabla^2 f^k(x) &= I + \gamma^k X^{-2}\end{aligned}$$

respectively. Note that the Hessian of $h^k(x)$ is independent of the proximal parameter ϵ^k, while the Hessian of $g^k(x)$ which is dependent on the value of α^k.

The Newton direction p^k at the kth iteration is the solution of the following quadratic minimization problem

$$\min_p \;\; gp + \frac{1}{2} pHp \;\; subject\ to \;\; Ap = 0 \tag{23}$$

where $g = \nabla f^k(x^k)$ and $H = \nabla^2 f^k(x^k)$. The equality constraint $Ap = 0$ is included in the problem to guarantee that each iterates x^k satisfies the equality constraints $Ax = b$. The Karush-Kuhn-Tucker conditions for the above quadratic minimization problem are

$$g + Hp - A^t u = 0 \tag{24}$$

$$Ap = 0 \tag{25}$$

Since H is an invertible diagonal matrix, we can solve equations (24) and (25) by first solving the m by m linear system

$$AH^{-1}A^t u = AH^{-1} g \tag{26}$$

for u, and then compute

$$p = H^{-1}(A^t u - g) \tag{27}$$

In our implementation, the linear system (26) is solved by using the Yale Sparse Matrix Package [Eisenstat, 1977 & 1982].

An initial strictly feasible point for the algorithm is obtained by a Phase I scheme similar to that described in [Gill et al, 1986]. The following are the the starting values we use

$$x_i^0 = \begin{cases} 2\,\|b\| & \text{if } c_i > 0 \\ \|b\| & \text{if } c_i = 0 \\ 0.5\,\|b\| & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots n$ and

$$x_{n+1}^0 = \left\| b - Ax^0 \right\|$$

The initial values of the interior penalty and the proximal point parameters are as follows

$$\begin{aligned} \epsilon^0 &= \begin{cases} \min(1.d+12, 1000x_{n+1}^0) & \text{if } x_{n+1}^0 > 1.d+05 \\ x_{n+1}^0 & \text{otherwise} \end{cases} \\ \gamma^0 &= \epsilon^0 \end{aligned}$$

At each iteration, the value of ϵ is updated

$$\epsilon^{k+1} = \min(1.d+18, 6\epsilon^k)$$

The interior penalty parameter is updated *every 3 iterations* as follows

$$\gamma^{k+1} = \begin{cases} \gamma^k/1.0d1 & \text{if } x_{n+1}^k < 1.d2 \\ \gamma^k/1.2d0 & \text{otherwise} \end{cases}$$

Phase I is terminated as soon as the value of the artificial variable x_{n+1} becomes less than $1.d-05$.

For both Phase I and Phase II, x^{i+1} is computed as follows

$$x^{k+1} = x^k + \min\{1, 0.98\lambda_{max}\}(\overline{x}^k - x^k)$$

where $\overline{x}^k$ is the solution of the quadratic minimization problem

$$\min_x \nabla f^k(x - x^k) + \frac{1}{2}(x - x^k)\nabla^2 f^k(x - x^k)$$

$$\textit{subject to } \; Ax = b$$

and λ_{max} is the maximum value of λ such that

$$x^k + \lambda(\overline{x}^k - x^k) \geq 0$$

At the start of Phase II, the value of interior penalty parameter used is

$$\gamma^0 = \min(\max(\|x\|^2, 1.d4), 1.d12)$$

and the value of the proximal point parameter is

$$\epsilon^0 = 5.d1\gamma^0$$

If $\gamma^k > \gamma_{min} = 1.d02$, then it is updated as follows

$$\gamma^{k+1} = \begin{cases} \gamma^k/1.0d1 & \text{if } |(cx^k - bu^k)/cx^k| < 1.d-4 \\ \gamma^k/1.5d0 & \text{if } |(cx^k - bu^k)/cx^k| < 1.d-3 \\ \gamma^k/1.2d0 & \text{if } |(cx^k - bu^k)/cx^k| < 1.d-1 \end{cases}$$

The value of ϵ is updated

$$\epsilon^{k+1} = \min(1.d+24, 2\epsilon^k)$$

Phase II is terminated if one of the following stopping conditions is satisfied

- $$\left|\frac{cx^{k+1} - cx^k}{cx^k}\right| \leq 3.d-08$$
  and
  $$\left|\frac{cx^{k+1} - bu^{k+1}}{cx^{k+1}}\right| \leq 1.d-07$$
- $$\left|\frac{cx^{k+1} - bu^{k+1}}{cx^{k+1}}\right| \leq 3.d-08$$
- $$\left|\frac{cx^{k+1} - cx^k}{cx^k}\right| \leq 1.d-09$$

When one of the above stopping conditions is satisfied, an iterative scheme to improve the feasibility of the solutions is employed. This is done by guessing at the basis of an optimal solution of the linear program (1). This refining procedure is due to Gay [Gay, 1989] and we shall describe it briefly here.

Let us denote the solution obtained after the termination of IPP by x^*. Since the interior point algorithm will not allow any of the components of x^* to equal zero, we say that x_j^* is in the basis if and only if $x_j^* \geq \delta$ for some small $\delta > 0$. Let the set $\mathcal{B}$ be defined as $\mathcal{B} := \{j | x_j^* \geq \delta\}$. If we assume that the cardinality of $\mathcal{B}$ is equal to m and that the m columns of the constraint matrix A corresponding to the index set $\mathcal{B}$ are linearly independent, the m variables obtained by solving the m by m linear system of equations along with $n - m$ zeros form a basic solution $\overline{x}$ to the linear system $Ax = b$. If $\overline{x} \geq 0$, then it is a basic feasible solution of the linear program (1), which will be taken as the optimal solution of the problem.

The data structures in our implementation of the interior point algorithm have been set up such that the linear system of the form $ADA^t z = r$ can be solved efficiently for different diagonal matrix D. Specifically, the diagonal matrix D in the IPP algorithm is the matrix $(X^i)^2$ where $x^i > 0$ is the current approximate solution. A basic feasible solution x can therefore be efficiently computed from the relation

$$x = ZA^t(AZA^t)^{-1}b$$

where Z is a diagonal matrix such that

$$Z_{ii} = \begin{cases} X_{ii}^2 & \text{if } i \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

We have used the following iterative scheme to refine the solution of the IPP algorithm for all the computational results reported here.

- $it = 0$
- while $it \leq itmax$ do
- - if $x_i \geq 1.d - 08$ then $D_{ii} = x_i^2$
    else $D_{ii} = 1.d - 12$.
  - compute $x = DA^t(ADA^t)^{-1}b$
  - if $\|(-x)_+\| \leq \sigma_1$ and $\|Ax - b\| \,/\, \|b\| \leq \sigma_2$ stop
    else $it = it + 1$
- endwhile

The values of $itmax$, σ_1 and σ_2 that we used are

$$itmax = 8, \quad \sigma_1 = 1.d - 10 \quad \sigma_2 = 5.d - 8$$

We tested the algorithm on 66 linear test problems, 63 of which are from the Netlib collection. The dimension of these 66 problems are given in Tables 1 and 2. In columns 3, 4 and 5 of these tables we list the number of rows (including the objective row), columns and nonzeros of matrix A of the linear program in its original MPS format. The next 3 columns show the size of the linear programs after the data is preprocessed so that these linear programs can be written in standard format (1).

The algorithm was implemented using FORTRAN 77 and run on a DECstation 3100. For comparison purpose, we solved these problems using MINOS 5.3 [B.A. Murtagh & M.A. Saunders, 1983] which is a linear programming package based on the simplex method. MINOS was run using the default parameter setting. The results that we obtained on the 66 test problems are listed in Tables 3 and 4. IPP could not find a feasible point for problem D2q06c and failed in Phase I. On the other 65 test problems, by refining the solutions of IPP we obtained the following results. On 8 problems the relative error of the objective function value is greater than 5.d-10, and on 7 problems the relative primal feasibility is greater than 5.d-10. IPP solved 35 of the 65 problems faster than MINOS 5.3. The total time taken by IPP to solve these 65 problems is 4849 seconds, while the total time for MINOS 5.3 is 6024 seconds giving a total time speedup of 1.24.

The relative error of the objective function value is computed as follows

$$Relative\ \ Error := \left|\frac{cx - cx^*}{cx^*}\right|$$

where cx^* is the optimal objective value reported by MINOS and the primal infeasibility is computed as follows

$$Primal\ Infeasibility = \frac{\|Ax - b\|}{\|b\|}$$

where $X := diag(x)$. In all 65 problems solved, the optimal solutions obtained by our algorithm are such that

$$\|(-x)_+\|_\infty \leq 1.d - 10$$

5 Summary

We have presented an algorithm based on the logarithmic barrier function perturbed by a proximal point term. The results from our implementation showed that the speed of this algorithm is as good as an unperturbed interior point algorithm, e.g. [Setiono, 1990]. However, perturbing the objective function by a proximal point term enabled us to obtain solutions of the linear programs with better primal feasibility.

The feasibility of the solutions obtained by IPP is on the average one order of magnitude better than the feasibility of the solutions obtained from the implementation of an interior point method without perturbation, e.g. [Setiono, 1990]. Similar to other interior point algorithms, we need to solve at each iteration the system of linear equations of the form

$$ADA^t z = r$$

where D is a positive diagonal matrix, and A is the constraint matrix of the linear program. The matrix D in our algorithm is $H = I + \gamma X^{-2}$. For the interior point algorithm without the perturbation on the objective function, the matrix D is $H = \gamma X^{-2}$, which can be very ill-conditioned as some of the components of x will become large, while some other components become close to zero. The presence of the identity matrix in the Hessian is a stabilizing factor that has enabled us to solve the system of linear equations more accurately. This in turn improved the primal feasibility of the solutions of the linear programs.

Our preliminary test results on a set of 66 linear programs indicate that this algorithm could be a viable alternative to the simplex method for solving linear programs.

Pr. No.	Problem Name	Original rows	Original columns	Original nonzeros	Adjusted rows	Adjusted columns	Adjusted nonzeros
1	25fv47	822	1571	11127	820	1876	10705
2	Adlittle	57	97	465	56	138	424
3	Afiro	28	32	88	27	51	102
4	Agg	489	163	2541	488	615	2862
5	Agg2	517	302	4515	516	758	4750
6	Agg3	517	302	4531	516	758	4756
7	Bandm	306	472	2659	305	472	2494
8	Beaconfd	174	262	3476	173	295	3408
9	Blend	75	83	521	74	114	522
10	Bnl1	644	1175	6129	642	1586	5532
11	Bnl2	2325	3489	16124	2324	4486	14996
12	Bore3d	234	315	1525	246	346	1473
13	Brandy	221	249	2150	193	303	2202
14	Capri	272	353	1786	446	641	2230
15	Cre-a	3517	4067	19054	3428	7248	18168
16	Cre-c	3069	3678	16922	2986	6411	15977
17	Czprob	930	3523	14173	1158	3562	10937
18	D2q06c	2172	5167	35674	2171	5831	33081
19	Degen2	445	534	4449	444	757	4201
20	Degen3	1504	1818	26230	1503	2604	25432
21	E226	224	282	2767	223	472	2768
22	Fffff800	525	854	6235	524	1028	6401
23	Finnis	498	614	2714	619	1141	2959
24	Gfrd-pnc	617	1092	3467	876	1420	2965
25	Grow15	301	645	5665	900	1245	6820
26	Grow22	441	946	8318	1320	1826	10012
27	Grow7	141	301	2633	420	581	3172
28	Israel	175	142	2358	174	316	2443
29	Kb2	44	41	291	52	77	331
30	Lotfi	154	308	1086	153	366	1136
31	Pilot.we	723	2789	9218	1256	3384	10255
32	Rabo	391	576	5510	317	560	5201
33	Recipe	92	180	752	211	300	903

Table 1: **LP dimensions**

Pr. No.	Problem Name	Original			Adjusted		
		rows	columns	nonzeros	rows	columns	nonzeros
34	Sc105	106	103	281	105	163	340
35	Sc205	206	203	552	205	317	665
36	Sc50a	51	48	131	50	78	160
37	Sc50b	51	48	119	50	78	148
38	Scagr25	472	500	2029	471	671	1725
39	Scagr7	130	140	553	129	185	465
40	Scfxm1	331	457	2612	330	600	2732
41	Scfxm2	661	914	5229	660	1200	5469
42	Scfxm3	991	1371	7846	990	1800	8206
43	Scorpion	389	358	1708	388	466	1534
44	Scrs8	491	1169	4029	490	1275	3288
45	Scsd1	78	760	3148	77	760	2388
46	Scsd6	148	1350	5666	147	1350	4316
47	Scsd8	398	2750	11334	397	2750	8584
48	Sctap1	301	480	2052	300	660	1872
49	Sctap2	1091	1880	8124	1090	2500	7334
50	Sctap3	1481	2480	10734	1480	3340	9734
51	Share1b	118	225	1182	117	253	1179
52	Share2b	97	79	730	96	162	777
53	Ship04l	403	2118	8450	360	2166	6380
54	Ship04s	403	1458	5810	360	1506	4400
55	Ship08l	779	4283	17085	712	4363	12882
56	Ship08s	779	2387	9501	712	2467	7194
57	Ship12l	1152	5427	21597	1042	5533	16276
58	Ship12s	1152	2763	10941	1042	2869	8284
59	Stocfor1	118	111	474	117	165	501
60	Stocfor2	2158	2031	9492	2157	3045	9357
61	Truss1	201	1602	6586	200	1602	4984
62	Truss2	501	4312	17896	500	4312	13584
63	Truss3	1001	8806	36642	1000	8806	27836
64	Vtp.base	199	203	914	347	477	1331
65	Wood1p	245	2594	70216	244	2595	70216
66	Woodw	1099	8405	37478	1098	8418	37487

Table 2: **LP dimensions (continued)**

Pr. No.	Problem Name	Iterations	Rel. Error	Primal Infeas.	MINOS (seconds)	IPP (seconds)	Time Ratio
1	25fv47	54	1.98E-12	3.67E-08	339.47	138.83	2.45
2	Adlittle	32	4.88E-13	1.98E-14	0.97	0.75	1.29
3	Afiro	24	0.00E+00	2.62E-15	0.31	0.70	0.54
4	Agg	69	1.25E-12	5.62E-13	4.43	45.02	0.10
5	Agg2	47	4.93E-14	3.74E-14	7.71	52.33	0.15
6	Agg3	48	0.00E+00	2.83E-15	7.76	50.45	0.15
7	BandM	44	6.31E-14	9.03E-13	9.64	8.56	1.13
8	Beaconfd	43	0.00E+00	2.85E-13	3.45	9.17	0.38
9	Blend	25	3.25E-14	3.46E-13	1.20	1.43	0.84
10	Bnl1	62	2.37E-07	3.90E-10	42.18	36.00	1.17
11	Bnl2	77	6.59E-09	8.10E-08	609.54	1124.26	0.54
12	Bore3d	68	7.27E-14	1.80E-13	3.01	7.78	0.39
13	BrandY	44	6.59E-14	2.13E-12	6.43	7.86	0.82
14	Capri	44	1.00E-12	1.80E-16	4.46	12.44	0.36
15	Cre-a	68	1.73E-07	8.17E-10	592.03	143.23	4.13
16	Cre-c	69	3.56E-13	3.81E-13	665.41	124.60	5.34
17	CzProb	82	0.00E+00	2.20E-14	75.20	39.79	1.89
18	D2q06c	-	-	-	-	-	-
19	Degen2	40	2.37E-12	6.11E-12	29.19	38.79	0.75
20	Degen3	53	3.04E-14	9.62E-13	720.18	954.51	0.75
21	E226	50	4.53E-12	6.39E-07	7.59	9.11	0.83
22	Fffff800	77	7.75E-09	4.33E-13	27.37	80.88	0.34
23	Finnis	62	5.79E-07	3.23E-11	10.75	16.82	0.64
24	Gfrd-Pnc	48	1.44E-14	1.40E-14	18.17	8.13	2.23
25	Grow15	35	2.99E-10	1.05E-12	18.37	20.15	0.91
26	Grow22	37	1.81E-09	3.24E-12	34.51	32.35	1.07
27	Grow7	33	2.10E-14	1.20E-14	4.98	8.36	0.60
28	Israel	37	1.39E-10	5.52E-14	4.09	32.83	0.12
29	Kb2	39	1.14E-13	5.14E-11	0.67	0.92	0.73
30	Lotfi	35	0.00E+00	1.84E-13	3.83	2.83	1.35
31	Pilot.we	70	1.22E-06	3.47E-13	229.08	85.66	2.67
32	Rabo	32	2.39E-09	6.77E-15	16.52	73.19	0.23
33	Recipe	34	3.75E-14	1.02E-14	1.04	1.99	0.52

Table 3: **Comparison between Minos 5.3 and IPP (DECstation 3100)**

Pr. No.	Problem Name	Itera-tions	Rel. Error	Primal Inf.	MINOS (seconds)	IPP (seconds)	Time Ratio
34	Sc105	36	0.00E+00	2.45E-14	0.88	1.16	0.76
35	Sc205	40	0.00E+00	1.48E-13	1.99	1.95	1.02
36	Sc50a	25	1.54E-14	6.35E-15	0.45	0.42	4.74
37	Sc50b	25	0.00E+00	1.15E-15	0.43	0.44	4.52
38	Scagr25	35	2.03E-13	2.98E-13	8.49	4.01	2.12
39	Scagr7	58	4.27E-14	3.67E-14	1.27	1.48	0.86
40	Scfxm1	48	0.00E+00	1.18E-12	7.75	9.38	0.83
41	Scfxm2	56	3.54E-13	9.10E-12	22.99	22.43	1.02
42	Scfxm3	65	1.82E-14	1.17E-11	45.64	40.34	1.13
43	Scorpion	33	1.60E-13	1.28E-12	4.57	3.61	1.27
44	Scrs8	58	1.84E-12	2.03E-07	18.71	13.39	1.40
45	ScSd1	24	1.04E-13	1.47E-13	4.62	2.88	1.60
46	ScSd6	24	0.00E-12	1.80E-14	17.51	5.05	3.47
47	ScSd8	21	0.00E-14	6.07E-13	97.11	10.21	9.51
48	ScTap1	32	0.00E+00	2.23E-13	4.55	3.92	1.16
49	ScTap2	29	0.00E+00	2.85E-14	32.03	25.00	1.28
50	ScTap3	33	0.00E+00	1.01E-13	61.23	36.99	1.66
51	Share1b	72	2.66E-12	3.28E-10	1.64	3.53	0.46
52	Share2b	23	4.81E-13	1.86E-11	2.90	1.53	1.90
53	Ship04l	43	9.48E-13	3.13E-12	12.00	12.25	0.98
54	Ship04s	39	1.00E-12	3.91E-12	7.49	7.96	0.94
55	Ship08l	44	0.00E+00	1.19E-13	31.08	27.55	1.13
56	Ship08s	41	5.20E-14	6.48E-14	16.45	14.64	1.12
57	Ship12l	44	6.81E-14	5.70E-13	67.34	36.74	1.83
58	Ship12s	43	6.71E-14	6.82E-13	31.66	19.02	1.66
59	Stocfor1	38	2.42E-14	2.54E-12	1.07	1.60	0.67
60	Stocfor2	165	2.57E-14	1.89E-10	182.63	154.17	1.18
61	Truss1	26	1.75E-13	9.00E-13	23.43	9.82	2.39
62	Truss2	28	1.92E-13	5.69E-13	176.07	58.53	3.01
63	Truss3	30	2.18E-14	6.88E-13	930.90	160.87	5.49
64	Vtp.base	50	0.00E+00	5.31E-16	2.25	6.34	0.35
65	Wood1p	52	5.76E-11	1.94E-08	165.14	711.35	0.23
66	Woodw	56	1.89E-11	1.73E-08	542.64	270.54	2.01
-	**TOTAL**	-	-	-	6024.46	4848.82	1.24

Table 4: **Comparison between Minos 5.3 and IPP (DECstation 3100) (continued)**

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