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# PARTIALLY AND TOTALLY ASYNCHRONOUS ALGORITHMS FOR LINEAR COMPLEMENTARITY PROBLEMS

by

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# PARTIALLY AND TOTALLY ASYNCHRONOUS ALGORITHMS FOR LINEAR COMPLEMENTARITY PROBLEMS *

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**Abstract.** A unified treatment is given for partially and totally asynchronous parallel successive overrelaxation (SOR) algorithms for the linear complementarity problem. Convergence conditions are established and compared to previous results. Convergence of the partially asynchronous method for the symmetric linear complementarity problem can be guaranteed if the relaxation factor is sufficiently small. Unlike previous results this relaxation factor interval does not depend explicitly on problem size.

**Key words.** Linear complementarity, asynchronous methods, successive overrelaxation.

## 1. Introduction.

We consider here parallel iterative algorithms for the (not necessarily symmetric) linear complementarity problem

$$Mx + q \geq 0 \quad x \geq 0 \quad x^T(Mx + q) = 0 \tag{1.1}$$

where $M$ is a real $n$-dimensional matrix and $q$ is a given vector in $\mathbb{R}^n$. Various parallel synchronous as well as asynchronous methods based on a successive overrelaxation scheme have been proposed and discussed in [7, 8, 5, 12]. A principal aim of this work is to provide a unified treatment for these parallel schemes and establish their convergence. General asynchronous iterative algorithms for fixed points have been obtained by Bertsekas and Tsitsiklis [2, 3]. However, the emphasis of this paper will be on the linear complementarity problem, as a special but important case of the fixed point problem.

We give now a brief summary of the paper. We begin in Section 2 by defining the general Asynchronous Successive Overrelaxation Algorithm and we consider a parallel implementation of the algorithm. Concepts of totally and partially asynchronous schemes are precisely defined. Section 3 deals with convergence conditions for the Totally Asynchronous SOR Algorithm and its relationship to previously established results. In particular we will show that our sufficient conditions for the convergence of the totally asynchronous algorithm imply the convergence conditions of the Jacobi Overrelaxation Algorithm. The same result will hold also for the partially asynchronous case. In Sections 4 and 5, conditions for the convergence of the Partially Asynchronous SOR Algorithm are obtained. More specifically, Section 5 deals with partially asynchronous algorithms for the symmetric linear complementarity problem for which convergence is established under the assumption that the matrix $M$ is positive semidefinite and under a relaxation factor interval which is an attenuated (0,2) interval. The bound (5.6) improves the bound obtained in [2] which explicitly depends on the size of the problem. Our bound, instead, depends on the maximum number of components updated at the same instant. This quantity is bounded above by the number of different processors used in the computation. We will discuss in more detail the meaning of this new bound at the end of Section 5.

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We briefly describe our notation now. For a vector $x$ in the $n$-dimensional real space $\mathbb{R}^n$, $x_+$ will denote the vector with components $(x_+)_i = \max(x_i, 0)$, $i = 1, \ldots, n$. The scalar product of two vectors $x$ and $y$ in $\mathbb{R}^n$ will be denoted by $x^T y$. For $1 \leq p \leq +\infty$ the $p$-norm of a vector $x$ in $\mathbb{R}^n$ will be denoted by $\|x\|_p$. For the 2-norm the subscript will be dropped. For $A$ in $\mathbb{R}^{mxn}$, $A_i^T$ will denote the $i$th row of A. The spectral radius of a matrix will be indicated by $\rho(A)$. If $A$ is a square matrix, $\|A\|_p$ will denote the matrix norm of $A$ induced by the vector norm $\|.\|_p$. For a vector $x \in \mathbb{R}^n$, $|x|$ will denote the vector with components $|x_i|$, $i = 1, \ldots, n$, and similarly for a real $mxn$ matrix $A$, $|A|$ will denote the matrix with absolute value elements $|A_{ij}|$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Here and throughout the symbols $:=$ and $=:$ denote definition of the term on the left and right sides of each symbol respectively.

**2. Asynchronous Successive Overrelaxation Algorithm.** We begin by making the observation that (see [6]) the linear complementarity problem (1.1) is equivalent to the fixed point problem:

$$x = (x - \omega E(Mx + q))_+ \tag{2.1}$$

for positive $\omega$ and an arbitrary positive diagonal matrix $E$. We will use this as the basis for the following Asynchronous SOR Algorithm.

**Asynchronous SOR Algorithm**

For each $i \in \{1, \ldots, n\}$ let $\mathcal{T}_i$ be the instants of time at which the $i^{th}$ component of $x$ is updated and

$$x_i^{t+1} := \begin{cases} (1-\lambda)x_i^{\tau_{ii}(t)} + \lambda \left( x_i^{\tau_{ii}(t)} - \omega E_{ii} \left( \sum_{j=1}^{n} M_{ij} x_j^{\tau_{ji}(t)} + q_i \right) \right)_+ & \text{if } t \in \mathcal{T}_i \\ x_i^t & \text{otherwise.} \end{cases}$$

where $\tau_{ji}(t)$ are instants of times, $\lambda \in (0, 1]$ and $\omega$ and $E_{ii}$ are positive scalars.

**Remark** The previous algorithm can be realized by the following parallel implementation. Assume that p processors are available with each processor responsible for updating a subset of the components of $x$. At time $t$ one or more components of $x$ are updated using (eventually outdated) values of $x_j$, $j = 1, \ldots, n$ computed at some time in the past. The following natural assumptions we will be used to distinguish different degree of asynchronism:

**A1** $0 \leq \tau_{ji}(t) \leq t \;\; \forall t \in \mathcal{T}_i$, $\forall i$ and $\forall j = 1, \ldots, n$.
**A2** The sets $\mathcal{T}_i$ are infinite.
**A3** If $\{t_k\} \subset \mathcal{T}_i$ and $t_k \mapsto +\infty$ then $\lim_k \tau_{ji}(t_k) = +\infty$ for every $j = 1, \ldots, n$.

Following [2], an asynchronous algorithm satisfying **A1**, **A2** and **A3** will be referred as a **totally** asynchronous algorithm. A **partially** asynchronous algorithm satisfies **A1** as well as the following conditions involving a time-frame limit $B$:

**A4** For every $i \in \{1, \ldots, n\}$ and $t \geq 0$

$$\mathcal{T}_i \cap \{t, t+1, \ldots, t+B-1\} \neq \emptyset.$$

**A5** $t - B \leq \tau_{ji}(t) \leq t \;\; \forall t \in \mathcal{T}_i, \forall i$ and $\forall j = 1, \ldots, n$.
**A6** $\tau_{ii}(t) = t \;\; \forall t \in \mathcal{T}_i$ and $\forall i = 1, \ldots, n$.

As noted in [3], asynchronous algorithms can converge under any of the following degree of asynchronism:

1. total asynchronism;
2. partial asynchronism, for every value $B$ (but not under total asynchronism);
3. partial asynchronism, but only if $B$ is sufficiently small.

In the next sections we will derive conditions that guarantee convergence for the asynchronous SOR in each of these cases. In order to do that, we need to introduce the following operators that will be needed in the sequel:

$$h(x) := (x - \omega E(Mx + q))_+ \tag{2.2}$$

$$f(x) := (1 - \lambda)x + \lambda h(x) \tag{2.3}$$

**3. The Totally Asynchronous Algorithm.** The main advantage of totally asynchronous algorithms is that the processors need not have access to a global clock and they can continue updating the components assigned to them without waiting for messages from the other processors. In addition, each processor need not immediately communicate or receive the newly computed values, but is only required to transmit them once in a while. Convergence of general asynchronous methods has been established by various authors starting with Chazan and Miranker [4] and Baudet [1]. In [5] convergence has been established for a totally asynchronous algorithm for the linear complementarity problem. In particular it has been proved in [5] that $h$ is a contracting operator if the spectral radius of $(1-\omega)I + \omega D^{-1}|L+U|$ is less than 1, where $D$ is the diagonal part of $M$ ( assumed to be positive) and $L$ (resp. $U$) is the strict lower (resp. upper) triangular part of $M$. Therefore, under these assumptions the convergence of the Totally Asynchronous SOR Algorithm was proved with $\lambda = 1$. Moreover, the convergence proof can be easily modified to establish convergence (under the same assumptions) for all values of $\lambda$ in the interval $(0, 1]$.

We now discuss in more detail the convergence conditions for the Totally Asynchronous SOR Algorithm. Using arguments similar to those of [5, Lemma 2.3] it is easy to show that

$$|h(x) - h(y)| \leq |I - \omega E(L + D + U)| \;\; |x - y|$$

Now, let

$$A_\omega = |I - \omega E(L + D + U)|$$

and assume that

$$(I - \omega ED)_{ii} > 0 \quad \forall i = 1, \ldots, n$$

then $\rho(A_\omega) < 1$ if and only if $D - |L+U|$ is an M-matrix (that is it has a nonnegative inverse). In fact $\rho(A_\omega) < 1$ if and only if [14, Theorem 3.8] $I - A_\omega$ is a M-matrix. But

$$I - A_\omega = I - |I - \omega E(L + D + U)| = I - |(I - \omega ED) - \omega E(L + U)|$$

$$= \omega ED - \omega E|L + U| = \omega E(D - |L + U|)$$

where the third equality follows from the fact that the diagonal of $A_\omega$ is strictly positive. Hence $D - |L + U|$ must be an M-matrix. From the above discussion we can conclude that the convergence of the Totally Asynchronous SOR Algorithm can be guaranteed if the following conditions are satisfied:

**C1** $C(M) := D - |L + U|$ is an M-matrix.
**C2** $(I - \omega ED)_{ii} > 0 \; \forall i = 1, \ldots, n$

The following theorem summarizes these results.

THEOREM 3.1. *Assume that conditions* **C1** *and* **C2** *are satisfied, where $D$ is the diagonal part of $M$, and $L$ (resp. $U$) is the strict lower (resp. upper) triangular part of $M$. Then the sequence $\{x^t\}$ generated by the Totally Asynchronous SOR Algorithm satisfying the assumptions* **A1-A3** *converges to the unique solution of the linear complementarity problem (1.1).* □

Now, since

$$2(\omega E)^{-1} - D - |L + U| = 2\left((\omega E)^{-1} - D\right) + D - |L + U|$$

$$= 2(\omega E)^{-1}(I - \omega ED) + D - |L + U|$$

the matrix $2(\omega E)^{-1} - D - |L + U|$ is the sum of an M-matrix and a positive diagonal matrix and therefore [9, p. 109] is an M-matrix. This implies that the matrix $2(\omega E)^{-1} - M$ is positive definite hence each accumulation point of the Projected Jacobi Overrelaxation Algorithm solves the linear complementarity problem(see [6]). This is to be expected, since the Projected Jacobi Overrelaxation Algorithm is a special case of the Asynchronous SOR algorithm, where

$$\mathcal{T}_i = \mathbb{N} \quad \text{and} \quad \tau_{ji}(t) = t \;\; \forall i \text{ and } j = 1, \ldots, n.$$

Note that the conditions stated in the above theorem are much stronger than the conditions required for the convergence of the Projected Jacobi Algorithm, namely the matrix $2(\omega E)^{-1} - M$ being positive definite [6].

**4. The Partially Asynchronous Algorithm.** General asynchronous methods are discussed in [10] and convergence conditions for asynchronous methods for quadratic problems are considered. In this section we will concentrate on Partially Asynchronous SOR Algorithm for not necessarily symmetric linear complementarity problems and we will derive conditions that guarantee convergence of the algorithm for every value of the constant $B$. We will show that the following conditions are sufficient for convergence:

**C3** $\|I - \omega EM\|_\infty \leq 1$.
**C4** $M$ irreducible.
**C5** The solution set $X^*$ of the linear complementarity problem (1.1) is not empty.

Condition similar to **C3-C5** are required [2] for the convergence of the Partially Asynchronous Algorithm for solving system of linear equations.

In order to establish our result, the following lemma that is a special case of [10, Proposition 3.1] is needed.

LEMMA 4.1. *If conditions* **C3-C5** *are satisfied then the function* $h$ *is pseudo–nonexpansive, that is:*

$$\|h(x) - x^*\|_\infty \leq \|x - x^*\|_\infty \quad \forall x \in \mathbb{R}^n \text{ and } \forall x^* \in X^*.$$

*and for every* $x \in \mathbb{R}^n$ *and* $x^* \in X^*$ *such that*

$$\|x - x^*\|_\infty = \min_{y \in X^*} \|x - y\|_\infty > 0$$

*there exists* $i \in I(x, x^*) := \{i : |x_i - x_i^*| = \|x - x^*\|_\infty\}$ *such that*

$$h_i(y) \neq y_i \;\; \forall y \in U(x, x^*)$$

*where*

$$U(x, x^*) := \left\{ y \in \mathbb{R}^n : \begin{array}{l} y_i = x_i \text{ for } i \in I(x, x^*) \\ |y_i - x_i^*| < \|x - x^*\|_\infty \text{ for } i \notin I(x, x^*) \end{array} \right\}$$

□

We are now ready to establish our convergence results for the Partially Asynchronous SOR Algorithm.

THEOREM 4.2. *Let conditions* **C3-C5** *hold. The sequence* $\{x^t\}$ *generated by the Partially Asynchronous SOR Algorithm satisfying assumptions* **A1** *and* **A4-A6** *with* $\lambda \in (0, 1)$ *converges to some element in* $X^*$.

*Proof.* From [2, Proposition 2.2, p. 493] we have that the mapping $f$ defined in (2.3) satisfies the following condition (for $\lambda \in (0, 1)$):

**C6** If $x_i \neq f_i(x)$ and $x^* \in X^*$ then $|f_i(x) - x_i^*| < \|x - x^*\|_\infty$

Now, since

$$\|I - \omega EM\|_\infty \leq 1$$

we have that $\forall i = 1, \ldots, n$

$$|1 - \omega E_{ii} M_{ii}| + \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{ij}| \leq 1$$

The previous inequality and the irreducibility of the matrix $M$ imply that $M$ is diagonally dominant with positive diagonal elements. Therefore [11] $M$ is column sufficient and hence [13] the solution set of the linear complementarity problem(1.1) is convex. Note that the convexity of $X^*$ can also be inferred using Lemma 2.4 in [10].

The result now follows from [2, Proposition 2.3, p.495]. □

The following example shows a case for which a partially asynchronous but not a totally asynchronous SOR algorithm converges. Nonconvergence of the totally asynchronous algorithm is caused by the fact that the two processors do not communicate often enough.

Consider the linear complementarity problem (1.1) where

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad q = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (n = 2)$$

The updating formula (when $\lambda = 1$) is:

$$\begin{cases} x_1 := (x_1 - \omega(x_1 - x_2))_+ = ((1-\omega)x_1 + \omega x_2))_+ \\ x_2 := (x_2 - \omega(-x_1 + x_2))_+ = (\omega x_1 + (1-\omega)x_2))_+ \end{cases}$$

We consider the following scenario (see [2, p.484-485] for a similar example): two processors $P_1$ and $P_2$ are available and processor $P_i$ updates component $x_i$ at each instant, $i = 1, 2$. At instants $t_1, t_2, \ldots$ each processor broadcasts the newly computed value and this information is used by the other processor. Starting with positive values for $x_1$ and $x_2$ and choosing $\omega \in (0, 1]$, the updating formula becomes:

$$\begin{cases} x_1^{t+1} := (1-\omega)x_1^t + \omega x_2^{t_k} \\ x_2^{t+1} := \omega x_1^{t_k} + (1-\omega)x_2^t \end{cases}$$

for $t \in [t_k, t_{k+1}]$. The following iterative scheme has been analyzed in [2] for the case $\omega = \frac{1}{2}$ and it has been shown that convergence cannot be guaranteed in the totally asynchronous case but it is obtained if $t_{k+1} - t_k \leq B$ for any fixed $B$.

The next example shows the importance of the parameter $\lambda < 1$. For the same problem, consider the following iterative scheme:

$$\begin{cases} x_1 := (1-\lambda)x_1 + \lambda\,(x_1 - \omega(x_1 - x_2))_+ \\ x_2 := (1-\lambda)x_2 + \lambda\,(x_2 - \omega(-x_1 + x_2))_+ \end{cases}$$

where $\lambda \in (0, 1]$ and $\omega \in (0, 1]$. Starting with positive values for $x_1$ and $x_2$, the updating formula becomes:

$$\begin{cases} x_1 := (1-\lambda\omega)x_1 + \lambda\omega x_2 \\ x_1 := \lambda\omega x_1 + (1-\lambda\omega)x_2 \end{cases}$$

In this case (using $E = D^{-1} = I$) we have that

$$I - \omega EM = \begin{bmatrix} 1-\omega & \omega \\ -\omega & 1-\omega \end{bmatrix}$$

and for $\omega = 1$, $\|I - \omega EM\|_\infty = 1$. But choosing $\omega = 1$ and $\lambda = 1$ we have

$$x_1 := (x_2)_+ \quad x_2 := (x_1)_+$$

and if $x_1^0 \neq x_2^0$ the Projected Jacobi Overrelaxation Algorithm does not converge to a solution of the linear complementarity problem. In order to guarantee convergence (when $\omega = 1$) a value of $\lambda$ less than 1 must be used.

We now investigate the relationship between the sufficient conditions for convergence of the partially asynchronous method and the sufficient conditions for the totally asynchronous and the Projected Jacobi Overrelaxation algorithms. Let

$$A_\omega = I - \omega E M \tag{4.1}$$

Condition **C3** requires that $\|A_\omega\|_\infty \leq 1$ and hence $\|(A_\omega)_i\|_1 \leq 1 \; i = 1, \ldots, n$. If for some $i$ we have that $\|(A_\omega)_i\|_1 < 1$ then [9, p. 109, 6.2.16] $\rho(A_\omega) < 1$ and the convergence proof follows from that for the Totally Asynchronous SOR algorithm (in fact, in this case the conditions **C1** and **C2** are satisfied). Hence the interesting case is

$$\|(A_\omega)_i\|_1 = 1 \;\; \forall i = 1, \ldots, n$$

i.e.

$$|1 - \omega E_{ii} M_{ii}| + \omega E_{ii} \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{ij}| = 1 \tag{4.2}$$

In this case

$$\frac{2}{\omega E_{ii}} - M_{ii} \geq 0$$

(in fact if $\omega E_{ii} M_{ii} > 2$ the condition (4.2) cannot be satisfied) and for $\lambda \in (0, 1)$

$$\frac{2}{\lambda \omega E_{ii}} - M_{ii} > 0.$$

Moreover the matrix $2(\lambda \omega E)^{-1} - M$ is strictly diagonally dominant. In fact either $1 - \omega E_{ii} M_{ii} \geq 0$ or $1 - \omega E_{ii} M_{ii} < 0$. In the first case, from (4.2) we have that

$$-\omega E_{ii} M_{ii} + \omega E_{ii} \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{ij}| = 0$$

and hence

$$M_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{ij}|$$

and

$$\frac{2}{\lambda \omega E_{ii}} - M_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{ij}| = 2 \left( \frac{1}{\lambda \omega E_{ii}} - M_{ii} \right) > 0.$$

In the second case, again from (4.2), we have that

$$-1 + \omega E_{ii} M_{ii} + \omega E_{ii} \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{ij}| = 1$$

which implies that

$$\frac{2}{\omega E_{ii}} - M_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |M_{ij}| = 0.$$

Hence in this case, $\forall\lambda \in (0,1)$

$$\frac{2}{\lambda\omega E_{ii}} - M_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |M_{ij}| > 0$$

and hence

$$\frac{2}{\lambda\omega E_{ii}} - M_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{n} |M_{ij}|.$$

Thus, the conditions of Theorem 4.2 imply that the matrix $2(\lambda\omega E)^{-1} - M$ is positive definite [6, Remark 3.1] and hence any accumulation point of the the Projected Jacobi Overrelaxation Algorithm solves the linear complementarity problem (1.1)

**5. Partially Asynchronous Algorithm for symmetric linear complementarity problems.** In this section we concentrate on the important case of partially asynchronous methods for symmetric linear complementarity problems. In this case conditions (1.1) are necessary optimality conditions for

$$\underset{x\geq 0}{\text{minimize}}\ F(x) := \underset{x\geq 0}{\text{minimize}}\ \frac{1}{2}x^T M x + q^T x \tag{5.1}$$

Similar results to those obtained here for gradient–like optimization algorithms and more general Lipschitzian functions $F(x)$ can be found in [2, §7.5].

We need the following lemmas:

LEMMA 5.1. *For each $t$, let $s^t := x^{t+1} - x^t$, where $\{x^t\}$ is the sequence constructed by the Partially Asynchronous SOR algorithm with $\lambda = 1$. Then*

$$s_i^t \left(M_i^T x^{i,t} + q_i\right) \leq -\frac{1}{\omega E_{ii}} \left(s_i^t\right)^2 \tag{5.2}$$

*for all $i = 1, \ldots, n$ and for all $t$, where the vector $x^{i,t}$ is defined as*

$$x^{i,t} := \left[x_1^{\tau_{1i}(t)}, x_2^{\tau_{2i}(t)}, \ldots, x_n^{\tau_{ni}(t)}\right]^T \tag{5.3}$$

*Proof.* If $t \notin \mathcal{T}_i$ then $s_i^t = 0$ and hence (5.2) is satisfied. If, instead, $t \in \mathcal{T}_i$ then

$$s_i^t = \left(x_i^t - \omega E_{ii}\left(M_i^T x^{i,t} + q_i\right)\right)_+ - x_i^t$$

and from the definition of projection:

$$\left(\left(x_i^t - \omega E_{ii}\left(M_i^T x^{i,t} + q_i\right)\right)_+ - x_i^t\right)\left(\left(x_i^t - \omega E_{ii}\left(M_i^T x^{i,t} + q_i\right)\right)_+ - x_i^t + \omega E_{ii}\left(M_i^T x^{i,t} + q_i\right)\right) \leq 0.$$

Hence

$$\omega E_{ii} s_i^t \left( M_i^T x^{i,t} + q_i \right) \leq - \left( s_i^t \right)^2$$

□

LEMMA 5.2. *Under the same assumptions of Lemma 5.1 we have:*

$$\left\| x^t - x^{i,t} \right\| \leq \sum_{k=t-B}^{t-1} \left\| s^k \right\| \tag{5.4}$$

*for all* $i = 1, \ldots, n$ *and for all* $t$, *where* $B$ *is the constant defined in* **A4-A5**

*Proof.* The proof of this lemma follows from simple modification of [2, Proposition 5.1, p. 529] □

LEMMA 5.3. *Under the same assumptions of Lemma 5.1:*

$$\sum_{i=1}^{n} s_i^t M_i^T \left( x^t - x^{i,t} \right) \leq \frac{1}{2} \rho(M) \left( B\sqrt{\mathcal{L}} \sum_{i=1}^{n} \left( s_i^t \right)^2 + \sqrt{\mathcal{L}} \sum_{k=t-B}^{t-1} \left\| s^k \right\|^2 \right) \tag{5.5}$$

*where* $\mathcal{L}$ *is the maximum number of components of* $x$ *updated at the same instant.*

*Proof.* Let $I(t) := \{ i \text{ such that } s_i^t \neq 0 \}$; we have:

$$\sum_{i=1}^{n} s_i^t M_i^T \left( x^t - x^{i,t} \right) \leq \rho(M) \sum_{i=1}^{n} | s_i^t | \left\| x^t - x^{i,t} \right\|$$

$$\leq \rho(M) \sum_{i=1}^{n} | s_i^t | \sum_{k=t-B}^{t-1} \left\| s^k \right\| \qquad \text{( from (5.4) )}$$

$$= \rho(M) \sum_{k=t-B}^{t-1} \sum_{i \in I(t)} | s_i^t | \left\| s^k \right\| = \rho(M) \sum_{k=t-B}^{t-1} \sum_{i \in I(t)} (\mathcal{L}^{\frac{1}{4}} | s_i^t |)(\mathcal{L}^{-\frac{1}{4}} \left\| s^k \right\|)$$

$$\leq \frac{1}{2} \rho(M) \sum_{k=t-B}^{t-1} \sum_{i \in I(t)} \left( \sqrt{\mathcal{L}} (s_i^t)^2 + \left\| s^k \right\|^2 \frac{1}{\sqrt{\mathcal{L}}} \right) = \frac{1}{2} \rho(M) \left( B\sqrt{\mathcal{L}} \sum_{i=1}^{n} (s_i^t)^2 + \sum_{k=t-B}^{t-1} \sum_{i \in I(t)} \left\| s^k \right\|^2 \frac{1}{\sqrt{\mathcal{L}}} \right)$$

$$\leq \frac{1}{2} \rho(M) \left( B\sqrt{\mathcal{L}} \sum_{i=1}^{n} (s_i^t)^2 + \sqrt{\mathcal{L}} \sum_{k=t-B}^{t-1} \left\| s^k \right\|^2 \right)$$

□

By using the above lemmas, we obtain our convergence theorem for the symmetric linear complementarity problem.

THEOREM 5.4. *Let* $M$ *be symmetric and* $F(x)$ *be bounded below on* $\mathbb{R}_+^n$ *and suppose that*

$$0 < \omega E_{ii} < \frac{2}{(2B\sqrt{\mathcal{L}} + 1)\rho(M)} \quad \textit{for every } i = 1, 2, \ldots, n. \tag{5.6}$$

*Any accumulation point of the sequence* $\{x^t\}$ *generated by the Asynchronous SOR Algorithm satisfying the assumptions* **A1** *and* **A4-A6** *solves the linear complementarity problem (1.1).*

*Proof.* We first show that

$$F(x^{t+1}) \leq F(x^0) + \sum_{i=1}^{n} \left( \frac{(2B\sqrt{\mathcal{L}}+1)\rho(M)}{2} - \frac{1}{\omega E_{ii}} \right) \sum_{k=0}^{t} \left(s_i^k\right)^2 \qquad (5.7)$$

We will then show that (5.7) and (5.6) imply that every accumulation point of the sequence $\{x^t\}$ solves the linear complementarity problem (1.1) provided that $F(x)$ is bounded below. We have

$$F(x^{t+1}) = F(x^t + s^t) = F(x^t) + \sum_{i=1}^{n} s_i^t \left(M_i^T x^t + q_i\right) + \frac{1}{2} s^{tT} M s^t$$

$$= F(x^t) + \sum_{i=1}^{n} s_i^t \left(M_i^T x^{i,t} + q_i\right) + \sum_{i=1}^{n} s_i^t M_i^T \left(x^t - x^{i,t}\right) + \frac{1}{2} s^{tT} M s^t$$

$$\leq F(x^t) + \sum_{i=1}^{n} \left( \frac{\rho(M)}{2} - \frac{1}{\omega E_{ii}} \right) \left(s_i^t\right)^2 + \sum_{i=1}^{n} s_i^t M_i^T \left(x^t - x^{i,t}\right) \text{ (from (5.2))}$$

$$\leq F(x^t) + \sum_{i=1}^{n} \left( \frac{\rho(M)}{2} - \frac{1}{\omega E_{ii}} \right) \left(s_i^t\right)^2 + \frac{1}{2}\rho(M) \left( B\sqrt{\mathcal{L}} \sum_{i=1}^{n} \left(s_i^t\right)^2 + \sqrt{\mathcal{L}} \sum_{k=t-B}^{t-1} \left\|s^k\right\|^2 \right) \text{ (from (5.5))}$$

$$= F(x^t) + \sum_{i=1}^{n} \left( \frac{(B\sqrt{\mathcal{L}}+1)\rho(M)}{2} - \frac{1}{\omega E_{ii}} \right) \left(s_i^t\right)^2 + \frac{\sqrt{\mathcal{L}}\rho(M)}{2} \sum_{k=t-B}^{t-1} \left\|s^k\right\|^2$$

By adding the inequalities from 0 to $t$, we obtain:

$$F(x^{t+1})$$

$$\leq F(x^0) + \sum_{i=1}^{n} \left( \frac{(B\sqrt{\mathcal{L}}+1)\rho(M)}{2} - \frac{1}{\omega E_{ii}} \right) \sum_{k=0}^{t} \left(s_i^k\right)^2 + \frac{B\sqrt{\mathcal{L}}\rho(M)}{2} \sum_{k=0}^{t} \sum_{i=1}^{n} \left(s_i^k\right)^2$$

$$= F(x^0) + \sum_{i=1}^{n} \left( \frac{(2B\sqrt{\mathcal{L}}+1)\rho(M)}{2} - \frac{1}{\omega E_{ii}} \right) \sum_{k=0}^{t} \left(s_i^k\right)^2$$

Note that condition (5.6) implies that

$$\frac{(2B\sqrt{\mathcal{L}}+1)\rho(M)}{2} - \frac{1}{\omega E_{ii}} < 0$$

for all $i = 1, \ldots, n$. The remaining of the proof follows by simple modifications of the proofs of Proposition 5.3 (p. 535) and Proposition 5.1 (p. 529-531) of [2]. Since $F$ is bounded below on $\mathbb{R}^n_+$, then $\forall i = 1, \ldots, n$

$$\sum_{k=0}^{\infty} \left(s_i^k\right)^2 < \infty$$

and hence it follows that

$$\lim_k \mid s_i^k \mid = 0 \;\; \forall i = 1, \ldots, n$$

Since $x^{t+1} = x^t + s^t$, we obtain that

$$\lim_t \left\| x^{t+1} - x^t \right\| = 0.$$

But for $t \in \mathcal{T}_i$ and for all $j$

$$\mid x_j^{i,t} - x_j^t \mid \leq \sum_{k=t-B}^{t} \mid s_j^t \mid,$$

and thus

$$0 \leq \lim_t \left\| x^{i,t} - x^t \right\| \leq \lim_t \sum_{k=t-B}^{t} \left\| s^k \right\| = 0.$$

Now let $x^*$ be an accumulation point of $\{x^t\}$ and

$$\lim_l x^{t_l} = x^*.$$

For each $t_l$ let $r_l$ be such that

$$\mid t_l - r_l \mid \leq B \quad r_l \in \mathcal{T}_i$$

(the existence of the sequence $\{r_l\}$ is guaranteed by the assumptions **A4** and **A5**). Since

$$\lim_l \left\| x^{t_l} - x^{r_l} \right\| = 0 \;\text{ and }\; \lim_l \left\| x^{i,r_l} - x^{r_l} \right\| = 0$$

we obtain

$$\lim_l x^{r_l} = x^* = \lim_l x^{i,r_l} = \lim_l x^{r_l+1}$$

But

$$x_i^{r_l+1} = \left( x_i^{r_l} - \omega E_{ii} \left( M_i^T x^{i,r_l} + q_i \right) \right)_+$$

and taking the limit on both sides we have

$$x_i^* = \left( x_i^* - \omega E_{ii} \left( M_i^T x^* + q_i \right) \right)_+ .$$

We can repeat that for each i and therefore

$$x^* = (x^* - \omega E(Mx^* + q))_+ .$$

□

We comment now on the meaning of the bound (5.6). For a given fixed problem, the quantity $\mathcal{L}$ is an upper bound on the number of components updated at each instant by the Partially Asynchronous SOR Algorithm. We notice that Assumption **A4** holds only if $B\mathcal{L} \geq n$ and hence this bound still depends (implicitly) on the size of the problem. On the other hand, the number of processing elements used in the computation also provides an upper bound for $\mathcal{L}$. Therefore, this bound reflects the fact that the time-frame $B$ for the asynchronous algorithm, the size of the problem and the number of processing elements used in the computation are not independent quantities but are related to each other. However, if the number of processing elements in the computation is much smaller than $n$ but sufficiently large to allow updating of all components every $B$ instants, then (5.6) provides a much tighter bound than [2, (5.16) p. 553].

As a final remark, we note that when $B = 0$ ( as for the Projected Jacobi Overrelaxation), the condition (5.6) becomes

$$0 < \omega E_{ii} < \frac{2}{\rho(M)}$$

which implies that the matrix $2(\omega E)^{-1} - M$ is positive definite and hence the sufficient conditions for the Projected Jacobi Overrelaxation Algorithm are satisfied also in this case.

**6. Conclusion.** We have studied sufficient conditions for convergence of totally and partially asynchronous SOR algorithms. For symmetric positive semidefinite linear complementarity problems we have shown convergence of a partially asynchronous SOR algorithm under a relaxation factor $\omega$ interval $(0, \omega)$. The value $\omega$ is inversely proportional to the measure of asynchronism $B$ and to the maximum number of components updated at the same instant.

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