ON THE MULTIGRID F-CYCLE

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Abstract. In a recent paper [4], a bound was derived on the convergence of the multigrid V-cycle for the case when the solution is in the Sobolev space $H^{1+\alpha}$ but not in $H^{1+\alpha'}$, $\alpha' > \alpha$, showing that the convergence factor approaches one only as $1 - O(k^{(\alpha-1)/\alpha})$ for a large number of levels k. We now extend the technique to obtain the asymptotically better bound $1 - O(k^{-(1-\alpha)^2/\alpha})$ on the multigrid F-cycle. We also show that in many cases, for practical values of k, one gets the same bound for the F-cycle as for the V-cycle with $\alpha = 1$.

1. Introduction. Multigrid methods are by now well established as fast and general solution methods for systems arising by discretization of elliptic equations. In this paper, we are concerned with theoretical bounds on convergence of multigrid methods. For more information and additional references, see the monographs [8, 5].

The basic multigrid schemes are the V-cycle and W-cycle, obtained by solving the coarse grid problem by one or two recursive applications of the multigrid method itself, respectively.

Convergence bounds for the W-cycle bounded away from one independently on the number of levels can be obtained under rather general circumstances, see [5]. The first convergence proof for the V-cycle was obtained in [2]. Like all later V-cycle convergence proofs, see [1, 6, 9, 10], it required a variational setting of the method (as in this paper) and full elliptic regularity, which means for second order systems that the solution is in the Sobolev space $H^2(\Omega)$ for the right-hand side in $L^2(\Omega)$. If, in general, the solution is only in $H^{1+\alpha}(\Omega)$, $\alpha < 1$, all known V-cycle proofs give bounds which approach one as the number of levels grows [7, 8]. This effect has been also observed in computational experiments. Nevertheless, the V-cycle is often used in practice because of its lower computational cost. An intermediate scheme, the F-cycle has just slightly higher computational cost than the V-cycle and much better convergence properties, and it is the method of choice in many practical situations [11].

In [4], it was shown that for $\alpha < 1$, the convergence factor of the V-cycle can approach one only very slowly. The same result was derived independently in [3]. Since then, our attention has been attracted to the F-cycle, and the question arose: is the F-cycle bound bounded away from one, as in the case of the W-cycle, or goes to one, as in the case of the V-cycle? In this paper, we answer this question by showing that the bound derived using the techniques of [7, 8] does converge to one but even more slowly than for the V-cycle. We also show that the F-cycle bound often remains constant for a finite number of levels.

The paper is organized as follows: In Section 2, we formulate the cycling schemes and recall the basic results of [7, 8]. Section 3 is concerned with the asymptotics of V-cycle and F-cycle bounds. Section 4 gives bounds for a small number of levels and explicit evaluations of the V, F, and W-cycle bounds in several representative situations.

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Our result can be formulated that the asymptotic bounds ρ_k obtained in Section 3 for the energy convergence factor squared ρ_k of the F-cycle with k levels satisfy inequality of the form

$$\rho_k \leq 1 - C(k+k_0)^{-\gamma}, \quad \gamma > 0.$$

We would like to remark that attempts to "identify" the quantities C, k_0, and γ from computational results so that

$$\bar{\rho}_k \approx 1 - C(k+k_0)^{-\gamma}, \tag{1.1}$$

where $\bar{\rho}_k$ are observed convergence factors squared, are somehow futile for the following reason: (1.1) can be written as

$$\log(1-\rho_k) \approx \log C + \gamma \log(k+k_0).$$

Then it is evident that the problem of determining C, γ and k_0 simultaneously is very poorly determined when one has the data for just few consecutive values of k, because the term $\gamma \log(k+k_0)$ behaves approximately like the same linear function over a fixed range of k for vastly different pairs (γ, k_0). We therefore refrain from such attempts.

2. Preliminaries. We first state the multigrid algorithm and convergence bounds, following [7, 8]. Let

$$V_k = \mathbf{R}^{n_k}, \quad k = 1, 2, \ldots,$$

and

$$I_{k-1}^k : V_{k-1} \to V_k$$

be full-rank linear operators (matrices), and

$$A_k : V_k \to V_k$$

be symmetric, positive definite linear operators such that the variational conditions

$$(I_{k-1}^k)^T A_k I_{k-1}^k = A_{k-1}$$

hold. The operators $(I_{k-1}^k)^T : V_k \to V_{k-1}$ are denoted by I_k^{k-1}. We are interested in the solution of the system of n_k linear equations

$$A_k u_k = f_k. \tag{2.1}$$

Define the energy norm

$$||u_k|| = \sqrt{u_k^T A_k u_k}, \qquad u_k \in V_k.$$

Consider the following generic two-level algorithm for the solution of the problem (2.1), starting from an initial approximation $u_k \in V_k$. Let $u_k \leftarrow u_k - B_{k,1}^{-1}(A_k u_k - f_k)$ and $u_k \leftarrow u_k - B_{k,2}^{-1}(A_k u_k - f_k)$ be linear stationary consistent iterative methods for the solution of (2.1), called smoothers.

ALGORITHM 2.1.

a) If $k = 1$, solve (2.1) by a direct method.

b) If $k > 1$, then set $u_k^1 = u_k$ and do the following:

Step 1 Let $u_k^2 = u_k^1 - B_{k,1}^{-1}(A_k u_k^1 - f_k)$ *(pre-smoothing).*

Step 2 Let $u_{k-1}^1 = 0, f_{k-1} = I_k^{k-1}(f_k - A_k u_k^2)$ *and use some iterative method with initial approximation* $u_{k-1}^1 = 0$ *to solve approximately the problem*

$$A_{k-1}u_{k-1} = f_{k-1}. \tag{2.2}$$

Let v_{k-1} *be the result.*

Step 3 Let $u_k^3 = u_k^2 + I_{k-1}^k v_{k-1}$.

Step 4 Let $v_k = u_k^3 - B_{k,2}^{-1}(A_k u_k^3 - f_k)$ *(post-smoothing).*

Define the *convergence factor* of an iterative method as the maximum ratio of the norm of errors before and after one iteration; that is, for the method above, the convergence factor is

$$\sup_{u_k \in V_k} \frac{||v_k - u_k^*||}{||u_k - u_k^*||},$$

where $u_k^* = A_k^{-1} f_k$, u_k is the initial and v_k the resulting approximate solution.

We then have the following bound on the convergence factors. Let

$$T_k = I - I_{k-1}^k A_{k-1}^{-1} I_k^{k-1} A_k$$

be the orthogonal projection (in the energy norm) onto the A-orthogonal complement of the range of $I_{k-1}^k = V_k/V_{k-1}$.

THEOREM 2.2. *[7, 8] Suppose there is $\alpha > 0$ and $\delta < +\infty$ such that for all $k = 1, 2, \ldots$*

$$\rho(A_k^\alpha)\rho(T_k A_k^{-\alpha}) \le \delta.$$

Let the square of the convergence factor of the iterative method in Step 2 of Algorithm 2.1 be at most $\bar{\varepsilon}$. Then the square of the convergence factor of Algorithm 2.1 is at most

$$p(\beta_1, \varepsilon_1)p(\beta_2, \varepsilon_2),$$

where ε_1 and ε_2 are arbitrary such that $\varepsilon_1\varepsilon_2 = \bar{\varepsilon}$ with $\varepsilon_1, \varepsilon_2 \in (0, 1]$, and

$$p(\beta, \varepsilon) = \max_{0 \le t \le 1} \frac{t + \varepsilon(1-t)}{1 + \beta t^{1/\alpha}}, \quad \beta_i = \frac{b_i}{\delta},$$

with $b_i, i = 1, 2$, depending only on algebraic properties of the smoothers. (For example, for $B_{k,i} = \frac{1}{\rho(A_k)} A_k$, one has $b_i = 1$.) If the pre-smoothing or post-smoothing step is absent, then the theorem holds with $\beta_1 = 0$ or $\beta_2 = 0$, respectively. (For more details and extensions, see [7, 8].)

We only look at the case when only post-smoothing or only pre-smoothing is present and the approximate solver in Step 2 of Algorithm 2.1 is the same method applied μ times to (2.2) on level $k-1$. Then Theorem 2.2 gives the following recursive bound ε_k on the convergence factor squared of Algorithm 2.1:

$$\varepsilon_1 = 0, \quad \varepsilon_k = p(\beta, \varepsilon_{k-1}^{\mu}), \tag{2.3}$$

where β is the one nonzero β_i. The resulting algorithm is called V-cycle if $\mu = 1$ and W-cycle if $\mu = 2$. It is known that for the V-cycle, $\varepsilon_k = \frac{1}{1+\beta}$, $k > 1$, if $\alpha = 1$, and $\varepsilon_k = 1 - O(k^{-\frac{1-\alpha}{\alpha}})$, $k \to \infty$, if $\alpha < 1$, see [4]. For the W-cycle, $\varepsilon_k \leq C < 1$ for all k and any $\alpha > 0$, with $C = C(\alpha, \beta)$, see [7, 8]. If both pre-smoothing and post-smoothing is present, the bound on the squared convergence factor is obtained as the product $\varepsilon_{k,1}\varepsilon_{k,2}$, where $\varepsilon_{k,i}$ satisfy recursions analogous to (2.3), cf. [8].

Obviously, the V-cycle has lower computational cost than the W-cycle and it is often used in practice in spite of the fact that the bounds ε_k on the squared convergence factors satisfy $\varepsilon_k \to 1$ as $k \to \infty$. This deterioration of convergence is actually observed in practice for even medium sized k. On the other hand, the W-cycle convergence factor has been observed not to deteriorate with growing k, exactly as predicted by the theory.

The purpose of this paper is to analyze the F-cycle, a cycling scheme between the V-cycle and the W-cycle. The F-cycle is defined as Algorithm 2.1 where the approximate solver for the system (2.2) in Step 2 is defined as *one application of V-cycle and one application of the F-cycle itself.* We then get from Theorem 2.2 the following recursion for the bound ρ_k on the F-cycle convergence factor squared and the bound σ_k on the convergence factor squared of the V-cycle. For simplicity, assume that only pre-smoothing or only post-smoothing is present, and let β be the one nonzero β_i:

$$\sigma_1 = \rho_1 = 0, \quad \sigma_k = f(\sigma_{k-1}), \quad \rho_k = f(\rho_{k-1}\sigma_{k-1}), \quad k > 1, \tag{2.4}$$

$$f(\varepsilon) = \max_{0 \leq t \leq 1} \frac{t + \varepsilon(1-t)}{1 + \beta t^{1/\alpha}}. \tag{2.5}$$

The rest of the paper is concerned with study of the recursive bounds σ_k and ρ_k defined by (2.4) and (2.5).

3. Asymptotic behavior of F and V-cycle bounds. We start with the following lemma. Its proof is a revised version of the proof in [4].

LEMMA 3.1. *Let $\varepsilon \geq \frac{1}{1+\beta}$, $\alpha < 1$, $\beta > 0$, and f be defined by (2.5). Then*

$$f(\varepsilon) \leq \varepsilon + C_1(1-\varepsilon)^{\frac{1}{1-\alpha}},$$

with $C_1 = C_1(\alpha, \beta)$.

Proof. Since the function $g(t) = t^{\alpha}$ is concave on $[0,1]$, from the inequality $g(t) \leq g(c) + (t-c)g'(c)$, we have for any $c \in [0,1]$ and for all $t \in [0,1]$, that

$$t^{\alpha} \leq c^{\alpha} + (t-c)\alpha c^{\alpha-1} = \alpha c^{\alpha-1} t + (1-\alpha)c^{\alpha}.$$

Define t_c as the solution of $\alpha c^{\alpha-1} t_c + (1-\alpha)c^{\alpha} = 1$. Then

$$t_c = \frac{1-(1-\alpha)c^{\alpha}}{\alpha c^{\alpha-1}} \geq c^{1-\alpha} \tag{3.1}$$

because $c \le 1$, and $\alpha \le 1$. Writing (2.5) as

$$f(\varepsilon) = \max_{0 \le t \le 1} \frac{\varepsilon + (1-\varepsilon)t^\alpha}{1+\beta t},$$

we have

$$f(\epsilon) \le \max_{0 \le t \le 1} \frac{\varepsilon + (1-\varepsilon)\min\left\{1, \alpha c^{\alpha-1} t + (1-\alpha)c^\alpha\right\}}{1+\beta t}$$

The maximum may be attained only at $t = t_c$ or $t = 0$ and using the inequality $\varepsilon + (1-\varepsilon)(1-\alpha)c^\alpha < 1$, we have that

$$f(\varepsilon) \le \max\left\{\frac{1}{1+\beta t_c}, \varepsilon + (1-\varepsilon)(1-\alpha)c^\alpha\right\} < 1.$$

Choosing

$$c = \left(\frac{1-\varepsilon}{\beta\varepsilon}\right)^{\frac{1}{1-\alpha}},$$

we have $c \le 1$ because $\varepsilon \ge \frac{1}{1+\beta}$, and from (3.1),

$$\frac{1}{1+\beta t_c} \le \frac{1}{1+\beta c^{1-\alpha}} = \varepsilon,$$

and

$$\varepsilon + (1-\varepsilon)(1-\alpha)c^\alpha = \varepsilon + \tau(1-\varepsilon)^{1+\frac{\alpha}{1-\alpha}}, \quad \tau = \frac{1-\alpha}{(\beta\varepsilon)^{\frac{\alpha}{1-\alpha}}},$$

where $\tau \le C_1(\alpha, \beta)$ because we have assumed that $\epsilon \ge \frac{1}{1+\beta}$. □

The next lemma isolates an argument from [4], which we will need later.

LEMMA 3.2. *There is a constant* $C_2 = C_2(\alpha, \beta) > 0$ *such that if*

$$\frac{1}{1+\beta} \le \varepsilon \le 1 - C_2 m^{-\frac{1-\alpha}{\alpha}}, \quad m \ge 1, \tag{3.2}$$

then

$$f(\varepsilon) \le 1 - C_2(m+1)^{-\frac{1-\alpha}{\alpha}}. \tag{3.3}$$

Proof. By Lemma 3.1, $f(\varepsilon) \le \varepsilon + C_1(1-\varepsilon)^{\frac{1}{1-\alpha}}$. Assume that (3.2) holds with some $C_2 > 0$. Then, because $f(\varepsilon)$ is monotone in ε,

$$f(\varepsilon) \le f(1 - C_2 m^{-\frac{1-\alpha}{\alpha}}) \le 1 - C_2 m^{-\frac{1-\alpha}{\alpha}} + C_1\left(C_2 m^{-\frac{1-\alpha}{\alpha}}\right)^{\frac{1}{1-\alpha}}$$

and to prove (3.3), it is sufficient to have

$$C_1 C_2^{\frac{1}{1-\alpha}} m^{-\frac{1}{\alpha}} \leq C_2 \left(m^{-\frac{1-\alpha}{\alpha}} - (m+1)^{-\frac{1-\alpha}{\alpha}} \right). \tag{3.4}$$

Because the function $h(x) = x^{-\frac{1-\alpha}{\alpha}}$ is convex on $[m, m+1]$, we have

$$h(m) - h(m+1) \geq -h'(m+1),$$

giving

$$m^{-\frac{1-\alpha}{\alpha}} - (m-1)^{-\frac{1-\alpha}{\alpha}} \geq \frac{1-\alpha}{\alpha}(m+1)^{-\frac{1}{\alpha}}.$$

So, (3.4) and thus (3.3) will be satisfied if

$$C_1 C_2^{\frac{1}{1-\alpha}} m^{-\frac{1}{\alpha}} \leq C_2 \frac{1-\alpha}{\alpha}(m+1)^{-\frac{1}{\alpha}}.$$

This is equivalent to

$$C_2^{\frac{\alpha}{1-\alpha}} \leq \frac{1-\alpha}{\alpha} \left(\frac{m}{m+1} \right)^{\frac{1}{\alpha}} \frac{1}{C_1},$$

which is satisfied for all sufficiently small C_2 because of the restriction $m \geq 1$. □

We can now easily obtain the V-cycle result of [4].

THEOREM 3.3. *There is a constant k_0 such that the V-cycle bound σ_k defined by (2.4),(2.5) satisfies*

$$\sigma_k \leq 1 - C_2(k_0 + k)^{-\frac{1-\alpha}{\alpha}}.$$

Proof. Obviously, $\sigma_k \geq \frac{1}{1+\beta}$, $k \geq 2$. Let k_0 be such that $k_0 + 2 \geq 1$ and

$$\frac{1}{1+\beta} \leq \sigma_2 \leq 1 - C_2(k_0 + 2)^{-\frac{1-\alpha}{\alpha}}.$$

Then the theorem follows immediately by induction from Lemma 3.2. □

We now proceed to the main result of this section.

THEOREM 3.4. *Let $0 < \alpha < 1$ and $\beta > 0$. Then the F-cycle bound ρ_k, defined by (2.4) and (2.5), satisfies*

$$\rho_1 = 0, \quad \rho_2 \geq \frac{1}{1+\beta}, \quad \lim_{k \to \infty} \rho_k = 1,$$

and there exist constants k_0, k_1, and C_4, depending only on α and β, so that for all $k \geq k_1$, it holds that

$$\rho_k \leq 1 - C_4(k_0 + k)^{-\frac{(1-\alpha)^2}{\alpha}}.$$

Proof. Because $f(0) \geq \frac{1}{1+\beta}$, we have immediately that $\sigma_2 = \rho_2 \geq \frac{1}{1+\beta}$. From the definition of f, it follows that $f(\varepsilon) > \varepsilon$ for all $\varepsilon \in [0,1)$, because

$$\frac{d}{dt}\left(\frac{t+\varepsilon(1-t)}{1+\beta t^{1/\alpha}}\right)\Bigg|_{t=0} > 0.$$

Therefore, $\sigma_k = f(\sigma_{k-1}) > \sigma_{k-1}$ and $\sigma_k \to \sigma \in [0,1]$, $k \to \infty$. From the continuity of f, $f(\sigma) = \sigma$. It follows that $\sigma = 1$ and so $\sigma_k \to 1$. Now we are ready to prove that $\rho_k \to 1$. Suppose

$$\limsup_{k\to\infty} \rho_k = \rho < 1. \tag{3.5}$$

Then there exists a convergent subsequence $\rho_{k_l} \to \rho$, $l \to \infty$, and

$$\rho_{k_l+1} = f(\rho_{k_l}\sigma_{k_l}) \to f(\rho) > \rho,$$

a contradiction to (3.5).

Define the numbers p and m_k by

$$\rho_k = 1 - C_2 m_k^{-p}, \quad p = \frac{1-\alpha}{\alpha} \tag{3.6}$$

with $C_2 > 0$ from Lemma 3.2. Because $\rho_k \to 1$, it holds that $m_k \to +\infty$. From (3.6) and Theorem 3.3,

$$\rho_k\sigma_k \leq (1 - C_2 m_k^{-p})\left(1 - C_2(k_0+k)^{-p}\right) = 1 - C_2\tilde{m}_k^{-p}, \quad \tilde{m}_k = g(k, m_k),$$

where the function g is defined by

$$g(k,m) = \left((k_0+k)^{-p} + m^{-p} - C_2(k_0+k)^{-p}m^{-p}\right)^{-1/p}.$$

Note that the term in brackets is nonnegative for all $k \geq 2$. Define

$$k_1 = \min\{k \in \mathbf{N} : \; k \geq 2, \; \rho_k\sigma_k \geq \frac{1}{1+\beta}, \; g(k, m_k) \geq 1\}. \tag{3.7}$$

It follows from $\rho_k \to 1$, $\sigma_k \to 1$, and $g(k, m_k) \to +\infty$, as $k \to \infty$, that k_1 is well defined. Now from (3.6) and Lemma 3.2, we have for all $k \geq k_1$, that

$$\rho_{k+1} = f(\rho_k\sigma_k) \leq 1 - C_2(g(k, m_k) + 1)^{-p},$$

so

$$m_{k+1} \leq 1 + g(k, m_k), \quad k \geq k_1. \tag{3.8}$$

It follows from $0 \leq \sigma_k \leq 1 - C_2(k_0+k)^{-p}$, cf., Theorem 3.3, that

$$\frac{\partial g}{\partial m} = -\frac{1}{p}\left((k_0+k)^{-p} + m^{-p} - C_2(k_0+k)^{-p}m^{-p}\right)^{-\frac{1}{p}-1} \tag{3.9}$$

$$\left(1 - C_2(k_0+k)^{-p}\right)(-p)m^{-p-1}$$

$$\geq 0$$

for all $k \geq 2$ and $m \geq 0$. We will show that

$$m_{k_1} \leq C_3(k_0+k_1)^{1-\alpha} \tag{3.10}$$

and

$$1 + g\Big(k, C_3(k_0+k)^{1-\alpha}\Big) \leq C_3(k_0+k+1)^{1-\alpha} \quad \text{for all } k \geq k_1. \tag{3.11}$$

Then, using (3.8) and (3.9) a simple induction shows that $m_k \leq C_3(k_0+k)^{1-\alpha}$ and the theorem immediately follows.

Denote

$$Q = \frac{1 + g\Big(k, C_3(k_0+k)^{1-\alpha}\Big)}{C_3(k_0+k)^{1-\alpha}}.$$

Then $Q \leq 1$ implies (3.11). From the definition of g,

$$\frac{g(k,m)}{m} = \left(m^p(k_0+k)^{-p} + 1 - C_2(k_0+k)^{-p}\right)^{-1/p},$$

so

$$Q = \frac{1}{C_3(k_0+k)^{1-\alpha}} + \left(1 + C_3^p(k_0+k)^{(1-\alpha)p-p} - C_2(k_0+k)^{-p}\right)^{-1/p}.$$

Using the facts that $(1-\alpha)p - p = -\alpha p = \alpha - 1$ and $-p-1+\alpha = \frac{-1+\alpha^2}{\alpha} < 0$, we have

$$\begin{aligned} Q &= \frac{1}{C_3(k_0+k)^{1-\alpha}} + \left(1 + (C_3^p - C_2(k_0+k)^{-p-1+\alpha})(k_0+k)^{\alpha-1}\right)^{-1/p} \\ &\leq \frac{1}{C_3(k_0+k)^{1-\alpha}} + \left(1 + \frac{C_3^p - C_2'}{(k_0+k)^{1-\alpha}}\right)^{-1/p}, \end{aligned}$$

for all $k \geq k_1$, where $C_2' = (k_0+k_1)^{-p-1+\alpha}$. Define $h(x) = (1+x)^{-1/p}$ and let $x_1 > 0$. Because h is convex on $[0, x_1]$, it holds that

$$h(x) \leq h(0) + \frac{x}{x_1}(h(x_1) - h(0)), \quad 0 \leq x \leq x_1.$$

Using this inequality with

$$x = \frac{C_3^p - C_2'}{(k_0+k)^{1-\alpha}}, \quad x_1 = \frac{C_3^p - C_2'}{(k_0+k_1)^{1-\alpha}},$$

we obtain

$$Q \leq \frac{1}{C_3(k_0+k)^{1-\alpha}} + 1 + \frac{(k_0+k_1)^{1-\alpha}}{(k_0+k)^{1-\alpha}} \left[\left(1 + \frac{C_3^p - C_2'}{(k_0+k_1)^{1-\alpha}} \right)^{-1/p} - 1 \right].$$

Consequently,

$$Q \leq 1 + (k_0+k)^{-(1-\alpha)} F(C_3),$$

where

$$\lim_{C_3 \to +\infty} F(C_3) < 0.$$

So, $Q \leq 1$ for all $k \geq k_1$ and all sufficiently large C_3. □

4. F-cycle bounds for small number of levels. In this section, we show that the F-cycle bounds are, in fact, often constant for the number of levels that occur in practice. It was proved in [7, 8] that the W-cycle bounds $\tau_k = f(\tau_{k-1}^2)$ satisfy $\tau_k = \frac{1}{1+\beta}$ for $\alpha \geq (\beta+1)/(\beta+2)$. We prove a related result for the F-cycle.

THEOREM 4.1. *Let $\tilde{\sigma}_1 = 0 \leq \tilde{\sigma}_2 \leq \tilde{\sigma}_3 \leq \ldots$ be bounds on the convergence factor squared of the V-cycle. Let $0 < \alpha < 1$, and $\beta > 0$. Then $\tilde{\rho}_k$, defined by*

$$\tilde{\rho}_1 = 0, \quad \tilde{\rho}_k = f(\tilde{\rho}_{k-1}\tilde{\sigma}_{k-1}), \quad k > 1, \tag{4.1}$$

where f is given by (2.5), is an upper bound on the convergence factor squared of the F-cycle, and

$$\frac{1 - \tilde{\sigma}_{k-1}}{(1-\alpha)(1+\beta-\tilde{\sigma}_{k-1})} \geq 1 \tag{4.2}$$

implies

$$\tilde{\rho}_k = \frac{1}{1+\beta}.$$

Proof. First, (4.1) follows immediately from Theorem 2.2. We proceed by induction. Assume that

$$\tilde{\rho}_{k-1} \leq \frac{1}{1+\beta}$$

and that (4.2) holds. Denote $\varepsilon = \frac{1}{1+\beta}$ and $\sigma = \tilde{\sigma}_{k-1}$. We show that

$$\frac{t + \varepsilon\sigma(1-t)}{1+\beta t^{1/\alpha}} \leq \varepsilon, \quad \text{for all } t,\ 0 \leq t \leq 1, \tag{4.3}$$

which will conclude the proof. But (4.3) is equivalent to

$$\varepsilon(1+\beta t^{1/\alpha}) \geq \varepsilon\sigma(1-t) + t,$$

for all t, $0 < t \leq 1$, which is the same as

$$\beta \geq \Phi(t), \quad \text{for all } t,\ 0 < t \leq 1,$$

where

$$\Phi(t) = \varepsilon^{-1} t^{-1/\alpha}(\varepsilon\sigma(1-t) + t - \varepsilon) = t^{-1/\alpha}(\sigma - 1) + t^{-\frac{1}{\alpha}+1}(1 - \sigma\varepsilon)/\varepsilon.$$

Now $\Phi'(t_m) = 0$ is equivalent to

$$-\frac{1}{\alpha} t_m^{-\frac{1}{\alpha}-1} \varepsilon(\sigma - 1) + (-\frac{1}{\alpha} + 1) t_m^{-\frac{1}{\alpha}}(1 - \sigma\varepsilon) = 0$$

which gives

$$t_m = \frac{\varepsilon(1-\sigma)}{(1-\alpha)(1-\sigma\varepsilon)}.$$

If $t_m > 1$, then the maximum of Φ on (0,1] is attained at $t = 1$ and (4.3) holds. □

COROLLARY 4.2. *A necessary condition for (4.2) to hold for* some $\tilde{\sigma}_k > 0$ *is*

$$(1-\alpha)(1+\beta) \leq 1. \tag{4.4}$$

If we use in Theorem 4.1 $\tilde{\sigma}_k = \sigma_k$, *where* σ_k *is the V-cycle bound from (2.4), then a necessary condition for* $\rho_k = \frac{1}{1+\beta}$ *for some* $k > 2$ *is*

$$(1-\alpha)(2+\beta) \leq 1; \tag{4.5}$$

in particular, one must have $\alpha > \frac{1}{2}$.

Proof. (4.4) is immediate from (4.2). For (4.5) note that $\sigma_{k-1} \geq \frac{1}{1+\beta}, k > 2$, and that (4.2) then implies

$$1 \leq \frac{1 - \frac{1}{1+\beta}}{(1-\alpha)\left(1 + \beta - \frac{1}{1+\beta}\right)} = \frac{1}{(1-\alpha)(2+\beta)}.$$

□

In Fig. 1 to Fig. 5, we have plotted the values of the V-cycle bounds σ_k (full line), the F-cycle bounds ρ_k (dashed line) and the W bounds τ_k (dashdot line), $\tau_k = f(\tau_{k-1}^2)$. We see that indeed for $\alpha > \frac{1}{2}$ and not too large β, the W-cycle and F-cycle bounds coincide for small k. In general, the F-cycle bound grows, albeit more slowly than the V-cycle bound. (Because larger β corresponds to smaller bound, the higher curves are for smaller β). Because we were interested in the asymptotics of σ_k, ρ_k, and τ_k, we have plotted their values for admittedly unrealistically large values of k. In reality, one never uses more that $k = 8$ levels.

We should note that good smoothing corresponds to large β. Thus, for good smoothing, β is large and (4.2) is not likely to occur. Small β may give the constant or essentially constant ρ_k. But one is, in practice, of course still better off smoothing a few times more, or generally getting better smoothing to get large beta, because even though the convergence factor decays with k, it is still smaller than the essentially constant, but larger, factor for small β.

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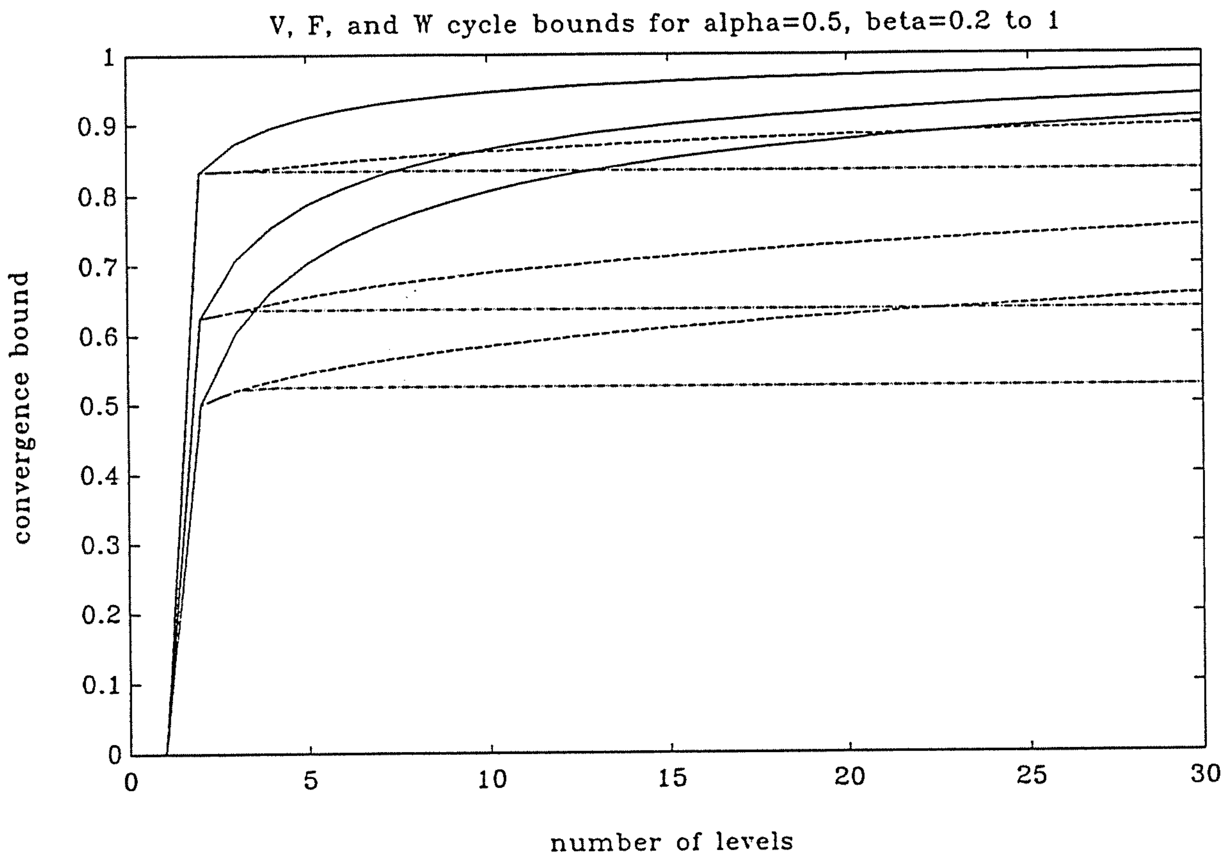
V, F, and W cycle bounds for alpha=0.5, beta=0.2 to 1
convergence bound
number of levels
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
5
10
15
20
25
30

Figure 1

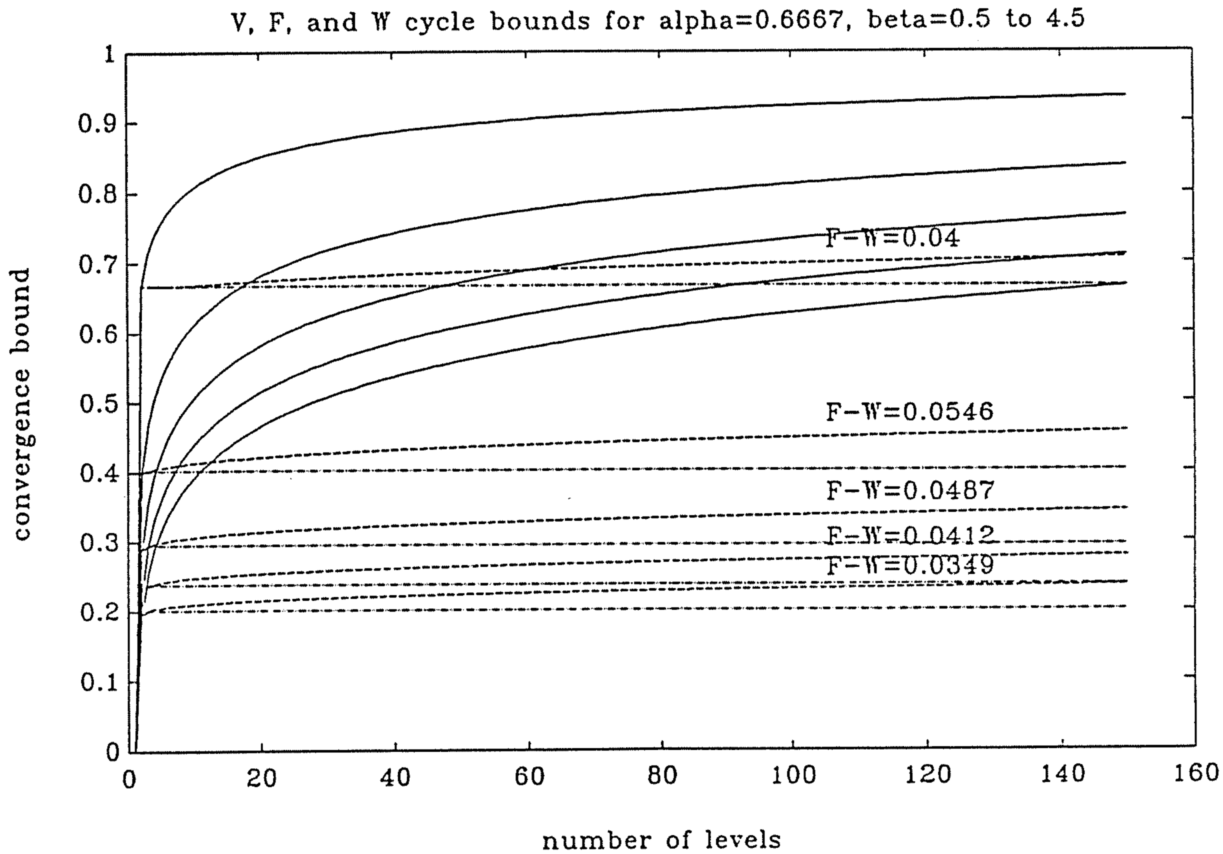
V, F, and W cycle bounds for alpha=0.6667, beta=0.5 to 4.5
convergence bound
number of levels
F−W=0.04
F−W=0.0546
F−W=0.0487
F−W=0.0412
F−W=0.0349
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
20
40
60
80
100
120
140
160

Figure 2

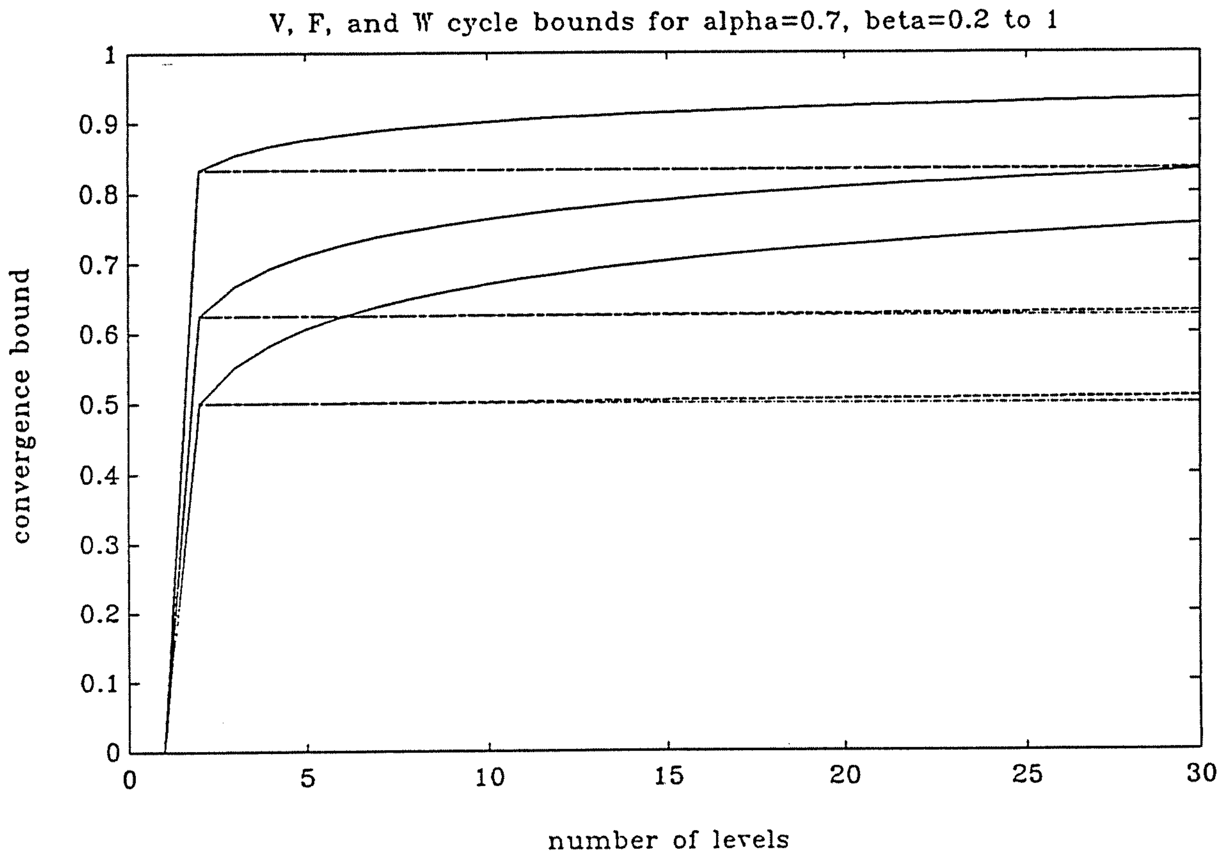
V, F, and W cycle bounds for alpha=0.7, beta=0.2 to 1
convergence bound
number of levels
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25
30

Figure 3

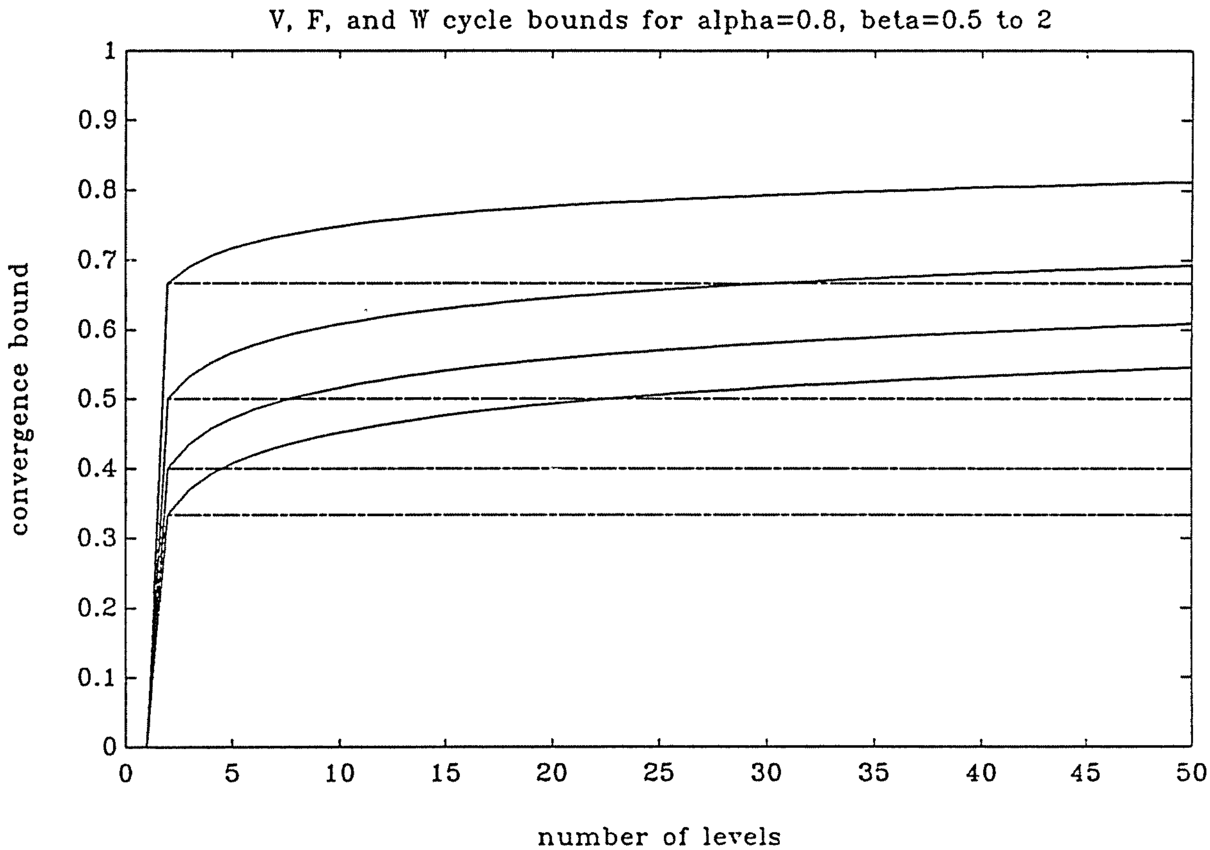
V, F, and W cycle bounds for alpha=0.8, beta=0.5 to 2
convergence bound
number of levels
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25
30
35
40
45
50

Figure 4

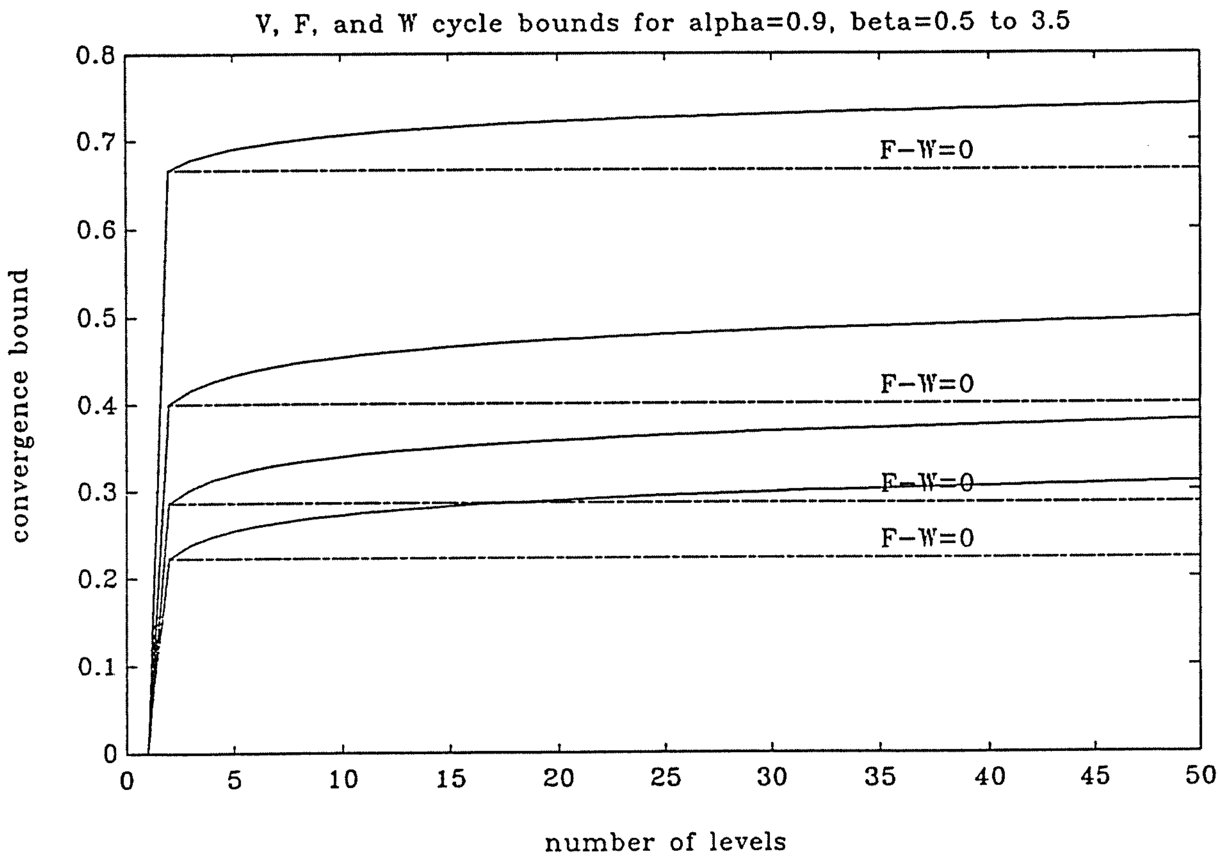
V, F, and W cycle bounds for alpha=0.9, beta=0.5 to 3.5
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
F−W=0
F−W=0
F−W=0
F−W=0
0
5
10
15
20
25
30
35
40
45
50
convergence bound
number of levels

Figure 5