ON THE CONVOLUTION OF A BOX SPLINE WITH A COMPACTLY SUPPORTED DISTRIBUTION: THE EXPONENTIAL-POLYNOMIALS IN THE LINEAR SPAN

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Computer Sciences Technical Report #813

January 1989

UNIVERSITY OF WISCONSIN-MADISON

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November 1988

ABSTRACT

For a given box spline B and a compactly supported distribution μ, we examine in this note the convolution $B * \mu$ and the space $H(B * \mu)$ of all exponential-polynomials spanned by its integer translates. The main result here provides a necessary and sufficient condition for the equality $H(B * \mu) = H(B)$. This condition is given in terms of the distribution of the zeros of the Fourier-Laplace transform of $B * \mu$ and allows us to reduce the above equality to much simpler settings.

The importance of this result is for the determination of the approximation properties of the space spanned by the integer translates of $B * \mu$. A typical example is discussed.

AMS (MOS) Subject Classifications: primary 41A15, 41A63.

Key Words: exponentials, polynomials, multivariate, box splines, exponential box splines, integer translates, splines, multivariate splines, compactly supported functions.

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1. Introduction

The basic model in multivariate splines on a uniform mesh (= multivariate splines on a regular grid) consists of a compactly supported function ϕ defined on $\mathbb{R}^s$ and the space $S(\phi)$ spanned by its integer translates. Two of the most important criteria for a favourable choice of ϕ are the linear independence of the integer translates of ϕ, and the local approximation properties of the space $H(\phi) :=$ the set of all exponential-polynomials that lie in $S(\phi)$. The significance of this last space is due to the fact that in most circumstances the local approximation power of $H(\phi)$ can be shown, with the aid of the so-called quasi-interpolation schemes, to provide a lower bound on the approximation power corresponding to $S(\phi)$ and appropriate scaled-versions of it. However, these two basic properties (the linear independence of the integer translates and the good local approximation power of the space $H(\phi)$) are highly competitive properties, a fact which will be illustrated later on.

In many of the practical examples of ϕ, the compactly supported function is constructed by convolving together several functions or distributions. A tentative justification for such an approach would emphasize the fact that the functions of $H(\phi)$ are determined by the distribution of the zeros of the Fourier-Laplace transform of ϕ together with the multiplicities of these zeros; such a property can be more efficiently treated when ϕ is expressed as a convolution of simple factors.

Exponential box (EB-)splines, introduced in [R_1], generalize the well-known polynomial box splines ([BD], [BH_1]) and provide a wide selection of choices for the function ϕ. To introduce a typical EB-spline, let Γ be a finite multiset (to be referred later as a *defining set*) with cardinality $\#\Gamma$ consisting of elements of the form

$$\gamma = (x_\gamma, \lambda_\gamma), \tag{1.1}$$

where $x_\gamma \in \mathbb{Z}^s \backslash 0$ and $\lambda_\gamma \in \mathbb{C}$. The EB-spline corresponding to Γ, $B(\Gamma)$, is defined via its Fourier transform by

$$\widehat{B}(\Gamma|x) := \prod_{\gamma\in\Gamma} \widehat{B}(\gamma|x) := \prod_{\gamma\in\Gamma} \left(\int_0^1 e^{(\lambda_\gamma - i x_\gamma \cdot x)t}\, dt \right). \tag{1.2}$$

Note that indeed the exponential box spline can be expressed as a convolution of lower order ones. In fact if $\Gamma = \Gamma_1 \cup \Gamma_2$ it follows from (1.2) that

$$B(\Gamma) = B(\Gamma_1) * B(\Gamma_2). \tag{1.3}$$

In case

$$\langle\Gamma\rangle := \mathrm{span}\{x_\gamma\}_{\gamma\in\Gamma} = \mathbb{R}^s, \tag{1.4}$$

$B(\Gamma)$ gives rise to a compactly supported function $B(\Gamma|\cdot)$; otherwise the EB-spline is merely a distribution (actually a measure) supported in $\langle\Gamma\rangle$. For more information about EB-splines we refer the reader to [$R_{1,2}$], [BR], [DM] and [DR].

Only few other examples of a function ϕ can be found in the literature, and most of these examples consists of bivariate piecewise-polynomials. In fact some of these functions are obtained by convolving a (polynomial) box spline with a certain (and simple) compactly supported function. (cf. e.g., [BH_2] and [CH]). Stimulated by these latter functions, we became interested in the properties of a function ψ obtained as a convolution of an EB-spline and an arbitrary compactly supported distribution. For that model, the question of linear independence of the integer translates has been thoroughly discussed in [CR]. In this note we compare $H(\psi)$ (with ψ as above) with $H(B(\Gamma))$. Our main (and essentially only) result here provides a necessary and sufficient condition for the equality

$$H(\psi) = H(B(\Gamma)). \tag{1.5}$$

The statement as well as the proof of the main result are presented in section 3. In section 2 we collect the notations and preliminaries needed for this proof. Finally, we discuss in section 4 an example which demonstrates the efficiency and usefulness of the main result.

This paper together with [CR] allows us to determine clear criteria for a "good" choice of μ. The various applications of these results will be studied in a subsequent paper of C.K. Chui and the auther.

2. Notations and Preliminaries

Throughout this paper we use π for the space of all s-dimensional polynomials, e_θ for the exponential $\exp(i\theta\cdot)$, and $\widehat{\phi}$ for the Fourier-Laplace transform of the compactly supported distribution ϕ, i.e., the analytic continuation of the Fourier transform of ϕ.

Given a subset K of the defining set Γ, we find it convenient to refer to linear properties of $\{x_\gamma\}_{\gamma\in K}$ in terms of K. Thus, we say that K is *linearly independent* and mean that the vectors $\{x_\gamma\}_{\gamma\in K}$ are linearly independent. Also we use

$$\langle K\rangle$$

for the *real* span of $\{x_\gamma\}_{\gamma\in K}$, and

$$K^\perp$$

for the *complex* set

$$\{x\in C^s \mid x\cdot x_\gamma = 0,\ \forall\gamma\in K\}.$$

For $x\in I\!R^s\backslash 0$, let D_x be the directional derivative in the x-direction. Given $K\subset\Gamma$, we set

$$D^K := \prod_{\gamma\in K} D^\gamma := \prod_{\gamma\in K}(D_{x_\gamma}-\lambda_\gamma).$$

The differential operators of the form D^K play an important role in box spline theory; particularly we have (cf. [R_1;Th. 2.2])

(2.1)Proposition. *For $K\subset\Gamma$*

$$D^K B(\Gamma) = \nabla^K B(\Gamma\backslash K),$$

where ∇^K is the difference operator

$$\nabla^K = \prod_{\gamma\in K}(E^0 - e^{\lambda_\gamma}E^{x_\gamma}),$$

and E^x is the shift operator

$$E^x f = f(\cdot - x).$$

The above differential operators are also important in the analysis of $H(B(\Gamma))$. To discuss this part let us first introduce the following collection of subsets of Γ

$$I\!K(\Gamma) := \{K\subset\Gamma \mid \langle\Gamma\backslash K\rangle \neq I\!R^s\}.$$

We have (cf. e.g., [BR;Th. 6.2])

(2.2)Proposition. *Let θ be in C^s, $p \in \pi$, and assume $\widehat{B}(\Gamma|\theta) \neq 0$. Then*

$$e_\theta p \in H(B(\Gamma))$$

if and only if

$$D^K(e_\theta p) = 0, \quad \forall K \in \mathbb{K}(\Gamma).$$

We now discuss the Fourier analysis elements which are needed in the sequel. First note that (1.2) implies that

$$\widehat{B}(\gamma|x) = 0 \Leftrightarrow \lambda_\gamma - ix \cdot x_\gamma \in 2\pi i \mathbb{Z}\backslash 0. \tag{2.3}$$

The next result is a straightforward generalization of a well-known one (see e.g., [B;§2]). It provides, for an arbitrary compactly supported ϕ, a characterization of $H(\phi)$ in terms of the distribution of the zeros of $\widehat{\phi}$:

(2.4)Result. *Assume that $\widehat{\phi}(\theta) \neq 0$ for some $\theta \in C^s$, and let $p \in \pi$. Then*

$$e_\theta p \in H(\phi)$$

if and only if for all $q \in \pi$

$$(q(D)p)(-iD)\widehat{\phi}(\theta + 2\pi\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^s\backslash 0, \tag{2.5}$$

where $q(D)$ is the differential operator with constant coefficients associated with q. In case $p = 1$ this condition is reduced to

$$\widehat{\phi}(\theta + 2\pi\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^s\backslash 0. \tag{2.6}$$

Related to the above result is the fact that the condition (2.6) is always necessary for $e_\theta \in H(\phi)$. In this case one has

$$\phi *' e_\theta = \widehat{\phi}(\theta) e_\theta, \tag{2.7}$$

where here and later

$$\phi *' f := \sum_{\alpha \in \mathbb{Z}^s} f(\alpha)\phi(\cdot - \alpha). \tag{2.8}$$

It follows [R3], that the condition

$$\widehat{\phi}(\theta + 2\pi\alpha) = 0, \ \forall\alpha \in \mathbb{Z}^s$$

is sufficient for the linear dependence of the integer translates of ϕ. A comparison of this last condition with (2.6) demonstrates the competition between the properties of a rich $H(\phi)$ on the one hand, and linear independence of the integer translates of ϕ on the other hand.

3. The Main Result

Throughout this section μ is a fixed compactly supported distribution and $B(\Gamma)$ denotes an exponential box spline whose defining set Γ satisfies

$$\langle\Gamma\rangle = \mathbb{R}^s. \tag{3.1}$$

Given a subset $M \subset \Gamma$ we set

$$\psi_M := B(M) * \mu. \tag{3.2}$$

The following theorem provides a necessary and sufficient condition for the equality

$$H(B(\Gamma)) = H(\psi_\Gamma). \tag{3.3}$$

Theorem. *Let θ be in C^s, and assume that $\widehat{\psi}_\Gamma(\theta) \neq 0$. Then the following conditions are equivalent:*

(a) For some $p \in \pi$,

$$e_\theta p \in H(\psi_\Gamma)\backslash H(B(\Gamma)). \tag{3.4}$$

(b) There exists some linearly independent subset $M \subset \Gamma$ of cardinality $< s$ such that

$$e_\theta \in H(\psi_M). \tag{3.5}$$

(c) There exists some linearly independent subset $M \subset \Gamma$ of cardinality $< s$ such that

$$\lambda_\gamma - i\theta \cdot x_\gamma = 0, \ \forall\gamma \in M, \tag{3.6}$$

and

$$\widehat{\mu}(\theta + 2\pi\alpha) = 0, \ \forall\alpha \in \mathbb{Z}^s \cap M^\perp\backslash 0. \tag{3.7}$$

Proof: We start the proof by showing that (b) $\Longleftrightarrow$ (c). This equivalence is the content of the following two claims.

Claim 1. *Let M be any subset of Γ that satisfies (3.6) and (3.7). Then (3.5) holds with respect to this M.*

Proof of Claim 1. Since we assume $\widehat{\psi}_\Gamma(\theta) \neq 0$, it follows that $\widehat{\psi}_M(\theta) \neq 0$. Therefore, in view of (2.4)Result the claim will be proved as soon as we show that

$$(3.8) \qquad \widehat{B}(M|\theta + 2\pi\alpha)\widehat{\mu}(\theta + 2\pi\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^s \backslash 0.$$

For $\alpha \in \mathbb{Z}^s \cap M^\perp \backslash 0$, (3.8) is guaranteed by (3.7). Otherwise, there exists $\gamma \in M$ such that $\alpha \cdot x_\gamma \neq 0$; this means, in view of (3.6) (and since $x_\gamma \in \mathbb{Z}^s$) that

$$(3.9) \qquad \lambda_\gamma - i(\theta + 2\pi\alpha) \cdot x_\gamma = (\lambda_\gamma - i\theta \cdot x_\gamma) - i2\pi\alpha \cdot x_\gamma \in 2\pi i \mathbb{Z} \backslash 0,$$

and thus, by (2.3), $\widehat{B}(\gamma|\theta + 2\pi\alpha) = 0$. We conclude that

$$(3.10) \qquad \widehat{B}(M|\theta + 2\pi\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^s \backslash M^\perp,$$

and hence (3.8) is verified and the claim is thus proved.

Claim 2. *Assume that M is a minimal subset of Γ for which $e_\theta \in H(\psi_M)$. Then M is necessarily linearly independent and satisfies (3.6), (3.7).*

Proof of Claim 2. The assumptions here together with (2.4)Result allows us to conclude that (3.8) holds with respect to the given M. Now, the minimality of M implies the existence of $\{\alpha_\gamma\}_{\gamma \in M} \subset \mathbb{Z}^s \backslash 0$ such that

$$\widehat{B}(\gamma|\theta + 2\pi\alpha_\gamma) = 0, \ \gamma \in M,$$

which by (2.3) implies that

$$(3.11) \qquad \lambda_\gamma - i(\theta + 2\pi\alpha) \cdot x_\gamma \in 2\pi i \mathbb{Z}, \ \forall \alpha \in \mathbb{Z}^s, \ \ \gamma \in M.$$

In particular

$$(3.12) \qquad \lambda_\gamma - i\theta \cdot x_\gamma \in 2\pi i \mathbb{Z}, \ \forall \gamma \in M.$$

Utilizing the fact that the assumption $\widehat{\psi}_\Gamma(\theta) \neq 0$ implies $\widehat{B}(\gamma|\theta) \neq 0$ for all $\gamma \in \Gamma$, we may combine (3.12) together with (2.3) to conclude

$$(3.13) \qquad \lambda_\gamma - i\theta \cdot x_\gamma = 0, \ \forall \gamma \in M,$$

that is (3.6) holds.

Now, let $\alpha \in M^{\perp}\backslash 0$; then (3.13) shows that

$$\lambda_{\gamma} - i(\theta + 2\pi\alpha)\cdot x_{\gamma} = 0, \ \forall \gamma \in M,$$

which yields (in view of (2.3)) that for such an α

$$\widehat{B}(M|\theta + 2\pi\alpha) \neq 0,$$

and thus (3.8) forces

$$\widehat{\mu}(\theta + 2\pi\alpha) = 0,$$

which proves (3.7).

To complete the proof of the claim, it remains to show that M is necessarily linearly independent. Let $M_1 \subset M$ be a linearly independent set that spans $\langle M \rangle$. Since (3.6) and (3.7) hold with respect to M, they hold with respect to M_1, and therefore Claim 1 implies that (3.5) is valid with M_1 replacing M. The minimality of M then ensures that $M_1 = M$, so M is indeed linearly indepednent and Claim 2 is thus established.

Let us now prove the implication (a) $\Longrightarrow$ (b). Assuming that (a) holds, we define $g := \psi_{\Gamma} *' e_{\theta}p$ (see (2.8)). Now, for an arbitrary compactly supported function ϕ, it is known, [B], that the assumption $\widehat{\phi}(\theta) \neq 0$ implies that the semi-discrete convolution operator $\phi *'$ is 1-1 on $e_{\theta}\pi$ and induces an automorphism on the space $\{e_{\theta}q \in H(\phi) | \ q \in \pi\}$. It thus follows that g lies in $H(\psi_{\Gamma})\backslash H(B(\Gamma))$ and admits the form $g = e_{\theta}q_1$ for some polynomial q_1. We now invoke (2.2)Proposition to conclude that for some $K \in \mathbb{K}(\Gamma)$

$$D^K g \neq 0.$$

Consequently, (2.1)Proposition leads to

$$D^K g = D^K(B(\Gamma) * \mu *' (e_{\theta}p)) = B(\Gamma\backslash K) * \mu *' \nabla^K(e_{\theta}p). \tag{3.14}$$

Since $D^K g$ is clearly of the form $e_{\theta}q$ for some polynomial $q \neq 0$, we conclude from (3.14) that for $M_1 := \Gamma\backslash K$ we have

$$e_{\theta}q \in H(\psi_{M_1}). \tag{3.15}$$

Since the space $H(\psi_{M_1})$ is shift-invariant (i.e., closed under integer translates), (3.15) readily implies that

$$e_{\theta} \in H(\psi_{M_1}),$$

while by the definition of $\mathbb{K}(\Gamma)$, $\langle M_1 \rangle \neq \mathbb{R}^s$. Thus, (b) will be obtained as soon as we prove that M_1 can be replaced by one of its linearly independent subsets: let M be a minimal subset of M_1 with respect to the property $e_\theta \in H(\psi_M)$; application of Claim 2 thus ensures that such M is necessarily linearly independent, and (b) is therefore obtained.

It remains to prove that (c) implies (a). Here, let M be the set appearing in (c) and let ξ be any non-trivial vector in $M^\perp$. Define

$$p(x) := (\xi \cdot x)^k, \tag{3.16}$$

where k is the least non-negative integer satisfying

$$f := e_\theta p \notin H(B(\Gamma)).$$

We contend that f satisfies (a), i.e., $f \in H(\psi_\Gamma)$.

Let us show that indeed $f \in H(\psi_\Gamma)$. Since we assume $\widehat{\psi}_\Gamma(\theta) \neq 0$, application of (2.4)Result yields that this will be established as soon as we prove that

$$(D_\xi)^j \widehat{\psi}_\Gamma(\theta + 2\pi\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^s \backslash 0, \quad j = 1, \ldots, k. \tag{3.17}$$

In the verification of (3.17) we consider two types of points:

(1) $\alpha \in \mathbb{Z}^s \backslash M^\perp$: for such an α choose $\gamma \in M$ such that $\alpha \cdot x_\gamma \neq 0$. By appealing to (3.6) we obtain

$$\lambda_\gamma - i(\theta + 2\pi\alpha) \cdot x_\gamma \in 2\pi i \mathbb{Z} \backslash 0,$$

hence the Fourier transform of $B(\gamma)$ vanishes at $\theta + 2\pi\alpha$. On the other hand, by (1.2), this transform is constant along any direction orthogonal to x_γ. Since $\gamma \in M$ and $\xi \in M^\perp$, it follows that $x_\gamma \perp \xi$. We conclude that $\widehat{B}(\gamma)$, and hence $\widehat{\psi}_\Gamma$, vanishes on the line

$$\{(\theta + 2\pi\alpha + t\xi)\}_{t \in \mathbb{R}}. \tag{3.18}$$

Now it is clear that for such an α (3.17) holds (even without any restriction on j).

(2) Let $\alpha \in \mathbb{Z}^s \cap M^\perp \backslash 0$. Since we assume that $e_\theta(x)(\xi \cdot x)^{k-1} \in H(B(\Gamma))$, then an application of (2.4)Result yields that

$$(D_\xi)^j \widehat{B}(\Gamma | \theta + 2\pi\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^s \backslash 0, \ j = 1, \ldots, k-1,$$

which means that $\widehat{B}(\Gamma)$ has a k-fold zero at $\theta + 2\pi\alpha$. Since by (3.7) $\widehat{\mu}(\theta + 2\pi\alpha) = 0$ as well, we conclude that ψ_Γ has a $(k+1)$-fold zero at this point, and (3.17) thus holds for this case as well.

The implication (c) $\Longrightarrow$ (a) is now established, and the proof of the Theorem therefore came to its end. ♠

4. An Example

We discuss here an example which illustrates the efficiency and usefulness of the Theorem in the analysis of special cases.

Let $B(\Gamma)$ be a bivariate three-directional exponential box spline, that is for all γ $x_\gamma \in \{(1,0),(0,1),(1,1)\}$. Let μ be the characteristic function of the triangle with vertices at $(0,0),(1,0),(1,1)$. Define, as before $\psi := B(\Gamma) * \mu$. (Certain smooth piecewise-polynomials of minimal support are obtained in this way; cf. [CH] and the references therein). We contend that

$$H(\psi) = H(B(\Gamma)), \tag{4.1}$$

which means (at least for piecewise-polynomials with their scaled version obtained by dialations) that the approximation properties of $B(\Gamma)$ are not improved in the smoothing process $B(\Gamma) \mapsto B(\Gamma) * \mu$.

To prove (4.1) we make use of the implication (a) $\Longrightarrow$ (b) of the Theorem, which reduces (4.1) to proving that for every $\gamma \in \Gamma$

$$H(\psi_\gamma) = \{0\}, \tag{4.2}$$

where $\psi_\gamma = B(\gamma) * \mu$.

To prove (4.2) we fix $\gamma \in \Gamma$ and assume, for a contradiction that $e_\theta \in H(\psi_\gamma)$ for some $\theta \in C^s$. This implies (see (2.7)) that

$$\psi_\gamma *' e_\theta = ce_\theta,$$

for some c. Invoking (2.1)Proposition we obtain

$$D^\gamma(ce_\theta) = D^\gamma(\psi_\gamma *' e_\theta) = \mu *' \nabla^\gamma(e_\theta).$$

Since the supports of the integer translates of μ do not fill all of $\mathbb{R}^2$ and $D^\gamma(ce_\theta)$ is entire we conclude that $D^\gamma(e_\theta) = 0$, which is to say that

$$ix_\gamma \cdot \theta - \lambda_\gamma = 0. \tag{4.3}$$

This allows us to compute explicitly the semi-discrete convolution $B(\gamma) *' e_\theta$. For this purpose we use the fact that the distributional definition of $B(\gamma)$ is, [R$_1$],

$$B(\gamma)(f) = \int_0^1 e^{t\lambda_\gamma} f(tx_\gamma)\, dt, \tag{4.4}$$

for suitable test functions f, i.e., $B(\Gamma)$ is a distribution supported on $\{tx_\gamma\}_{0\le t\le 1}$ with its mass distributed like $e^{t\lambda_\gamma}$ on that support.

It now follows from (4.3) and (4.4) that $\chi := B(\gamma) *' e_\theta$ is a distribution supported on the lines

$$\{\alpha + tx_\gamma\}_{t\in\mathbb{R}},\ \alpha \in \mathbb{Z}^2, \tag{4.5}$$

and on each of these lines its mass is distributed proportionally to $e^{t\lambda_\gamma}$. So the definition of μ implies that

$$\psi_\gamma *' e_\theta = \mu * \chi$$

is non-trivial on the one hand and is either discontinuous or vanishes along all lines of the form (4.5) hence cannot be any multiple of e_θ. This means that (4.2) has been obtained and our claim has just established.

Alternatively, (4.2) could be obtained by computing the functions involved in the local structure of ψ_γ. Nevertheless, the argument exploited above can be applied in more general setting, e.g., when μ is the characteristic function of a subset of the above triangle .

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