ON THE NORM EQUIVALENCE OF
SINGULARLY PERTURBED ELLIPTIC
DIFFERENCE OPERATORS

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ABSTRACT

Consider the system of linear algebraic equations \( L_{h,k}U = f \) which arises from the finite-difference discretization of the singularly perturbed elliptic operator

\[
L_K u := - \left\{ \frac{\partial}{\partial x} a \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} b \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} b \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} c \frac{\partial u}{\partial y} \right\} + K^\sigma \left\{ d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} \right\} + K p u
\]

(1.5a)

where \( 1 \ll K < \infty , \ 0 \leq \sigma \leq 1 \). In this work we are concerned with preconditioning operators \( \tilde{L}_{K,h} \) so that

(i) The condition number \( [(\tilde{L}_{K,h})^{-1} L_{K,h}] \) and the condition number \( [L_{K,h}(\tilde{L}_{K,h})^{-1}] \) is bounded independently of \( h \) and

(ii) there are bounds on the growth of these condition numbers as a function of \( K \).
1. Introduction

The numerical solution of elliptic boundary-value problems leads to the related problem of actually obtaining the solution of a large, sparse system of linear equations

\[ A_n U = F \]  \hspace{1cm} (1.1)

where \( A_n \) is an \( n \times n \) matrix and \( n \) is the number of grid points. There is a large literature connected with the analysis of iterative methods for the solution of (1.1) - see [1], [2].

Almost all iterative methods, including the multigrid methods [3] can be cast in the framework of a preconditioning followed by iterative improvement. That is, we consider the system

\[ B_n^{-1} A_n U = B_n^{-1} F \]  \hspace{1cm} (1.1)

or the system

\[ (A_n B_n^{-1}) V = A_n(B_n^{-1} V) = F \] \hspace{1cm} (1.3a)

\[ U = B_n^{-1} V \] \hspace{1cm} (1.3b)

Then an iterative method is applied to this new problem. Of course, one chooses \( B_n \) so that \( B_n^{-1} \) is "easy" to compute. Furthermore, it is often advantageous to choose \( B_n \) to be positive definite symmetric. With the practical success of multigrid methods for uniformly elliptic problems with positive symmetric part there is a particular interest in preconditioned iterative methods for which the condition number of \( B_n^{-1} A_n \) [or \( A_n B_n^{-1} \)] is bounded independent of the dimension \( n \). It is easy to develop iterative methods that yield estimates of the form

\[ \|e^j\| \leq K_0 \left( \frac{k-1}{k+1} \right)^j \|e^0\| \] \hspace{1cm} (1.4)

where \( k = \sqrt{C} \) or \( C \) \([C = \text{condition number \( (B_n^{-1} A_n) \)}]\). Thus, in the case where \( C \) is independent of \( n \) one has an iterative method which is competitive with multigrid and whose convergence rate is independent of \( n \).

Several authors have studied this problem [4]-[6] in the special case where \( A_n \) (and \( B_n \)) is positive definite or has a positive definite symmetric part. In that case one can analyze the preconditioned iterative method using the concept of "Spectrally Equivalent Operators" introduced by D’Yakanov [7]. More recently there have been results for the
truly indefinite case [8]-[10]. These problems require the concept of “Norm Equivalence” (see [8] for a detailed discussion).

In this work we are concerned with singularly perturbed boundary-value problems of the form

\[ L_K u := - \left\{ \frac{\partial}{\partial x} a \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} b \frac{\partial u}{\partial y} + \frac{\partial}{\partial x} b \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} c \frac{\partial u}{\partial y} \right\} + K^\sigma \left\{ d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} \right\} + K pu = f \quad \text{in } \Omega \]  

(1.5a)

and

specified boundary conditions on \( \partial \Omega \)  

(1.5b)

where \( 1 < K < \infty, \quad 0 \leq \sigma \leq 1 \). The ellipticity of the system is expressed by the requirement that there are constants \( 0 < q \leq Q \) such that: for all \((x, y) \in \bar{\Omega}\) and all \( \xi = (\xi_1, \xi_2) \) we have

\[ q (\xi_1^2 + \xi_2^2) \leq a \xi_1^2 + 2b \xi_1 \xi_2 + c \xi_2^2 \leq Q (\xi_1^2 + \xi_2^2) \]  

(1.5c)

The domain \( \Omega \) is taken as the unit square

\[ \Omega := (x, y) = 0 < x, y < 1 \]  

(1.6a)

and the boundary conditions take the form

\[ u \text{ or } \frac{\partial u}{\partial \nu} = 0 \]  

(1.6b)

along an entire side. The operator \( L_K \) is approximated by a finite-difference operator \( L_{K,h} \). This is the situation discussed in [8, section 3]. In that work \( K \) is kept fixed (indeed it doesn’t appear in the discussion) and the authors concentrate on preconditionings \( B_n \) for which condition number \( (B_n^{-1} A_n) \) is bounded independently of \( n \). In this work we focus our attention on both \( h \) and \( K \) and discuss preconditioning operators \( \tilde{L}_{K,h} \) so that

(i) The condition number \( [(\tilde{L}_{K,h})^{-1} L_{K,h}] \) and the condition number \( [L_{K,h} (\tilde{L}_{K,h})^{-1}] \) is bounded independently of \( h \) and

(ii) there are bounds on the growth of these condition numbers as a function of \( K \).

Throughout this work the symbols \( M, M_0, \bar{M}, \) etc. will denote constants which depend on the coefficients \( a, b, c, d, e, p \) and the first derivatives of the coefficients \( a, b, c, d, e \) but not on \( h \) or \( K \).
Remark 1.1: All of our results are easily extended to the case of the Dirichlet problem in a convex polygonal domain $\Omega$ whose boundary agrees with the discrete boundary.

Remark 1.2: The problem (1.5) is frequently rewritten in terms of $K^{-1} := \varepsilon$. These problems arise in models of reaction-diffusion and convection-diffusion. To maintain accuracy, it is often necessary that $K$ and $h$ be constrained by a condition of the form $Kh = \text{constant as } K \to \infty \text{ or } h \to 0$.

Remark 1.3: The operator employed as a preconditioner is typically based on the positive definite symmetric part of the given operator. If $p \leq 0$ in (1.5a) this operator would correspond to the second order derivative terms, which do not depend on $K$. To see why it is important to consider $K$ as well as $h$ when preconditioning (1.5), consider the following simple stationary wave propagation model $L_Ku = -\Delta u - (K + i\delta K^r)u = f$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $\delta > 0$ and $1 \geq r \geq 0$. Suppose that $L_K$ is approximated by the discrete operator, $L_{K,h}$, and we use the discrete Laplacian, $L_{K,h}^0$, as a preconditioner. Since $L_{K,h}$ and $L_{K,h}^0$ have constant coefficients, it can be seen by a simple eigenvalue analysis that

$$C \left( (L_{K,h}^0)^{-1}L_{K,h} \right) = O(K^{2-r})$$

as $K \to \infty$ and this estimate is sharp. On the other hand, if we precondition by $L_{K,h}^+ = L_{K,h}^0 + K$, it can be readily seen that

$$C \left( (L_{K,h}^+)^{-1}L_{K,h} \right) = O(K^{1-r})$$

as $K \to \infty$. Hence $L_{K,h}^+$ is a much better preconditioner than $L_{K,h}^0$, even for moderate values of $K$. Preconditioners for more general variable coefficient wave propagation models in general domains are analyzed in [10] using finite element error analysis.

We briefly outline the remainder of the paper. In Section 2 we define our notation. In Section 3 we establish some basic estimates for the finite difference operators. We state and prove the main results in Section 4 (Theorems 4.1-4.4). The analysis uses the concept of norm equivalence (see (4.1)). The preconditioner can be any operator that is norm equivalent to $B_h$ defined by (4.19), uniformly in $K$ and $h$. It is shown in [11] how multigrid methods can be employed to give specific preconditioners of this kind. The
condition number estimates in Theorems 4.1-4.4 will typically be much worse for large $K$ when positive definite symmetric preconditioning operators other than (4.19) are used. This can be seen in general using the analysis in Section 4 and was demonstrated in Remark 1.3 above. Finally, it is shown in Section 5 how to transform Convection Diffusion type operators into a simpler operator. This new operator can then be preconditioned as in Section 4 with the resulting condition number bounded independently of $K$ and $h$ as $K \to \infty$. 
2. Preliminaries I: Notation

Let \( p \) and \( \ell \) be integers and set

\[
\Delta x = \frac{1}{p+1}, \quad \Delta y = \frac{1}{\ell+1}, \quad h = \max(\Delta x, \Delta y)
\]

(2.1)

\[
\Omega_h = \{(x_k, y_j) \in \Omega, \ x_k = k\Delta x, \ y_j = j\Delta y\}
\]

(2.2a)

\[
\partial \Omega_h = \{(x_k, y_j) \in \partial \Omega, \ x_k = k\Delta x, \ y_j = j\Delta y\}
\]

(2.2b)

\[
\bar{\Omega}_h = \Omega_h \cup \partial \Omega_h.
\]

(2.2c)

Note: If \((x_k, y_j) \in \partial \Omega_h\) then either \(k = 0\) or \(p+1\) or \(j = 0\) or \(\ell+1\).

Let \(S_h\) denote the set of grid vectors \(V = \{V_{k,j}\}\) defined on \(\bar{\Omega}_h\) that satisfy the appropriate discrete boundary conditions. Thus, if the boundary conditions associated with \(L_k\) require

(i) \(U(0, y) = 0\), then \(V_{0,j} = 0\), \(j = 0, 1, \ldots (\ell + 1)\),

(ii) \(U_y(x, 1) = 0\), then \(V_{k,\ell+1} = V_{k,\ell}\), \(k = 1, 2, \ldots p\), and so on.

Remark 2.1: In the case of the boundary condition (ii) above one would probably choose \(\Delta y\) differently so that

\[
y_\ell = 1 - \frac{1}{2}\Delta y, \quad y_{\ell+1} = 1 + \frac{1}{2}\Delta y.
\]

However, such a modification has no effect on our analysis. Hence, for the purposes of this discussion we formulate the discrete (finite-dimensional) spaces as above.

Let \(G(x, y)\) be a function defined on \(\bar{\Omega}\). We write

\[
G_{k,j} = G(x_k, y_j), \quad G_{k+\frac{1}{2},j} = G(x_k + \frac{1}{2}\Delta x, y_j), \text{ etc.}
\]

(2.3)

Let \(V \in S_h\); we denote the usual forward, backward, and centered difference quotients denoted by subscripts in the following manner

\[
[V_x]_{k,j} = \frac{1}{\Delta x}[V_{k+1,j} - V_{k,j}],
\]

(2.4a)

\[
[V_x]_{k,j} = \frac{1}{\Delta x}[V_{k,j} - V_{k-1,j}],
\]

(2.4b)

5
\[ [V_{\bar{z}}]_{k,j} = \frac{1}{2\Delta x} [V_{k+1,j} - V_{k-1,j}] . \] (2.4c)

with similar notation for difference quotients in the y directions. Let \( T_x, T_y \) denote the shift operators

\[ [T_x V]_{k,j} = V_{k+1,j}, \quad [T_y V]_{k,j} = V_{k,j+1} , \] (2.5a)

\[ [T_x^{-1} V]_{k,j} = V_{k-1,j}, \quad [T_y^{-1} V]_{k,j} = V_{k,j-1} . \] (2.5b)

With this notation, we are able to describe finite difference operators that correspond to the differential operator \( L_k \).

Let

\[ \tilde{a}(x, y) = a(x + \frac{1}{2}\Delta x, y) \] (2.6a)

\[ \tilde{c}(x, y) = c(x, y + \frac{1}{2}\Delta y) \] (2.6b)

and for \( U \in S_h \) we define

\[ [L_{0,h} U] = -\{(\tilde{a}U_x)_x + (bU_\bar{z})_y + (bU_\bar{y})_x + (\tilde{c}U_y)_y \} \] (2.7a)

\[ [L_{K,h} U] = [L_{0,h} U] + K^\sigma \{dU_\bar{z} + eU_\bar{y} \} + KpU . \] (2.7b)

One can think of (2.7b) as the “natural centered-difference” approximation of \( L_k \). However, there are other reasonable and useful difference approximations. Let

\[ [A_{1,h} U] = -\{aU_{xx} + 2bU_{x\bar{y}} + cU_{y\bar{y}} \} , \] (2.8a)

and

\[ [L^{(1)}_{K,h} U] = [A_{1,h} U] + \left( K^\sigma d - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} \right) U_x \]

\[ + \left( K^\sigma e - \frac{\partial c}{\partial y} - \frac{\partial b}{\partial x} \right) U_y + KpU . \] (2.8b)

Finally, we may also make use of the operator \( L^{(2)}_{K,h} \) given by

\[ [L^{(2)}_{K,h} U] = [L_{0,h} U] + \frac{K^\sigma}{2} \{(dU)_x + dU_\bar{z} + (eU)_y + eU_\bar{y} \} \]

\[ + \left( Kp - \frac{K^\sigma}{2} \frac{\partial d}{\partial x} - \frac{K^\sigma}{2} \frac{\partial e}{\partial y} \right) U . \] (2.9)
Each of these difference operators has certain advantages. For example the operator $L_{K,h}^{(2)}$ clearly exhibits its symmetric and skew-symmetric parts. These are

$$S_{K,h}U = L_{0,h}U + \left(Kp - \frac{K^2}{2} \frac{\partial d}{\partial x} - \frac{K^2}{2} \frac{\partial e}{\partial y}\right)U \tag{2.10a}$$

$$\tilde{S}_{K,h}U = \frac{K^2}{2} \left\{(dU)_x + dU_x + (eU)_y + eU_y\right\} \tag{2.10b}$$

We now introduce some norms and semi-norms defined on $S_h$. For every $V \in S_h$ we set

$$\|V\|_g = \left[\Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} |V_{k,j}|^2\right]^{\frac{1}{2}}, \tag{2.11a}$$

$$|V|_{g,1} = \left[\Delta x \Delta y \sum_{k=0}^{p} \sum_{j=0}^{\ell} [(V_x)^2_{k,j} + (V_y)^2_{k,j}]\right]^{\frac{1}{2}}, \tag{2.11b}$$

$$|V|_{g,2} = \{\|V_x\|_g^2 + 2\|V_{x,y}\|_g^2 + \|V_{y,y}\|_g^2\}^{\frac{1}{2}}, \tag{2.11c}$$

$$\|V\|_{g,1} = \{\|V\|_g^2 + |V|_{g,1}^2\}^{\frac{1}{2}}, \tag{2.11d}$$

$$\|V\|_{g,2} = \{\|V\|_{g,1}^2 + |V|_{g,2}^2\}^{\frac{1}{2}}. \tag{2.11e}$$

Finally, if $B_h$ is a linear operator acting in $S_h$, we define

$$\|B_h\|_g = \max_{0 \neq V \in S_h} \frac{\|B_hV\|_g}{\|V\|_g}.$$  

**Remark 2.2.** The operators in (2.7)-(2.9) may be defined analogously with the first order centered differences replaced by forward or backward differences (e.g. $dU_x$ replaced by $dU_x$, etc.). This is often the case with various singular perturbation models. It can be seen that the results of this paper (in particular, Theorems 4.1-4.4) hold using analogous arguments.
3. Preliminaries II: Estimates

In this section we collect some basic estimates.

**Lemma 3.1:** Let $V \in S_h$. Let $A_{1,h}$ be given by (2.8a). Let $0 < q \leq Q$ be the constants of (1.5c). Then

$$|V|_{l,2}^2 \leq \frac{2Q^2}{q^4} \|A_{1,h}V\|_{l}^2. \quad (3.1)$$

**Proof:** See Lemma A.1 of [8] and [12].

**Lemma 3.2:** Let $V \in S_h$. Let $L_{0,h}$ be given by (2.7a). Then, there are constants $M > 0$, $h_0 > 0$ such that, for $0 < h \leq h_0$ we have

$$|V|_{l,1}^2 \leq \frac{2}{q} \left[ \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} [L_{0,h}V]_{k,j}[V]_{k,j} \right]. \quad (3.2)$$

**Proof:** We give a rather detailed proof because this result seems to be a "folk theorem" but we have found no complete proof.

Let us observe that

$$q \leq \min \{a(x,y), c(x,y) : (x,y) \in \overline{\Omega}\}. \quad (3.3)$$

A straightforward summation-by-parts argument gives

$$J := \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} [L_{0,h}V]_{k,j}[V]_{k,j} = I_1 + 2I_2 + I_3 \quad (3.4)$$

where

$$I_1 := \Delta x \Delta y \sum_{k=0}^{p} \sum_{j=1}^{\ell} (\bar{a})_{k,j}(V_x)^2_{k,j}, \quad (3.5a)$$

$$I_2 := \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} [bV_x]_{k,j}, \quad (3.5b)$$

$$I_3 := \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=0}^{\ell} [\bar{c} (V_y)^2]_{k,j}. \quad (3.5c)$$
First let us rewrite $I_2$ in terms of $V_x$, $V_x^2$, $V_y$, $V_y^2$. We have

$$I_2 = \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} b [V_x V_y + V_x V_y^2 + V_x V_y + V_y V_y] .$$

(3.6)

Turning to $I_1$ and $I_3$ we write

$$\tilde{a} = a + \frac{\Delta x}{2} \left( \frac{\partial a}{\partial x} \right)' , \quad \tilde{a} = (T_x a) - \frac{\Delta x}{2} \left( \frac{\partial a}{\partial x} \right)'' .$$

(3.7)

Thus we have

$$I_1 = \frac{1}{2} \Delta x \Delta y \sum_{j=1}^{\ell} [a_{0,j}(V_x)_{0,j}^2 + a_{p+1,j}(V_x)_{p+1,j}^2]$$

$$+ \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} a [(V_x^2) + (V_y^2)] + F_1 ,$$

(3.8a)

$$I_3 = \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} [c_{k,0}(V_y)_{k,0}^2 + c_{k,\ell+1}(V_y)_{k,\ell+1}^2]$$

$$+ \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} c [(V_y^2) + (V_y^2)] + F_2 ,$$

(3.8b)

where

$$|F_1| + |F_2| \leq M_\alpha |V|_{g,1}^2 .$$

(3.8c)

Collecting these formulae yields

$$J = \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} [a(V_x)^2 + 2bV_x V_y + c(V_y)^2]$$

$$+ \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} [a(V_x)^2 + 2bV_x V_y + c(V_y)^2]$$

$$+ \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} [a(V_x)^2 + 2bV_x V_y + c(V_y)^2]$$

$$+ \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} [a(V_x)^2 + 2bV_x V_y + c(V_y)^2]$$

$$+ \frac{1}{2} \Delta x \Delta y \sum_{j=1}^{\ell} [a_{0,j}(V_x)_{0,j}^2 + a_{p+1,j}(V_x)_{p+1,j}^2]$$

$$+ \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} [c_{k,0}(V_y)_{k,0}^2 + c_{k,\ell+1}(V_y)_{k,\ell+1}^2] + F_1 + F_2 .$$

(3.9)
Observe that

\[(V_{x})_{k,j} = (V_{x})_{k-1,j}\]  \hspace{1cm} (3.10a)

and

\[a_{0,j} \geq q, \quad a_{p+1,j} \geq q, \quad c_{k,0} \geq q, \quad c_{k,e+1} \geq q.\]  \hspace{1cm} (3.10b)

Thus using (1.5c) we obtain

\[q |V|_{g,1}^{2} \leq J + |F_{1}| + |F_{2}|.\]  \hspace{1cm} (3.11)

That is

\[q |V|_{g,1}^{2} \leq J + Mh |V|_{g,1}^{2}.\]

Thus, the lemma follows for \( h_{0} = \frac{q}{2M} \).

**Lemma 3.3**: There are linear operators \( \hat{E}_{h}, \tilde{E}_{h} \) defined on \( S_{h} \) such that

\[\begin{align*}
L_{K,h}U &= L^{1}_{K,h}U + \hat{E}_{h}U \hspace{1cm} (3.12a) \\
L_{K,h}U &= L^{(2)}_{K,h}U + \tilde{E}_{h}U \hspace{1cm} (3.12b)
\end{align*}\]

and

\[\|\hat{E}_{h}U\|_{g} \leq M \|U\|_{g,1}, \quad \|\tilde{E}_{h}U\|_{g} \leq M (1 + K^\sigma)h \|U\|_{g,1}.\]  \hspace{1cm} (3.13)

**Proof**: These estimates follow from the estimates and formulae of lemma A.3 of [8]. For estimate (3.13) we require that \( d \) and \( e \) have bounded second derivatives.

**Lemma 3.4**: There are linear operators \( F_{h}, F_{h}^{(1)} \) defined on \( S_{h} \) such that: for every real \( \alpha \) and every \( V \in S_{h} \) we have

\[e^{-\alpha x}[L_{K,h}(e^{\alpha x}V)] = [L_{0,h}V] + K^{\sigma}[dV_{x} + eV_{g}] + [Kp + K^{\sigma}d\alpha]V + F_{h}V\]  \hspace{1cm} (3.14a)

\[e^{-\alpha x}[L_{K,h}^{(1)}(e^{\alpha x}V)] = [L_{0,h}V] + K^{\sigma}[dV_{x} + eV_{g}] + [Kp + K^{\sigma}d\alpha]V + F_{h}^{(1)}V\]  \hspace{1cm} (3.14b)

and

\[\|F_{h}V\|_{g} \leq M(1 + K^\sigma \alpha^{4}h)\|V\|_{g,1}\]  \hspace{1cm} (3.15a)
\[ \| F_h^{(1)} V \|_g \leq M(1 + K^\sigma \alpha^4 h) \| V \|_{g,1}. \] (3.15b)

**Proof:** These estimates follow from Lemma 3.3 and several straightforward computations. For example, it is easy to verify that

\[ e^{-\alpha x} [a(e^{\alpha x} V)_x]_z = (a V_x)_z + 0((\alpha^2 + \alpha^4 h) \| V \|_{g,1}), \] (3.16a)

\[ e^{-\alpha x} (e^{\alpha x} V)_z = V_z + \alpha V + 0((\alpha^2 + \alpha^4 h) \| V \|_{g,1}). \] (3.16b)

**Corollary:** There are linear operators \( \tilde{F}_h \), \( \tilde{F}_h^{(1)} \) defined on \( S_h \) such that: for every real \( \alpha \) and every \( V \in S_h \) we have

\[ e^{-\alpha x}[L_{K,h}(e^{\alpha x} V)] = L_{0,h} V + \frac{1}{2} K^\sigma [dV_z + (dV)_z] \]
\[ + \frac{1}{2} K^\sigma [eV_y + (eV)_y] + [K_p + K^\sigma d\alpha] V \] (3.17a)
\[ + F_h V + K^\sigma \tilde{F}_h V, \]

\[ e^{-\alpha x}[L_{K,h}^{(1)}(e^{\alpha x} V)] = L_{0,h} V + \frac{1}{2} K^\sigma [dV_z + (dV)_z] \]
\[ + \frac{1}{2} K^\sigma [eV_y + (eV)_y] + [K_p + K^\sigma d\alpha] V \] (3.17b)
\[ + F_h^{(1)} V + K^\sigma \tilde{F}_h^{(1)} V, \]

and

\[ \| \tilde{F}_h V \|_g \leq M(h \| V \|_{g,1} + \| V \|_g), \quad \| \tilde{F}_h^{(1)} V \|_g \leq M(h \| V \|_{g,1} + \| V \|_g). \] (3.17c)

**Proof:** Direct computation based on Lemma 3.4. \( \blacksquare \)
4. Preconditioning Estimates

In this section we are concerned with the condition numbers of families \( \{A_hB_h^{-1}\} \) and \( \{B_h^{-1}A_h\} \) where \( A_h \) denotes one of \( L_{k,h}, L_{k,h}^{(1)} \) or \( L_{k,h}^{(2)} \) and \( B_h \) is a suitable chosen invertible operator defined on \( S_h \). We first discuss the families \( \{A_hB_h^{-1}\} \) and then obtain results for the families \( \{B_h^{-1}A_h\} \) by the use of the adjoint relationships. One major tool is the following concept.

**Definition:** Let \( A_h, B_h \) be defined on \( S_h \) and assume that both are invertible. We say that \( \{A_h\} \) is uniformly norm equivalent to \( \{B_h\} \) if there exist positive constants \( 0 < \alpha < \beta \) such that,

\[
\alpha \|B_hU\|_g \leq \|A_hU\|_g \leq \beta \|B_hU\|_g , \quad \forall U \in S_h , \tag{4.1}
\]

\( \alpha \) and \( \beta \) are independent of \( h \). This concept was used in [10] and studied in depth in [8]. In the cases of interest in this work, \( A_h \) and \( B_h \) will depend on the parameter \( K >> 0 \). In some of these cases it is not possible to find \( \alpha \) and \( \beta \) independent of \( K \). Hence we will have occasion to deal with the case where \( \alpha \) and \( \beta \) depend on \( K \).

**Lemma 4.1:** Suppose (4.1) holds. Then

\[
\|A_hB_h^{-1}\|_g \leq \beta , \tag{4.2a}
\]

\[
\|B_hA_h^{-1}\|_g \leq \frac{1}{\alpha} , \tag{4.2b}
\]

so that

\[
C(A_hB_h^{-1}) \leq \beta/\alpha . \tag{4.3}
\]

**Proof:** To obtain (4.2a) we set \( U = B_h^{-1}V \). To obtain (4.2b) we set \( U = A_h^{-1}V \). Since \( (A_hB_h^{-1})^{-1} = B_hA_h^{-1} \) we obtain (4.3) from (4.2a), (4.2b).

In the work that follows \( B_h \) will either be another discrete elliptic operator \( \tilde{L}_{K,h} \) or an operator taken from a family \( \{B_h\} \) which is uniformly norm equivalent to \( \tilde{L}_{K,h} \), with respect to \( K \) as well as \( h \).
Lemma 4.2: Let $B_h$ be a discrete elliptic operator of the form $L_{K,h}$, $L_{K,h}^{(1)}$ or $L_{K,h}^{(2)}$. Let $\sigma = 1$. Assume that $d(x,y) \geq d_0 > 0$ and $Kh \leq M_0$. Then there is a $K_0 > 0$ and an $h_0 > 0$ such that, for all $K \geq K_0$ and all $h$, $0 < h \leq h_0$ we have the following estimates

\[ \|B_h^{-1}\|_g \leq M/K \]  
(4.4a)

\[ |B_h^{-1}f|_{g,1} \leq MK^{-\frac{1}{2}}\|f\|_g \]  
(4.4b)

\[ |B_h^{-1}f|_{g,2} \leq M(1 + K^{\frac{1}{2}})\|f\|_g \]  
(4.4c)

Proof: Let

\[ B_hU = f. \]  
(4.5)

Let $U = e^{\alpha x}V$, where the positive number $\alpha$ will be determined later. Invoking the Corollary to Lemma 3.4 we see that there is a constant $M_0$, depending only on the coefficients of $L_{K,h}$ but not on $h$ or $K$ so that

\[ L_{0,h}V + \frac{1}{2}K [dV_x + (dV)_x + eV_y + (eV)_y] \\
+ K [p + \alpha d]V = e^{-\alpha x}f + Q_hV \]  
(4.6a)

where

\[ \|Q_hV\|_g \leq M_0(1 + \alpha^4)\|V\|_{g,1} + KM_0\|V\|_g. \]  
(4.6b)

Multiplying by $V$ and summing over $\Omega_h$ we use Lemma 3.2 to obtain

\[ \frac{q}{2} |V|^2_{g,1} + K [\alpha d_0 - \|p\|_\infty]\|V\|_g^2 \leq \|f\|_g \cdot \|V\|_g \\
+ M_0(1 + \alpha^4)\|V\|_{g,1}\|V\|_g + KM_0\|V\|_g^2. \]  
(4.7)

Let

\[ \alpha = \alpha_0 = \frac{\|p\|_\infty + M_0 + 2}{d_0}. \]  
(4.8a)

Then (4.7) and elementary inequalities yield

\[ \frac{q}{2} |V|^2_{g,1} + 2K\|V\|_g^2 \leq \|f\|_g \cdot \|V\|_g + M_1\|V\|_g^2 + \frac{q}{4} |V|^2_{g,1} \]  
(4.9)

where

\[ M_1 = \frac{M_0^2(1 + \alpha_0^4)^2}{q} + M_0(1 + \alpha_0^4). \]
Thus, for $K \geq M_1$ we have

$$
\frac{q}{4} |V|_{g,1}^2 + K \|V\|_g^2 \leq \|f\|_g \cdot \|V\|_g
$$

(4.10)

which immediately implies

$$
\|U\|_g \leq e^{\alpha_0} \|V\|_g \leq e^{\alpha_0} K^{-1} \|f\|_g,
$$

and

$$
|U|_{g,1}^2 \leq e^{2\alpha_0}(1 + 2\alpha_0)^2 |V|_{g,1}^2 \leq e^{2\alpha_0}(1 + 2\alpha_0)^2 \frac{4}{q} \frac{1}{K} \|f\|_g^2.
$$

Thus, we have proven (4.4a) and (4.4b).

Using Lemma 3.3 and (2.8a), (2.8b) we see that

$$
A_{1,h}U + K(dU_x + eU_y) + KP = f + R_h U
$$

(4.11a)

where

$$
\|R_h U\|_g \leq M \|U\|_{g,1}.
$$

(4.11b)

Using (4.4a) and (4.4b) we see that

$$
A_{1,h}U = F
$$

where

$$
\|F\|_g \leq (1 + K^{\frac{1}{2}}) M \|f\|_g.
$$

Using Lemma 3.1 we see that 4.4c holds. ~

**Remark**: The estimates (4.4) remain valid if the assumption $d(x, y) \geq d_0 > 0$ is replaced by $-d(x, y) \geq d_0 > 0$, or $e(x, y) \geq e_0 > 0$, or $-e(x, y) \geq e_0 > 0$. In the first case $\alpha_0$ is replaced by the negative of (4.8a). In the latter two cases $e^{\alpha x}$ is replaced by $e^{\alpha y}$.

**Lemma 4.3**: Let $B_h$ be a discrete elliptic operator of the form $L_{K,h}$, $L_{K,h}^{(1)}$ or $L_{K,h}^{(2)}$. Assume that $K \leq M_0$. Let $\sigma < 1$ and $p(x, y)$ satisfy

$$
p(x, y) \geq p_0 > 0.
$$

(4.12)
Then there is a $K_0 > 0$ and an $h_0 > 0$ such that, for all $K \geq K_0$ and all $0 < h \leq h_0$ we have the estimates (4.4a), (4.4b) and

$$\|B_h^{-1}f\|_{g,2} \leq M(1 + K^{\sigma - \frac{1}{2}})\|f\|_g.$$  \hfill (4.4c')

**Proof:** As before, let

$$B_h U = f.$$  \hfill (4.5)

Using Lemma 3.3, we see that

$$L_{K,1}^{(2)} U = f + Q_h U$$  \hfill (4.13a)

where

$$\|Q_h U\|_g \leq M(1 + K^{\sigma h})\|U\|_{g,1}.$$  \hfill (4.13b)

Multiplying by $U$ and summing over $\Omega_h$ we use Lemma 3.2 to obtain

$$\frac{q}{2}|U|_{g,1}^2 + K\|U\|_g^2 \leq \|f\|_g \cdot \|U\|_g$$

$$+ MK^{\sigma}\|U\|_g^2 + M(1 + K^{\sigma h})\|U\|_{g,1} \cdot \|U\|_g,$$

Thus, for $h$ small enough we have

$$\frac{q}{4}|U|_{g,1}^2 + (\frac{1}{2}K\|Q_h - M(1 + K^{\sigma})\|U|_g^2 \leq \|f\|_g \|U\|_g,$$

and the estimates follow as in the previous lemma.

In Lemma 4.2 and Lemma 4.3 we were concerned with cases in which the symmetric part of $B_h$ is positive definite (for $K$ large enough). We now turn to the indefinite case.

**Lemma 4.4:** Let $B_h$ be a discrete elliptic operator of the form $L_{K,1}^{(2)}$, $L_{K,1}^{(1)}$ or $L_{K,1}^{(2)}$ with

$$Kh \leq M_0.$$ 

Suppose

$$\|B_h^{-1}\|_g \leq M_1.$$  \hfill (4.14)

Then there is an $h_0$, such that for all $h$, $0 < h \leq h_0$, we have

$$|B_h^{-1}f|_{g,1} \leq M(1 + K^{\frac{1}{2}})\|f\|_g \|B_h^{-1}\|_g$$  \hfill (4.15a)
\[ |B_h^{-1} f|_{g,2} \leq M[1 + (K^\sigma + \frac{1}{2} + K)\|B_h^{-1}\|_g] \cdot \|f\|_g \, . \quad (4.15b) \]

**Proof:** Let \( B_h U = f \). Then Using Lemma 3.3

\[ L_{k,h}^{(2)} U = f + Q_h U \quad (4.16a) \]

where

\[ \|Q_h U\|_g \leq M(1 + K^\sigma h)\|U\|_{g,1} \, . \quad (4.16b) \]

Multiplying by \( U \) and using Lemma 3.2, we have

\[ \frac{q}{2} |U|_{g,1}^2 \leq K\|U\|_g^2 + \|f\|_g \cdot \|U\|_g + M\|U\|_{g,1} \cdot \|U\|_g \, . \]

So that

\[ \frac{q}{4} |U|_{g,1}^2 \leq (K + \frac{M}{q})\|U\|_g^2 + \|f\|_g \cdot \|U\|_g \, . \]

Hence (4.15a) holds.

Using Lemma 3.3 and (2.8) we see that

\[ A_{1,h} U = f + R_h U \quad (4.17a) \]

where

\[ \|R_h U\|_g \leq M[K^\sigma \|U\|_{g,1} + K\|U\|_g] \, . \quad (4.17b) \]

Hence, Lemma 3.1 and (4.15a) yields (4.15b).

**Definition:** We say \( A_h \) is of class I if the estimates (4.4) hold. That is

\[ \|A_h^{-1}\|_g \leq M/K \, , \quad |A_h^{-1} f|_{g,1} \leq MK^{-\frac{1}{2}} \|f\|_g \quad (4.18a) \]

\[ |A_h^{-1} f|_{g,2} \leq M(1 + K^{\sigma - \frac{1}{2}})\|f\|_g \, . \quad (4.18b) \]

**Theorem 4.1:** Let \( A_h \) be of class I. Let

\[ B_h = -\Delta_h + K \, . \quad (4.19) \]

That is

\[ B_h V = -(V_{xx} + V_{yy}) + KV \, , \]
and $B_h$ satisfies the boundary conditions of $A_h$. Then

$$C(A_h B_h^{-1}) = \|A_h B_h^{-1}\|_g \cdot \|B_h A_h^{-1}\|_g \leq M(1 + K^\sigma - \frac{1}{2})^2.$$  \hspace{1cm} (4.20)

**Proof:** Use (4.18a), (4.18b) and apply Lemma 4.3 to $B_h$. \hfill \Box

**Theorem 4.2:** Let $A_h$ be an operator which satisfies the estimates (4.14) and (4.15). That is

$$\|A_h^{-1}\|_g \leq M_1$$  \hspace{1cm} (4.21a)

$$|A_h^{-1} f|_{g,1} \leq M M_1 K^\frac{1}{2} \|f\|_g$$  \hspace{1cm} (4.21b)

$$|A_h^{-1} f|_{g,2} \leq M[1 + (K^\sigma + \frac{1}{2} + K)M_1] \|f\|_g.$$  \hspace{1cm} (4.21c)

Let $B_h$ be the operator given by (4.19). Then

$$C(A_h B_h^{-1}) \leq M(1 + K^\sigma - \frac{1}{2})[1 + (K^\sigma + \frac{1}{2} + K)M_1].$$

**Proof:** Apply lemma 4.3 to $B_h$.

We now turn to estimates on $C(B_h^{-1} A_h)$. We observe that

$$\|B_h^{-1} A_h\|_g = \|(B_h^{-1} A_h)^*\|_g = \|A_h^*(B_h^{-1})\|_g,$$  \hspace{1cm} (4.22a)

$$\|A_h^{-1} B_h\|_g = \|(A_h^{-1} B_h)^*\|_g = \|B_h (A_h^{-1})^*\|_g.$$  \hspace{1cm} (4.22b)

However, the representation $L_{k,h}^{(2)}$ shows that $A_h^*$ satisfies exactly the same estimates as $A_h$. Hence we immediately obtain the following results which mirror theorems 4.1 and 4.2.

**Theorem 4.3:** Let $A_h$ be of class I. Let $B_h$ be given by (4.19). Then

$$C(B_h^{-1} A_h) \leq M(1 + K^\sigma - \frac{1}{2})^2.$$  \hspace{1cm} (4.23)

**Theorem 4.4:** Let $A_h$ be an operator which satisfies the estimates (4.21). Let $B_h$ be the operator given by (4.19). Then

$$C(B_h^{-1} A_h) \leq M (1 + K^\sigma - \frac{1}{2})[1 + (K^\sigma + \frac{1}{2} + K)M_1].$$  \hspace{1cm} (4.24)
5. An Alternative Method for Convection - Diffusion Type Models

Suppose that $A_h$ is of class I (see estimates (4.18)) and let $B_h$ be given by (4.19). If $\sigma \leq \frac{1}{2}$, it follows from Theorems 4.1 and 4.3 that

$$C(A_hB_h^{-1}) = O(1) \text{ and } C(B_h^{-1}A_h) = O(1) \text{ as } K \to \infty.$$  \hspace{1cm} (5.1)

On the other hand, these condition numbers can increase with $K$ if $\sigma > \frac{1}{2}$. This includes the important case of convection-diffusion models for which $\sigma = 1$.

In this section, we assume that $\sigma > \frac{1}{2}$ and show how to replace problem (1.5) by one of the same form for which (5.1) holds. Suppose that either

(i) $d(x, y) \geq d_0 > 0$ or (ii) $-d(x, y) \geq d_0 > 0$ in $\Omega$ \hspace{1cm} (5.2a)

or

(ii) $e(x, y) \geq e_0 > 0$ or (ii) $-e(x, y) \geq e_0 > 0$ in $\Omega$. \hspace{1cm} (5.2b)

If (5.2a) holds, set

$$U = e^{-\alpha K^\sigma x} u$$ \hspace{1cm} (5.3a)

and

$$F = e^{-\alpha K^\sigma x} f.$$ \hspace{1cm} (5.4a)

If (5.2b) holds, set

$$U = e^{-\alpha K^\sigma y} u$$ \hspace{1cm} (5.3b)

and

$$F = e^{-\alpha K^\sigma y} f.$$ \hspace{1cm} (5.4b)

It is easily seen that (1.5a) is transformed into the following equation assuming either (5.2a)-(5.4a) or (5.2b)-(5.4b) holds:

$$\hat{L}_K U = -\left\{ \frac{\partial}{\partial x} \left( a \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( b \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial x} \left( b \frac{\partial U}{\partial y} \right) + \frac{\partial}{\partial y} \left( c \frac{\partial U}{\partial y} \right) \right\}$$

$$+ K^{\sigma} \left( \delta \frac{\partial U}{\partial x} + \epsilon \frac{\partial U}{\partial y} \right) + K^{2\sigma} \rho U = F$$ \hspace{1cm} (5.5)
with suitable functions \( \hat{d}, \hat{e} \), and \( \hat{p} \). Furthermore, we assume that the boundary conditions for \( U \) take the form
\[
U \text{ or } \frac{\partial U}{\partial n} = 0
\]
along an entire side.

Now suppose we show that \( \alpha \) can be chosen such that
\[
\hat{p} \geq \hat{p}_0 > 0 \text{ for } K \text{ sufficiently large}. \tag{5.7}
\]

Then if we discretize \( \hat{L}_K \) using one of the difference schemes, (2.7), (2.8), or (2.9) and replace \( K \) by \( K' = K^{2\sigma} \), it follows from Lemma 4.3 that the resulting discrete operator, \( A_h \), is of class I for \( K \) sufficiently large. Hence we may apply Theorems 4.1 and 4.3 to see that (5.1) holds. We may thus iteratively solve for the approximate solution, \( U^h \), of (5.5) with the number of iterations bounded as \( K \to \infty \). The approximate solution, \( u^h \), of (1.1) is now given by either \( u^h = e^{K^\sigma} U^h \) or \( u^h = e^{K^\sigma} U^h \). Hence our goal is to prove that (5.7) holds.

**Theorem 5.1:** Suppose that either (5.2a)-(5.4a) or (5.2b)-(5.4b) holds. Then \( \alpha \) may be chosen such that (5.7) holds.

**Proof:** It suffices to assume (5.2a) since analogous arguments hold assuming (5.2b). In view of (5.3a) and (5.3b), we see using a straightforward calculation that (5.5) holds with
\[
\hat{p} = ad - \alpha^2 a + pK^{1-2\sigma} - \alpha \frac{\partial a}{\partial x} K^{-\sigma}. \tag{5.8}
\]

Now set
\[
\alpha = \beta = \frac{d_0}{2\|a\|_{L^\infty(\Omega)}} \text{ if condition (i) in (5.2a) holds} \tag{5.9a}
\]
and
\[
-\alpha = \beta = \frac{d_0}{2\|a\|_{L^\infty(\Omega)}} \text{ if condition (ii) in (5.2a) holds} \tag{5.9b}
\]

It is readily seen using (5.8) and (5.9) that
\[
\hat{p} \geq \beta d_0 - \beta^2 \|a\|_{L^\infty} - (K^{1-2\sigma} \|p\|_{L^\infty(\Omega)} + \beta K^{-\sigma} \|\frac{\partial a}{\partial x}\|_{L^\infty(\Omega)})
\]
\[
= \frac{d_0^2}{4\|a\|_{L^\infty(\Omega)}} - \left(K^{1-2\sigma} \|p\|_{L^\infty} + \frac{K^{-\sigma} d_0 \|\frac{\partial a}{\partial x}\|_{L^\infty(\Omega)}}{2\|a\|_{L^\infty(\Omega)}}\right).
\]
Since $2\sigma > 1$, we see that (5.7) holds.

**Note:** Computational difficulties can arise using transformation (5.3) for $K$ large since it may be necessary to deal with very large or very small numbers.
References


