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BASED ON RED-BLACK GAUSS-SEIDEL SMOOTHINGS

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ABSTRACT

The $MGR[\nu]$ algorithms of Ries, Trottenberg and Winter, the algorithms 2.1 and 6.1 of Braess and the Algorithm 4.1 of Verfürth are all multigrid algorithms for the solution of the discrete Poisson equation (with Dirichlet boundary conditions) based on red-black Gauss-Seidel smoothing. Both Braess and Verfürth give explicit numerical upper bounds on the rate of convergence of their methods in convex polygonal domains.

In this work we reconsider these problems and obtain improved estimates for the $h-2h$ Algorithm 4.1 as well as W-cycle estimates for both schemes in non-convex polygonal domains. The proofs do not depend on the strengthened Cauchy inequality.

1. Introduction

The $MGR[\nu]$ algorithms of Ries, Trottenberg and Winter [12], the algorithms 2.1 and 6.1 of Braess [2], [3] and the algorithm 4.1 of Verfürth [14] are all multigrid algorithms for the solution of the discrete Poisson equation (with Dirichlet boundary conditions) based on red-black Gauss-Seidel smoothing.

The analysis of [12] is based on Fourier Analysis and is limited to the case where Ω is a rectangle. The discussion in [2], [3] and [14] is for the case where Ω is a convex polygonal domain whose boundaries lie on the horizontal, vertical or diagonal lines of a uniform grid. Recently Kamowitz and Parter [5] and Parter [11] described a variant of the basic algorithm for the general diffusion equation

$$-\nabla \cdot p \nabla u = f \tag{1}$$

with Dirichlet boundary conditions.

One interesting point about all these papers is the fact that the error estimates are specific numbers. That is, if ε represents the error before a multigrid cycle and $\bar{\varepsilon}$ the error after that cycle, then these papers provide estimates of the form

$$\|\bar{\varepsilon}\|_A \leq \rho \|\varepsilon\|_A \tag{2}$$

where ρ is a specific number, e.g. $\frac{1}{2}, \frac{1}{3}, \frac{2}{27}$ etc. Such results should be contrasted with the recent general convergence results of [1], [4], [6], [7], [8], [9], [15] where

$$\rho = \frac{C}{C+k} \tag{3}$$

with k the number of smoothing steps and C a constant which is, in general, very difficult to estimate.

Clearly, such explicit numerical estimates are preferable to the general estimates (3). Nevertheless, it is worthwhile to keep in mind that the estimates of [3] and [14] are greater than Fourier Analysis estimates of [12] for the special case where Ω is a square. Moreover, experimental results [13] indicate that these estimates are far from sharp. In particular, the V-cycle results of [3] and [4] are limited to convex domains as described above while the experimental results of [13] show excellent V-cycle convergence for domains with re-entrant corners.

In this paper we reconsider the analysis of the algorithms of [3] and [14]. Their analysis is based on the following:

(i) A finite-element interpretation of the equations based on a particular triangularization of the domain.

(ii) "The strengthened Cauchy Inequality."

(iii) Certain "energy" estimates.

We use the finite-element interpretation and implicitly use another view of the process (i.e. the choice of the interpolation and projections operators I_H^h, I_h^H as described in [5] and [11]). We also rely on the basic energy estimates on smoothing of Braess [3]. However, we do <u>not</u> use the strengthened Cauchy inequality. In the case of convex domains as described above our results for algorithms 6.1 of [3] are identical with those of [3]. On the other hand our estimates for algorithm 4.1 of [14] are stronger than those of [14]. Indeed our estimates for this $h - 2h$ multigrid scheme are (essentially) the same as those for the $h - (\sqrt{2})h$ multigrid scheme of [3]. In addition we observe that while the estimate on smoothing and this approach are not strong enough to yield V-cycle convergence estimates for nonconvex domains they do give estimates for the W-cycle in such domains. This is unimportant for the $h - (\sqrt{2})h$ algorithm since the W-cycle is not efficient for such cases. However, for the $h - 2h$ algorithms the W-cycle is a viable option and it is worthwhile to see numerical estimates for its rate of convergence. In general, there are very few rigorous results for domains with re-entrant corners. Thus, these results for nonconvex polygonal domains are also of particular theoretical interest. Nevertheless, we recall the fact that the experimental evidence of [13] implies that these results are quite weak. Thus, it is important to continue to seek methods of analyses for multigrid schemes.

In section 2 we develop the finite element formulation of [3] in some detail. We do this both for the sake of completeness and because the discussions in [2], [3] and [14] are both correct and <u>terse</u>. In section 3 we describe the algorithms in detail. Finally, in section 5 we obtain the estimates. In order to avoid some of the confusion of notation we employ the "coloring" conventions of [2], [3] and [14]. Hence, it is not red-black Gauss-Seidel but black-white Gauss-Seidel.

We have chosen to limit the discussion to the Poisson equation, i.e. the constant coefficient case. Since the results depend on "energy estimates" the extension to the general diffusion equation (1) follows along the lines indicated in [5] and [11].

Finally, I must express my appreciation to Naomi Decker and David Kamowitz whose assistance and insightful discussions were a great help in the development of this work.

2. Preliminaries

Let $h_0 > 0$ be given. Let

$$R(h_0) = \{(x_k, y_j) = (kh_0, jh_0);\ k, j = 0, \pm 1, \pm 2, \ldots\} \tag{2.1}$$

be the associated mesh points in the (x, y) plane. Let

$$B(h_0) = \{(x_k, y_j) \in R(h_0);\ k + j \equiv 1 \pmod 2\}\ , \quad \text{(black points)}\ ,$$

$$W(h_0) = \{(x_k, y_j) \in R(h_0);\ k + j \equiv 0 \pmod 2\}\ , \quad \text{(white points)}\ .$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain whose sides have slope ± 1, 0 or ∞ . We assume that each corner point of Ω belongs to $W(h_0)$. Thus, the domain Ω "fits" the mesh. We define

$$\Omega_{h_0} := R(h_0) \cap \Omega\ , \tag{2.2a}$$

$$\partial\Omega_{h_0} := R(h_0) \cap \partial\Omega\ . \tag{2.2b}$$

Following Braess [2,3] we triangulate Ω as follows. At each point $(x_k, y_j) \in W(h_0) \cap \Omega_{h_0}$ we draw the four lines of slope ± 1, 0, ∞ and consider the eight triangles with vertices (x_k, y_j) and the eight nearest neighbors $(x_{k\pm 1}, y_j), (x_h, y_{j\pm 1}), (x_{k\pm 1}, y_{j\pm 1}), (x_{k\pm 1}, y_{j\mp 1})$. At each point $(x_k, y_j) \in B(h_0) \cap \Omega_{h_0}$ we draw only the two lines parallel to the coordinate axis. In this way we obtain four triangles at (x_k, y_j) whose vertices are (x_k, y_j) and the four nearest neighbors $(x_{k\pm 1}, y_j), (x_k, y_{j\pm 1})$ - see figure 1.

Figure 1: The Triangulation.

Let S_{h_0} be the space of continuous functions which are linear on each triangle of this triangulation and vanish outside Ω . These functions are piecewise differentiable and have constant gradient ∇u on each triangle. For $u, v \in S_{h_0}$ we define the inner product and norm

$$\|u\|_{h_0}^2 = \langle u, u \rangle_{h_0} , \tag{2.3a}$$

with

$$\langle u, v \rangle_{h_0} = \frac{1}{2} h_0^2 \sum (\nabla u) \cdot (\nabla v) \tag{2.3b}$$

where the sum on the right is taken over all triangles.* We also introduce the usual mesh inner product defined as follows. For each $u \in S_{h_0}$ let

$$u_{kj} = u(x_k, y_j) , \tag{2.4a}$$

and

$$(u, v)_{h_0} = h_0^2 \sum u_{kj} v_{kj} . \tag{2.4b}$$

Finally, we introduce some operators on S_{h_0}. Let

$$[L_{h_0} U]_{k,j} = \frac{1}{h_0^2} \{4U_{k,j} - U_{k-1,j} - U_{k+1,j} - U_{k,j-1} - U_{k,j+1}\} \tag{2.5}$$

$$[M_{h_0} U]_{k,j} = \frac{1}{2h_0^2} \{\alpha_{kj} U_{k,j} - U_{k+1,j+1} - U_{k+1,j-1} - U_{k-1,j+1} - U_{k-1,j-1}\} \tag{2.6}$$

where

(1) $\alpha_{kj} = 4$ if all four of the points $(x_{k+1}, y_{y+1}), (x_{k+1}, y_{j-1}), (x_{k-1}, y_{j+1}), (x_{k-1}, y_{j-1})$ belong to $\Omega_{h_0} \cup \partial\Omega_{h_0}$.

(2) $\alpha_{kj} = 4 + \phi_{kj}$ where ϕ_{kj} is the number of neighbors not in $\Omega_{h_0} \cup \partial\Omega_{h_0}$. Let

$$[W_{h_0} U]_{kj} = \frac{1}{4h_0^2} \{\beta_{kj} U_{k,j} - U_{k-2,j} - U_{k+2,j} - U_{k,j-2} - U_{k,j+2}\} \tag{2.7}$$

where β_{kj} is defined in a manner analogous to α_{kj}. That is

$$\alpha_{kj} = 4 + \text{number of } \{U_{k\pm 2,j}, U_{k,j\pm 2}\} \text{ outside } \bar{\Omega} .$$

Remark: These formulae for α_{kj} and β_{kj} in the definition of M_{h_0}, W_{h_0} are consistent with the requirement of Braess that one double the weights (in certain semi-norms) when a neighbor is outside $\bar{\Omega}$.

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* We insert the factor $\frac{1}{2}$ to keep our notation near the notation of [3] and [14].

It is convenient to introduce the "averaging operator"

$$[a(h_0)U]_{kj} = \frac{1}{4}\left[U_{k+1,j} + U_{k-1,j} + U_{k,j-1} + U_{k,j+1}\right]. \tag{2.8}$$

A basic result (which is easily verified by a direct computation) is

$$\langle u, v\rangle_{h_0} = (L_{h_0}u, v)_{h_0} . \tag{2.9}$$

Let $H_1 = \frac{1}{\sqrt{2}}\, h_0$. The mesh $R(H_1)$ includes all points of $R(h_0)$ plus the points of the form $((k+\frac{1}{2})h_0,\ (j+\frac{1}{2})h_0)$. We will describe such points as (x_σ, y_μ) with the understanding that $(\sigma,\mu) = (k,j)$ or $((k+\frac{1}{2}),\ (j+\frac{1}{2}))$, i.e. either a pair of integers or a pair of half odd integers. The points of $R(H_1)$ are viewed as points on the rotated grid (45°) and the points are separated by H_1 . The sets Ω_{H_1}, $\partial\Omega_{H_1}$ are defined in the obvious manner. We color the points of $R(H_1)$ as follows. The "old" points, i.e. those belonging to $R(h_0)$, are white - $W(H_1)$ while the "new" points are black. Hence, as before, the vertices of Ω are white. We also add the secondary color for white points as follows:

white/red $\Longleftrightarrow$ "old" black points : $B(H_0)$,

white/green $\Longleftrightarrow$ "old" white points : $W(h_0)$.

This grid is triangulated by the same algorithm (rotated by 45°) as was used to triangulate $R(h_0)$. The space S_{H_1} is now constructed analogously as are the inner products $\langle u,v\rangle_{H_1}$, $(u,v)_{H_1}$. We define

$$[a(H_1)U]_{\sigma,\mu} = \frac{1}{4}[U_{\sigma+\frac{1}{2},\mu+\frac{1}{2}} + U_{\sigma+\frac{1}{2},\mu-\frac{1}{2}} + U_{\sigma-\frac{1}{2},\mu+\frac{1}{2}} + U_{\sigma-\frac{1}{2},\mu-\frac{1}{2}}] \tag{2.10}$$

$$[L_{H_1}U]_{\sigma,\mu} = \frac{4}{H_1^2}\Big[U_{\sigma,\mu} - [a(H_1)U]_{\sigma,\mu}\Big] \tag{2.11}$$

$$[M_{H_1}U]_{\sigma,\mu} = \frac{1}{2H_1^2}[\alpha_{\sigma,\mu}U_{\sigma,\mu} - U_{\sigma+1,\mu} - U_{\sigma-1,\mu} - U_{\sigma,\mu+1} - U_{\sigma,\mu-1}] \tag{2.12}$$

$$[W_{H_1}U]_{\sigma\mu} = \frac{1}{4H_1^2}\{\beta_{\sigma,\mu}U_{\sigma,\mu} - U_{\sigma+1,\mu+1} - U_{\sigma+1,\mu-1} - U_{\sigma-1,\mu+1} - U_{\sigma-1,\mu-1}\} \tag{2.13}$$

where $\alpha_{\sigma,\mu}$ and $\beta_{\sigma,\mu}$ are given by (essentially) the same algorithm as are α_{kj} and β_{kj}.

The space $S_{h_0} \subset S_{H_1}$ and the natural injection $J_{h_0}^{H_1} : S_{h_0} \to S_{H_1}$ simply defines the identity map restricted to S_{h_0} and takes piecewise-linear function into piecewise-linear functions. The projection $J_{H_1}^{h_0} = (J_{h_0}^{H_1})^*$ where $*$ refers to the inner product $\langle \ , \ \rangle$. That is, if $u \in S_{H_1}$, $v \in S_{h_0}$ we have

$$\langle J_{H_1}^{h_0} u, v\rangle_{h_0} = \langle u, J_{h_0}^{H_1} v\rangle_{H_1} \tag{2.14}$$

However, there is another way to look at this imbedding which is related to the inner product $(\ , \ )_{h_0}$. The functions of S_{h_0}, S_{H_1} are completely determined by their values at the points of Ω_{h_0} and Ω_{H_1} respectively. Thus we can consider the projection $P_{H_1}^{h_0}$ defined by

$$(P_{H_1}^{h_0} u)_{kj} = u_{kj} \ , \quad (x_k, y_j) \in \Omega_{h_0} \ , \quad u \in S_{H_1} \ . \tag{2.15}$$

Actually, in practice we omit reference to this operator. For example, if $u \in S_{H_1}$ we write

$$L_{h_0} u \ , \quad M_{h_0} u \ \text{ or } \ W_{h_0} u$$

rather than $L_{h_0} P_{H_1}^{h_0} u$, $M_{h_0} P_{H_1}^{h_0}$ or $W_{h_0} P_{H_1}^{h_0} u$.

Let $h_1 = \left(\frac{1}{\sqrt{2}} H_1\right) = \left(\frac{1}{2}\right) h_0$. The mesh $R(h_1)$ is constructed precisely as was the mesh $R(h_0)$ - with h_1 replacing h_0. Observe that $R(H_1) \subset R(h_1)$ and we could have considered $R(h_1)$ to have been constructed from $R(H_1)$ just as $R(H_1)$ was constructed from $R(h_0)$. The points are now colored as above. That is

$$W(h_1) := R(H_1)$$

$$B(h_1) := R(h_1) \backslash R(H_1)$$

and

$$\text{White/red} := R(H_1) \backslash R(h_0) \ ,$$

$$\text{White/green} := R(h_0) \ .$$

The operators $L_{h_1}, M_{h_1}, W_{h_1}, a(h_1)$ are constructed as before.

Remark: In the notation of [3], [14]

$$(W_{H_1} u, u)_{H_1} = (|u|_{H_2, H_1})^2 \ . \quad \blacksquare \tag{2.16}$$

The process is continued to construct the spaces

$$S_{h_0} \subset S_{H_1} \subset S_{h_1} \subset \ldots S_{H_m} \subset S_{h_m}$$

and the operators $L_{h_j}, L_{H_j}, M_{h_j}, M_{H_j}, W_{h_j}, W_{H_j}$.

We close this section with some technical estimates required in our analysis.

Lemma 2.1: Let $u, v \in S_{h_0}$. Then

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_{h_0} . \tag{2.17}$$

Proof: We need only observe that each term of the sum on the right hand side, e.g. $(\nabla u) \cdot (\nabla v)$ occurs twice in the sum on the left hand side and

$$H_1^2 = \frac{1}{2} h_0^2 . \quad \blacksquare$$

Lemma 2.2: Let $h = h_j$ or H_j and let $u \in S_h$. Then

$$0 \leq (M_h u, u)_h \leq (L_h u, u)_h . \tag{2.18}$$

Proof: This estimate is actually proven in passing in [3, page 512]. However, because this estimate is central in our analysis we give the proof. For definiteness (and for consistency with the discussion in [3]) we take $h = h_j$. A direct "summation-by-parts" argument shows that

$$(L_h u, u)_h = \sum (u_{k,j+1} - u_{k,j})^2 + \sum (u_{k+1,j} - u_{k,j})^2 . \tag{2.19a}$$

Thus, in the notation of [3],

$$(L_h u, u)_h = \|u\|^2 .$$

Similarly,

$$(M_h u, u)_h = \frac{1}{2} \sum \{\theta_{k,j}(u_{k,j} - u_{k+1,j+1})^2 + \bar{\theta}_{k,j}(u_{k,j} - u_{k+1,j-1})^2\} . \tag{2.19b}$$

where $\theta_{k,j} = 1$ if both (x_k, y_j) and (x_{k+1}, y_{j+1}) are in $\bar{\Omega}$ and $\theta_{k,j} = 2$ if either (x_k, y_j) or (x_{k+1}, y_{j+1}) is not in $\bar{\Omega}$ (Note: if neither is in $\bar{\Omega}$ then $(u_{k,j} = u_{k+1,j+1} = 0.)$, and $\bar{\theta}_{k,j}$ is defined similarly. Thus, in the notation of [14]

$$(M_h u, u)_h = (|u|_{h,\sqrt{2}h})^2 .$$

Consider a square of side h whose sides are parallel to the x, y axes - see figure 2.

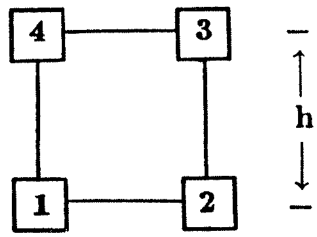

Figure 2

We suppose that at least one vertex (say "1") lies inside Ω. A direct computation yields

$$\frac{1}{2}(u_1-u_3)^2+\frac{1}{2}(u_2-u_4)^2 \leq \frac{1}{2}[(u_1-u_2)^2+(u_2-u_3)^2+(u_3-u_4)^2+(u_4-u_1)^2] \,. \quad (2.20)$$

We can think of the right-hand-side of (2.20) as being the contribution of this square to $(L_h u, u)_h$. The factor $\frac{1}{2}$ occurs because most sides parallel to the x or y axis will also appear as part of the contribution from another square. If we have a square in which all four sides will appear from another square, then the left-hand-side is precisely the contribution of this square to $(M_h u, u)_h$. If that is not the case we must modify the arguments. We deal with one case. Suppose point 3 is not in $\bar{\Omega}$. Then, since point 1 is in Ω both point 2 and point 4 are on $\partial\Omega_h$. Thus, the contribution of this square to $(M_h u, u)_h$ is $(u_1-u_3)^2=u_1^2$ and the contribution to $(L_h u, u)_h$ is $\frac{1}{2}(u_1-u_2)^2+\frac{1}{2}(u_1-u_4)^2=u_1^2$. Thus, there is a balance and the lemma is proven. ∎

The same argument yields

Lemma 2.3: Let $H=h_j$ or H_j . Let $u \in S_H$. Define

$$[\tilde{W}_H u]_{\sigma,\mu}=\begin{cases} 0, & (x_\sigma,y_\mu) \in W(H) \,, \\ [W_H u]_{\sigma,\mu}, & (x_\sigma,y_\mu) \in B(H) \,, \end{cases} \quad (2.21a)$$

$$[\tilde{M}_H u]_{\sigma,\mu}=\begin{cases} 0, & (x_\sigma,y_\mu) \in W(H) \,, \\ [M_H u]_{\sigma,\mu}, & (x_\sigma,y_\mu) \in B(H) \,, \end{cases} \quad (2.21b)$$

Then

$$(\tilde{W}_H u, u)_H \leq (\tilde{M}_H u, u)_H \,, \quad (2.22a)$$

$$(W_H u, u)_H \leq (M_H u, u)_H \; . \tag{2.22b}$$

Proof: The proof of both (2.22a) and (2.22b) follows exactly the same argument as the proof of Lemma 2.2. Actually, since $\tilde{M}_H$ and M_H also have special points at which the weights are doubled, the proof has some simplifying aspects. ∎

3. The Problems

In this section we describe the problem to be solved, several multigrid algorithms based on black/white Gauss-Seidel relaxation and the spaces and operators of section 2.

Let h denote an h_j or H_j $(j > 0)$ and let H denote the mesh size on the next coarser mesh, i.e., $H = H_j$ or h_{j-1}. Similarly $2h$ is h_{j-1} or H_{j-1}. Let $f \in S_h$ and consider the problem: find $U \in S_h$ such that

$$L_h U = f \,. \tag{3.1}$$

It is an easy matter, using (2.5) and the construction of section 2, to see that (3.1) is equivalent to the finite-element formulation

$$\langle U, v\rangle_h = (f, v)_h, \qquad \forall v \in S_h \,. \tag{3.2}$$

The black/white Gauss-Seidel relaxation is described by the half-steps G_h^b, G_h^w which are themselves described in terms of the averaging operator $a(h)$. Set

$$[G_h^b u]_{k,j} = \begin{cases} [a(h)u]_{k,j} + \frac{h^2}{4} f_{k,j}, & (x_k, y_j) \text{ black} \\ u_{k,j}, & (x_k, y_j) \text{ white} \end{cases} \tag{3.3a}$$

and

$$[G_h^w u]_{k,j} = \begin{cases} u_{k,j}, & (x_k, y_j) \text{ black} \\ [a(h)u]_{k,j} + \frac{h^2}{4} f_{k,j}, & (x_k, y_j) \text{ white} \end{cases} \tag{3.3b}$$

Let $G_{h,r}$ denote the composition of $r (r \geq 0)$ Gauss-Seidel half steps such that G_h^w and G_h^b alternate and a G_h^b operation is performed last. Let $G_{h,r}^*$ denote the execution of the same steps, but in reversed order (see [14, page 120]). Observe that $\bar{G}_h^w$ and $\bar{G}_h^b$ (the homogeneous parts of G_b^w and G_h^b respectively) are self-adjoint in the $\langle\ ,\ \rangle_h$ inner product. Therefore the homogeneous part of $G_{h,r}^*$ is the $\langle\ ,\ \rangle_h$ adjoint of the homogenous part of $G_{h,r}$.

We now describe two multigrid iterative schemes for the solution of (3.1). Let G_h be $G_{h,r}$ and $\bar{G}_h$ be its homogenous part. Let U denote the solution of (3.1) while u^0 is our current approximation for U. Set $\varepsilon_0 = U - u^0$.

Algorithm I:

(1) **Smoothing Step.** Given u^0. Compute

$$\tilde{u} = G_h u^0$$

(2) **Transfer Step.** Compute $r = f - L_h\tilde{u}$ and

$$r_H = \frac{1}{2}\, r \quad \text{on white points} .$$

(3) **Correction Step.** Let $U_H \in S_H$ denote the exact solution of

$$L_H U_H = r_H \; . \tag{3.4}$$

If $H = h_0$ set $V_H = U_H$.
If $H < h_0$ set $U_H^0 = 0$ and compute V_H by μ iterations of Algorithm I applied to (3.4).

(4) **Post smoothing step.** Set

$$\hat{u} = \tilde{u} + V_H \tag{3.5}$$

and

$$u^1 = G_h^* \hat{u} \; . \tag{3.6}$$

Repeat.

Observe that in equation (3.5) we add an element of S_H to an element of S_h. This causes no difficulty either because we think of $S_H \subset S_h$ and V_H is a function in S_H and hence a function in S_h or because it doesn't matter what values are given V_H at black points. The first step of (3.6) changes these new values without the use of those values.

Algorithm II: We assume $h = h_j$, $H = H_j$ and $j > 0$.

(1) **Smoothing Step.** Given u^0. Compute

$$\tilde{u} = G_h u^0 \; .$$

(2) **First Transfer Step.** Compute $r = f - L_h\tilde{u}$ and $r_H = \frac{1}{2}\, r$ on white points.

(3) **Intermediate Step.** Let $U_H \in S_H$ denote the exact solution of

$$L_H U_H = r_H = f_H \; . \tag{3.7a}$$

Set $U_H^0 = 0$ and, in terms of this problem compute

$$U_H^1 = G_H^b U_H^0 \ . \tag{3.7b}$$

(4) **Second Transfer Step.** Compute $\rho = r_H - L_H U_H^1$ and

$$r_{2h} = f_{2h} = \frac{1}{2}\rho \ . \tag{3.8}$$

(5) **Correction Step.** Let ϕ_{2h} denote the exact solution of

$$L_{2h}\phi_{2h} = f_{2h} \ . \tag{3.9}$$

If $2h = h_0$, set $Q_{2h} = \phi_{2h}$.
If $2h < h_0$ set $\phi_{2h}^0 = 0$ and compute Q_{2h} by μ iterations of Algorithm II applied to (3.9).

(6) **First Back Transfer.** Set

$$V_H = G_H^b(U_H^1 + Q_{2h}) \ .$$

(7) **Second Back Transfer and Post Smoothing Step.** Set

$$\hat{u} = \tilde{u} + V_H$$

and

$$u^1 = G_h^* \hat{u} \ .$$

Repeat.

Remarks: Algorithm I is Algorithm 6.1 of [3] while Algorithm II with $G_h = G_{h,r+2}$ is Algorithm 4.1 of [14]. When $\mu = 1$ we are discussing the V-cycle. When $\mu \geq 2$ we call this a W-cycle.

4. Analysis I

Both Braess [3] and Verfüth [14] use a "duality" argument to establish estimates on the rates of convergence of their multigrid schemes. This approach is also used by [1], [4], [15]. The following is our abstraction of this basic argument in a form sufficient for these problems (see [10] also).

Following McCormick and Ruge [7] we let T_h denote the $\langle\ ,\ \rangle$ orthogonal projection of S_h onto the nullspace $J_h^H L_h$ and let $\$_h$ denote the orthogonal projection onto the range of J_H^h. With each $u \in S_h$ (or equivalently, each $\bar{G}_h u$) we associate a value $\sigma(u) \in [0,1]$. Assume there are two functions $g(\sigma)$, $p(\sigma)$ defined on [0,1] and

$$\|\bar{G}_h u\|_h^2 \le g(\sigma)\|u\|_h^2 \ , \tag{4.1a}$$

$$\|T_h\bar{G}_h u\|_h^2 \le p(\sigma)g(\sigma)\|u\|_h^2 \ , \tag{4.1b}$$

$$0 \le g(\sigma) \le 1\ ,\ \ 0 \le p(\sigma) \le 1\ ,\ \ 0 \le \sigma \le 1\ . \tag{4.1c}$$

Theorem 4.1: Let ε_j be the error before a multigrid cycle and let $\bar{\varepsilon}_j$ be the error after that cycle.

Case 1: The V-cycle. Let δ , $0 \le \delta < 1$ be a fixed number which satisfies

$$(1-\delta)p(\sigma)g(\sigma) + \delta g(\sigma) \le \delta\ , \quad \sigma \in [0,1]\ . \tag{4.2a}$$

That is

$$\delta = \sup\left\{\frac{p(\sigma)g(\sigma)}{1+p(\sigma)g(\sigma)-g(\sigma)}\ ;\ \sigma \in [0,1]\right\}\ , \tag{4.2b}$$

then,

$$\|\bar{\varepsilon}_j\|_{h_j} \le \delta\|\varepsilon_j\|_{h_j}\ . \tag{4.3}$$

Case 2: The W-cycle. Let $\delta, 0 < \delta < 1$ be number which satisfies

$$(1-\delta^\mu)p(\sigma)g(\sigma) + \delta^\mu g(\sigma) \le \delta\ , \quad 0 \le \sigma \le 1\ . \tag{4.4}$$

Then

$$\|\bar{\varepsilon}_j\|_{h_j} \le \delta\|\varepsilon_j\|_{h_j}\ . \tag{4.5}$$

Proof: The general functions $p(\sigma)$, $g(\sigma)$ do not appear in the proof given in [3]. However, that proof which is based on a duality argument immediately yields this theorem. See [10] also. ▮

Remark: Consider the case $\mu = 2$ (standard W-cycle). Let

$$\eta = \max\ p(\sigma)g(\sigma)\ . \tag{4.6}$$

Then

$$\|\bar{\varepsilon}_j\|_{h_j} \leq \frac{\eta}{1-\eta}\ \|\varepsilon_j\|_{h_j}\ . \tag{4.7}$$

Remark: In order for this analysis to yield a convergence proof and/or a convergence estimate for the V-cycle it is necessary that

$$p(\sigma) = 0 \qquad \forall \sigma \,\epsilon\, D_1\ ,$$

where

$$D_1 = \{\sigma \epsilon [0,1];\ g(\sigma) = 1\}\ . \tag{4.8b}$$

Proof: Suppose there is a $\sigma_0 \,\epsilon\, D_1$ and $p(\sigma_0) \neq 0$. Then

$$\frac{p(\sigma_0)g(\sigma_0)}{1 + p(\sigma_0)g(\sigma_0) - g(\sigma_0)} = \frac{p(\sigma_0)}{p(\sigma_0)} = 1\ . \qquad ▮$$

It is this condition which limits the V-cycle analysis of this work and of [3] and [14] to the case of convex domains.

Remark: In [3] we find σ replaced by ρ and

$$g(\rho) = \rho^k\ , \quad 0 \leq \rho \leq 1\ ,$$

$$p(\rho) = \frac{1-\rho}{2-\rho}\ , \quad 0 \leq \rho \leq 1\ .$$

5. Analysis

Let h denote h_j or H_j and let H denote the mesh size on the next coarser mesh, i.e., $H = \sqrt{2}h = H_j$ or h_{j-1}. Similarly $2h = h_{j-1}$ or H_{j-1}. Let

$$N_h := \{u \in S_h, u = 0 \text{ on } W(h)\} , \tag{5.1a}$$

$$R_h^I := \{u \in S_h, u = a(h)u \text{ on } B(h)\} , \tag{5.1b}$$

$$R_h^{II} := \{u \in S_h, u = a(h)u \text{ on } W(h)\} . \tag{5.1c}$$

We will also make use of the spaces N_H, R_H^I and R_H^{II} which are defined as above with h replaced by H. Let Q^I, Q^{II} denote the projections

$$[Q^I u]_{k,j} = \begin{cases} [a(h)u]_{k,j}, & (x_k, y_j) \in B(h) , \\ u_{k,j}, & (x_k, y_j) \in W(h) . \end{cases} \tag{5.2a}$$

$$[Q^{II} u]_{k,j} = \begin{cases} u_{k,j}, & (x_k, y_j) \in B(h) , \\ [a(h)u]_{k,j}, & (x_k, y_j) \in W(h) . \end{cases} \tag{5.2b}$$

Remember that

$$Q_I^* = Q_I , \quad Q_{II}^* = Q_{II} . \tag{5.3}$$

If $G_{h,r}$ is the smoothing operator of section 3 then $\bar{G}_{h,r}$, the homogeneous part of $G_{h,r}$ is given by the alternating product

$$\bar{G}_{h,r} = \underbrace{Q^I Q^{II} \ldots Q'}_{r+1 \text{ factors}} \tag{5.4}$$

where $' = II$ if r is odd and $' = I$ if r is even.

Our first estimates are some results of Braess [3].

Lemma 5.1:

(i) If Ω is a convex polygonal domain, then

$$\|Q^{II} Q^I u\|_h^2 \leq (W_H Q^I u, Q^I u)_H . \tag{5.5a}$$

(ii) If Ω is a nonconvex polygonal domain, then

$$\|Q^{II} Q^I u\|_h^2 \leq 2(W_H Q^I u, Q^I u)_H . \tag{5.5b}$$

(iii) Let $\tilde{Q} = Q^I$ if r is odd and $\tilde{Q} = Q^{II}$ if r is even. Set

$$\varepsilon^\nu = Q^* \varepsilon^{\nu-1}, \quad \varepsilon^{\nu+1} = \tilde{Q} \varepsilon^\nu .$$

Then

$$\frac{\|\varepsilon^{\nu+1}\|_h}{\|\varepsilon^\nu\|_h} \geq \frac{\|\varepsilon^\nu\|_h}{\|\varepsilon^{\nu-1}\|_h} . \tag{5.6}$$

(iv) If Ω is a convex domain then

$$\frac{\|\bar{G}_{h,r}\varepsilon^0\|_h^2}{\|\varepsilon^0\|_h^2} \leq \left[\frac{(W_H \varepsilon^{r+1}, \varepsilon^{r+1})_H}{\|\varepsilon^{r+1}\|_h^2} \right]^r . \tag{5.7a}$$

(v) If Ω is not a convex domain then

$$\frac{\|\bar{G}_{h,r}\varepsilon^0\|_h^2}{\|\varepsilon^0\|_h^2} \leq \left\{ \max \left(\frac{2(W_H \varepsilon^{r+1}, \varepsilon^{r+1})_H}{\|\varepsilon^{r+1}\|_h^2} \, , \ 1 \right) \right\}^r . \tag{5.7b}$$

Proof: The basic estimates (5.5a), (5.5b) are actually proven in [3]. However, the statement in [3] is a weaker statement, see lemma 3.1 of [14] for the estimate (5.5a). The estimate (5.5b) follows from the same argument and the remarks on page 517 of [3]. Actually, this is a worst case estimate. For re-entrant corners of less than 135° one can find a constant smaller than 2. However, that is a tedious, lengthy calculation and we forego it. The estimate (5.6) is proven in [3]. We do not explicitly use this estimate but it is used with (5.5a), (5.5b) to establish the important estimates (5.7a), (5.7b). ∎

Lemma 5.2: Let $u \in R_h^I$. Then

$$L_h u = 0 \quad \text{on } B(h) , \tag{5.8a}$$

$$L_h u = L_H u + M_H u , \quad \text{on } W(h) \ = \ \Omega_H . \tag{5.8b}$$

Proof: Equation (5.8a) follows immediately from (5.1b) while (5.8b) follows from a direct computation (see [5]). ∎

Theorem 5.1: Let J_H^h and J_h^H be the natural injection and its adjoint as described in section 2. That is, for $u \in S_H$, $v \in S_H$ we have

$$\langle J_H^h u, v \rangle_h = \langle u, J_h^H v \rangle_H . \tag{5.9}$$

Let T and \$ be the orthogonal projections onto the nullspace $J_h^H L_h$ and the range of J_H^h . For each $u \in S_h$ let

$$\sigma = \frac{(M_H \bar{G}_{h,r}u, \bar{G}_{h,r}u)_H}{(L_H \bar{G}_{h,r}u, \bar{G}_{h,r}u)_H} . \tag{5.10}$$

Then

$$0 \le \sigma \le 1 . \tag{5.11}$$

Moreover, let

$$p(\sigma) = \frac{1}{2}(1-\sigma) . \tag{5.12a}$$

Case 1: Ω convex. Define

$$g(\sigma) = \left(\frac{2\sigma}{1+\sigma}\right)^r . \tag{5.12b}$$

Case 2: Ω non-convex. Define

$$g(\sigma) = \min\left\{\frac{4\sigma}{1+\sigma}, 1\right\}^r . \tag{5.12c}$$

Then

$$\|\bar{G}_{h,r}u\|_h^2 \le g(\sigma)\|u\|_h^2 \tag{5.13a}$$

and

$$\|T\bar{G}_{h,r}u\|_h^2 \le p(\sigma)g(\sigma)\|u\|_h^2 . \tag{5.13b}$$

Hence, in the case of Algorithm I we have the following results.

(i) If Ω is a convex domain then the V-cycle is convergent and, if ε is the error before a symmetric multigrid cycle and $\bar{\varepsilon}$ is the error after that cycle,

$$\|\bar{\varepsilon}\|_h \le \delta\|\varepsilon\|_h , \tag{5.14a}$$

where δ is given by (4.2b). Further the W-cycle is convergent with a δ satisfying (4.4). In particular

$$\delta \le \frac{p(\bar{\sigma})g(\bar{\sigma})}{1-p(\bar{\sigma})g(\bar{\sigma})} , \tag{5.14b}$$

where $p(\bar{\sigma})g(\bar{\sigma}) = \max\{p(\sigma)g(\sigma), 0 \le \sigma \le 1\}$.

(ii) If Ω is not a convex domain then we do not know if the V-cycle is convergent. However, if $r \ge 1$ the W-cycle is convergent.

Indeed (5.14a) holds with

$$\delta \leq \frac{1}{2} . \tag{5.14c}$$

Proof: Lemma 2.2 implies (5.11). Observe that $\bar{G}_{h,r}\ u\ \epsilon\ R_h^I$. Therefore, Lemma 5.2 asserts

$$\|\bar{G}_{h,r}u\|_h^2 = \frac{1}{2}\left[(L_H\bar{G}_{h,r}u, \bar{G}_{h,r}u)_H + (M_H\bar{G}_{h,r}u, \bar{G}_{h,r}u)_H\right] . \tag{5.15}$$

Thus, the estimate (5.13a) follows from (5.7a), (5.7b) and (2.22b). Once we have proven (5.13b) the convergence results contained in (5.14a), (5.14b) follow from Theorem 4.1. The estimate (5.14c) follows from the following argument. In the non-convex case

$$g(\sigma) = \begin{cases} \left(\frac{4\sigma}{1+\sigma}\right)^r , & 0 \leq \sigma \leq \frac{1}{3} , \\ 1 , & \frac{1}{3} \leq \sigma \leq 1 . \end{cases} \tag{5.16}$$

Clearly $g(\sigma)$ is bounded by the $g_1(\sigma)$ obtained for $r = 1$. Moreover, in the non-convex case, $g_1(\sigma)p(\sigma)$ assumes its maximum at $\sigma = \dfrac{1}{3}$. Hence

$$p(\bar{\sigma})g(\bar{\sigma}) \leq \frac{1}{3} .$$

Thus (5.14b) yields (5.14c).

Hence we turn our attention to the proof of (5.13b). Let

$$\bar{G}_{h,r}u = \bar{u} = T\bar{G}_{h,r}u + \$\bar{G}_{h,r}u . \tag{5.17}$$

Then, using Lemma 5.2 we see that

$$\|\bar{u}\|_h^2 = \frac{1}{2}[(L_H\bar{u}, \bar{u})_H + (M_H\bar{u}, \bar{u})_H] . \tag{5.18a}$$

Using Lemma 2.1 and (2.9) we see that $\$\bar{u}$ is determined by U_H, the solution of (3.4), and

$$\|\$\bar{u}\|_h^2 = (L_HU_H, U_H)_H . \tag{5.18b}$$

Lemma 5.2 and Step 2 of the algorithm yields

$$L_HU_H = \frac{1}{2}\left(L_H\bar{u} + M_H\bar{u}\right) . \tag{5.18c}$$

Thus

$$(L_HU_H, U_H)_H = \frac{1}{4}(L_H\bar{u} + M_H\bar{u}, \bar{u} + L_H^{-1}M_H\bar{u})_H ,$$

and

$$\|\$\bar{u}\|_h^2 = \frac{1}{4}\{(L_H\bar{u} + M_H\bar{u}, \bar{u})_H + (M_H\bar{u}, \bar{u})_H + (M_H\bar{u}, L_H^{-1}M_H\bar{u})_H\} .$$

Thus

$$\begin{aligned} \|T\bar{u}\|_h^2 &= \|\bar{u}\|_h^2 - \|\$\bar{u}\|_h^2 \\ &= \frac{1}{4}\ \{(L_H\bar{u}, \bar{u})_H - (M_H\bar{u}, L_H^{-1}M_H\bar{u})_H\} . \end{aligned} \tag{5.19}$$

Observe that

$$\begin{aligned} (M_H\bar{u}, \bar{u})_H &= (L_H^{-1}L_H\bar{u}, M_H\bar{u})_H \\ &\le (L_H^{-1}M_H\bar{u}, M_H\bar{u})_H^{\frac{1}{2}}(L_H^{-1}L_H\bar{u}, L_H\bar{u})_H^{\frac{1}{2}} . \end{aligned}$$

Thus

$$(M_H\bar{u}, L_H^{-1}M_H\bar{u})_H \ge \frac{(M_H\bar{u}, \bar{u})_H^2}{(L_H\bar{u}, \bar{u})_H} ,$$

and

$$\|T\bar{u}\|_h^2 \le \frac{1}{4}\ \left\{(L_H\bar{u}, \bar{u})_H - (M_H\bar{u}, \bar{u})_H^2 \Big/ (L_H\bar{u}, \bar{u})_H\right\} . \tag{5.20}$$

Since

$$\frac{\|T\bar{u}\|_h^2}{\|u\|_h^2} = \frac{\|T\bar{u}\|_h^2}{\|\bar{u}\|_h^2}\ \frac{\|\bar{u}\|_h^2}{\|u\|_h^2} , \tag{5.21}$$

we complete the proof of (5.13b) by noting that (5.20) and (5.10) give

$$\frac{\|T\bar{u}\|_h^2}{\|\bar{u}\|_h^2} \le \frac{1}{2}\ \frac{1-\sigma^2}{1+\sigma} = \frac{1}{2}\ (1-\sigma) . \quad \blacksquare$$

Remark: In the convex case we reobtain exactly the estimate of [3]. To see this we set

$$\rho = \frac{2\sigma}{1+\sigma} .$$

Then

$$\frac{1}{2}\ (1-\sigma) = \frac{1-\rho}{2-\rho} .$$

We now turn to the analysis of Algorithm II. Consider the equation (5.18c). Let us consider U_H and $P_h^H\bar{u} = \bar{u}$ as being written in the form

$$U_H = \xi + \tilde{U}_H, \qquad \bar{u} = \eta + \tilde{V}_H . \tag{5.22}$$

where $\xi, \eta \in N_H$ while $\tilde{U}_H$, $\tilde{V}_H \in R_H^I$.

Lemma 5.3: Consider U_H^1, the function in S_H computed in the intermediate Step (3) of Algorithm II. Then

$$U_H^1 = \xi \; . \tag{5.23}$$

Proof: Both U_H^1 and ξ are zero on $W(H)$. Hence both belong to N_H. By construction

$$L_H U_H^1 = r_H \qquad \text{on } B(H) \cap \Omega_H \tag{5.24a}$$

$$L_H(\xi + \tilde{U}_H) = r_H \qquad \text{on } B(H) \cap \Omega_H \; . \tag{5.24b}$$

But, $L_H \tilde{U}_H = 0$ on $B(H) \cap \Omega_H$. Since the equations (5.24) are explicit (with $U_H^1 = \xi = 0$ on $W(H)$) the lemma is proven.

Consider ϕ_{2h}, the solution of (3.9). While ϕ_{2h} is defined only on Ω_{2h} we may think of ϕ_{2h} as being in R_H^I. As we shall see, the values of ϕ_{2h} at points of $B(H) \cap \Omega_H$ never enter into our calculations. Observe that (3.8) yields

$$f_{2h} = \frac{1}{2}\, \rho = \frac{1}{2}\, (r_H - L_H U_H^1) = \frac{1}{2}\, L_H(U_H - U_H^1) \; .$$

Thus

$$L_{2h}\, \phi_{2h} = \frac{1}{2}\, L_H(U_H - U_H^1) = \frac{1}{2}\, L_H \tilde{U}_H \; . \tag{5.25}$$

Define the operators $L^{\#}$, $M^{\#}$, $\tilde{L}_{2h}$ as follows. For $v = \varsigma + f$ with $\varsigma \in N_H$, $f \in R_H^I$

$$L^{\#} v = \begin{cases} L_H \varsigma, & B(H) \cap \Omega_H \; , \\ L_H \varsigma + 2 L_{2h} f, & W(H) \cap \Omega_H \; . \end{cases} \tag{5.25a}$$

$$M^{\#} v = \begin{cases} M_H v, & B(H) \cap \Omega_H \; , \\ M_{2h} f, & W(H) \cap \Omega_H \; . \end{cases} \tag{5.25b}$$

$$\tilde{L}_{2h} v = \begin{cases} 0, & B(H) \cap \Omega_H \; , \\ L_{2h} f, & W(H) \cap \Omega_H \; . \end{cases} \tag{5.25c}$$

Then

$$(L^{\#} v, v)_H = (L_H \varsigma, \varsigma + f)_H + 2(\tilde{L}_{2h} f, f)_H \; , \tag{5.26a}$$

and

$$(L^{\#} v, v)_H = (L_H \varsigma, \varsigma)_H + (L_{2h} f, f)_{2h} \; . \tag{5.26b}$$

Remark: Since $\varsigma = 0$ on $W(H)$, we can replace $M_{2h}f$ by $M_{2h}v$ and $L_{2h}f$ by $L_{2h}v$ on $W(H)$. Moreover $L_H\varsigma = L_H v$ on $B(H) \cap \Omega_H$.

Lemma 5.4: Consider Algorithm II. Let $\varepsilon_0^0 = U - u^0$, $\tilde{\varepsilon} = U - \tilde{u}$. Let U_H^1 be given by (3.7b) and let ϕ_{2h} be the solution of (3.9). Then

$$L_h\tilde{\varepsilon} = L_H\tilde{\varepsilon} + M_H\tilde{\varepsilon} = L^{\#}\tilde{\varepsilon} + M^{\#}\tilde{\varepsilon} \text{ on } \Omega_H \ , \tag{5.27}$$

and

$$L^{\#}(U_H^1 + \phi_{2h}) = \frac{1}{2}\left(L^{\#}\tilde{\varepsilon} + M^{\#}\tilde{\varepsilon}\right) \text{ on } \Omega_H \ . \tag{5.28}$$

Proof: The first equality of (5.27) is merely the restatement of (5.8b). On $B(H) \cap \Omega_H$ the second equality follows from the definitions of $L^{\#}$ and $M^{\#}$, i.e. (5.25a) and (5.25b). On $W(H) \cap \Omega_H$ the second equality follows from two observations. First, lemma 5.2 with h replaced by H asserts that, if $f \in R_H^I$ then

$$L_H f = L_{2h} f + M_{2h} f \text{ on } W(H) \cap \Omega_H \ . \tag{5.29a}$$

Secondly, the definitions of section 2 show that

$$L_{2h}\tilde{\varepsilon} = M_H\tilde{\varepsilon} \text{ on } W(H) \cap \Omega_H = \Omega_{2h} \ . \tag{5.29b}$$

The proof of (5.28) follows from (5.24a), the definition of r_H, (5.25) and (5.27). ∎

Consider Algorithm II within the finite-element framework utilizing only the spaces S_{h_j} and the spaces S_{H_j} are merely intermediate tools. With this in mind T and \$ now denote the orthogonal projections onto nullspace $J_h^{2h}L_h$ and range J_{2h}^h where $J_{2h}^h = J_H^h J_{2h}^H$ and $J_h^{2h} = (J_{2h}^h)^*$. We think of the operation of the "smoother" as the application of $G_{h,r}$ and the addition of U_H^1. Thus, if G_h^{new} describes our smoother, then

$$G_h^{new}u^0 = \tilde{u} + U_H^1 \ . \tag{5.30}$$

Remark: A simple calculation shows that G_h^{new} is indeed an affine smoother. The estimate (5.37b) shows its homogeneous part $\bar{G}_h^{new}$ is of norm less than or equal to one.

Lemma 5.5: Let $u \in S_h$. Then

$$(W_H u, u)_H \le (M^{\#} u, u)_H \; . \tag{5.31}$$

Proof: Since W_H operates on points separated by $(2H)$, W_H splits into two independent operators. $\tilde{W}_H$ (see Lemma 2.3) which essentially acts on the points of $B(H)$ and another operator $\hat{W}_H$ which acts on the points of $W(H)$ and is zero on $B(H)$. However, it is easy to see that

$$\hat{W}_H = M_{2h} \quad \text{on } W(H) \; . \tag{5.32}$$

Hence,

$$(W_H u, u)_H = (\tilde{W}_H u, u)_H + (M_{2h} u, u)_H \; .$$

By Lemma 2.3 we have

$$(W_H u, u)_H \le (\tilde{M}_H u, u)_H + (M_{2h} u, u)_H = (M^{\#} u, u)_H \; . \quad \blacksquare$$

Lemma 5.6: Let $u \in S_h$. Then

$$0 \le \frac{(M^{\#} u, u)_H}{(L^{\#} u, u)_H} \le 1 \; . \tag{5.33}$$

Proof: We observe that (using definition (2.21b))

$$M_H u = \begin{cases} L_{2h} u & \text{on } W(H) \cap \Omega_H \; , \\ \tilde{M}_H u & \text{on } B(H) \cap \Omega_H \; . \end{cases} \tag{5.34}$$

Let $P_h^H u = \xi + f$. Using definition 5.29 and observing that $\xi = 0$ on $W(H)$,

$$(M^{\#} u, u)_H = (\tilde{M}_H u, u)_H + \frac{1}{2} \, (M_{2h} u, u)_{2h} \; .$$

Using Lemma 2.2 we have

$$(M^{\#} u, u)_H \le (\tilde{M}_H u, u)_H + \frac{1}{2} \, (L_{2h} u, u)_{2h} \; .$$

Using (5.34) and Lemma 2.2 again yields

$$(M^{\#} u, u)_H \le (M_H u, u)_H \le (L_H u, u)_H = (L_H \xi, \xi)_H + (L_H f, f)_H \; . \tag{5.35}$$

But

$$(L_H f, f)_H = (L_{2h} f, f)_H + (M_{2h} f, f)_H$$
$$\leq 2(L_{2h} f, f)_H$$

Substituting this estimate into (5.35) and using (5.26a) we obtain (5.33). ∎

Theorem 5.2: Consider Algorithm II. For $u \in S_h$, set

$$\sigma = \frac{(M^{\#} \bar{G}_{h,r} u, \bar{G}_{h,r} u)_H}{(L^{\#} \bar{G}_{h,r} u, \bar{G}_{h,r} u)_H} . \tag{5.36a}$$

Then Lemma 5.6 shows that

$$0 \leq \sigma \leq 1 . \tag{5.36b}$$

Define $p(\sigma)$ by (5.12a) and $g(\sigma)$ by (5.12b) and (5.12c). Then (5.13a), (5.13b) hold. Hence the conclusions of Theorem 1 also apply to this h to $2h$ algorithm.

Proof: Let $\tilde{u} = \bar{G}_{h,r} u$, let U_H^1 be the result of step of the smoothing in the homogeneous case, $f = 0$. That is

$$U_H^1 = \begin{cases} 0 & \text{on } B(h) \cup W(H) , \\ -\frac{h^2}{4} L_h \tilde{u} & \text{on } B(H) \cap \Omega_H . \end{cases} \tag{5.37a}$$

Set $\bar{u} = \tilde{u} + U_H^1 = \bar{G}_h^{new} u$. Then

$$\langle \bar{u}, \bar{u} \rangle_h = \langle \tilde{u}, \tilde{u} \rangle_h + 2\langle U_H^1, \tilde{u} \rangle_h + \langle U_H^1, U_H^1 \rangle_h$$
$$= \langle \tilde{u}, \tilde{u} \rangle_h + 2(U_H^1, L_H \tilde{u})_h + \langle U_H^1, U_H^1 \rangle_h .$$

Since we are in the homogeneous case

$$L_H U_H = r_H = -\frac{1}{2} L_h \tilde{u} \text{ on } \Omega_H .$$

Hence

$$\langle \bar{u}, \bar{u} \rangle_h = \langle \tilde{u}, \tilde{u} \rangle_h - 2(U_H^1, L_H U_H)_H + (L_H U_H^1, U_H^1)_H .$$

Using (5.24a) and the fact that $U_H^1 \in N_H$ we have

$$(U_H^1, L_H U_H)_H = (U_H^1, L_H U_H^1)_H = \langle U_H^1, U_h^1 \rangle_H .$$

Therefore

$$\langle \bar{u}, \bar{u} \rangle_h = \langle \tilde{u}, \tilde{u} \rangle_h - \langle U_H^1, U_H^1 \rangle_H \leq \langle \tilde{u}, \tilde{u} \rangle_h . \tag{5.37b}$$

Thus, (5.13a) follows immediately from (5.7a), (5.7b) and Lemma 5.5.

Turning to the proof of (5.13b) let

$$n = (L_h \bar{u}, \bar{u})_h - (L_{2h}\phi_{2h}, \phi_{2h})_{2h} \ , \tag{5.38a}$$

$$d = (L_h u, u)_h \ . \tag{5.38b}$$

Using Lemma 2.1 and 2.9 we see that

$$(L_{2h}\phi_{2h},\ \phi_{2h})_{2h} = \langle \phi_{2h},\ \phi_{2h} \rangle_{2h}$$

and hence

$$n = \|T\bar{u}\|_h^2 \ , \ \ d = \|u\|_h^2 \ .$$

Using (5.37b) and (5.26b) we see that

$$n = (L_h \tilde{u}, \tilde{u})_h - (L^{\#}(U_H^1 + \phi_{2h}),\ U_H^1 + \phi_{2h})_H \ . \tag{5.39}$$

Using (5.27) and (5.28) of Lemma 5.4 we see that the action is constrained to Ω_H and

$$L_h \tilde{u} = (L^{\#}\tilde{u} + M^{\#}\tilde{u}) \quad \text{on } \Omega_H \ ,$$

$$L^{\#}(U^1 + \phi_{2h}) = \frac{1}{2}\, L_h \tilde{u} \quad \text{on } \Omega_H \ .$$

A calculation completely analogous to the calculation in Theorem 5.1 (with L_H, M_H replaced by $L^{\#}$ and $M^{\#}$) yields

$$n \le \frac{1}{4}\left\{ (L^{\#}\tilde{u}, \tilde{u})_H - (M^{\#}\tilde{u}, \tilde{u})_H^2 \Big/ (L^{\#}\tilde{u}, \tilde{u})_H \right\} .$$

If $u \in R_n^I$ then

$$d = \frac{1}{2}\left\{ (L^{\#}u, u)_H + (M^{\#}u, u)_H \right\} .$$

Hence

$$\frac{n}{d} \le \frac{1}{2}\,(1-\sigma) g(\sigma) \ .$$

Thus, the theorem is proven. ∎

6. Concluding Remarks

We conclude with a table comparing our results with those of [3] and [14]. As we have pointed out in the introduction, our results for the V-cycle in convex domains are exactly those of [3] for Algorithm I and are stronger than those of [14] for Algorithm II*. Finally, we observe that our basic estimate for the V-cycle (and the estimate of [3, page 516]) is of the form

$$\delta(r) = \frac{1}{1+k}$$

which is consistent with the results of [6], [9], see [10] also.

r	**1**	**2**	**3**
Algorithm I See [3] also	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
Algorithm II	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
Algorithm 4.1 Results [14]	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$

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*Recall that Algorithm II is Algorithm 4.1 with r replaced by $r-2$.

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