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Computer Sciences Technical Report #610
August 1985
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Keywords: Diffusion Equation, Estimates.

AMS(MOS) Subject Classifications: 65N20.

Running Head: MGR MULTIGRID

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*Supported by the Air Force Office of Scientific Research under Contract No. AFOSR-82-0275.
ABSTRACT

The MGR[ν] algorithm of Ries, Trottenberg and Winter, Algorithm 2.1 of Braess and Algorithm 4.1 of Verfürth are all algorithms for the numerical solution of the discrete Poisson equation based on red-black Gauss-Seidel smoothing iterations. In this work we consider the extension of the MGR[0] method to the general diffusion equation \(-\nabla \cdot p\nabla u = f\). In particular, for the three grid scheme we extend an interesting and important result of Ries, Trottenberg and Winter whose results are based on Fourier analysis and hence intrinsically limited to the case where \(\Omega\) is a rectangle. Let \(\Omega\) be a general polygonal domain whose sides have slope \(\pm 1, 0\) and \(\infty\). Let \(\varepsilon^0\) be the error before a single multigrid cycle and let \(\varepsilon^1\) be the error after this cycle. Then \(\|\varepsilon^1\|_{L_h} \leq \frac{1}{2}(1+Kh)\|\varepsilon^0\|_{L_h}\) where \(\| \|_{L_h}\) denotes the energy or operator norm. When \(p(x,y) \equiv \text{constant}\), then \(K \equiv 0\).
1. Introduction

The MGR[ν] multigrid algorithms of Ries, Trottenberg and Winter [4], the Algorithm 2.1 of Braess [1], [2] and Algorithm 4.1 of Verfürth [5] are all algorithms for the numerical solution of the discrete Poisson equation (the usual 5-point difference equations with \( \Delta x = \Delta y = h \)) based on red-black Gauss-Seidel smoothing iterations. The analysis of [4] is based on Fourier Analysis and is restricted to the case where the basic domain \( \Omega \) is a square. The analysis of [1], [2] and [5] is for a bounded polygonal domain \( \Omega \) whose sides have slope \( +1,0 \) and \( -1 \) and is based on certain energy estimates and a particular interpretation of the matrix equations. While this is not explicitly stated, this interpretation can be viewed as a particular choice of \( I^h_H, I^H_H, I^2_h_H, I^{2h}_H \) etc., the operators which carry on the communication between the grids.

Recently, Kamowitz and Parter [3] considered a generalization of the algorithms of Ries, Trottenberg and Winter and Braess. They consider the general diffusion equation

\[
-\nabla \cdot \mathbf{p}(x, y) \nabla u = f \text{ in } \Omega ,
\]

\[
u = 0 \text{ on } \partial \Omega ,
\]

\[
p(x, y) \geq p_0 > 0 ,
\]

in general domains \( \Omega \). Using a different choice of \( I^h_H, I^H_H \) than Braess, i.e. imagining a different interpolation structure in the space \( S_H \), they employ other "Energy Estimates" to obtain the basic estimate - for a two grid scheme: let \( \varepsilon^0 \) denote the error before a single multigrid cycle and let \( \varepsilon^1 \) denote the error after that complete multigrid cycle, then
\begin{equation}
\| \varepsilon^l \|_{L^h} \leq \frac{1}{2} (1+K_h) \| \varepsilon^0 \|_{L^h}
\end{equation}

where the constant \( K \) depends only on \( p_0 \) and \( \| \nabla p \|_\infty \), the \( \infty \) norm of the gradient of \( p(x,y) \) and \( \| \cdot \|_{L^h} \) denotes the operator or energy norm. However, it is important to remark that despite the different interpretation of the problem, in the case of constant diffusion coefficient \( p(x,y) = 1 \) we are dealing with exactly the same problem and the same iterative method. The estimate \( 1.2 \) is thus a generalization of the estimates

\begin{equation}
\rho(MG) \leq \frac{1}{2}, \quad \rho(MGR[0]) = \frac{1}{2}
\end{equation}

of \[1\] and \[4\].

Another remarkable estimate of Ries, Trottenberg and Winter \[4\] is the fact that, in the case of Poisson equation in the square, if a third grid is introduced and one uses the MGR[0] method one obtains

\begin{equation}
\rho(MGR[0], 3 \text{ grid}) = \frac{1}{2}.
\end{equation}

In this report we obtain this estimate in the form \( 1.2 \) for the general diffusion equation \( 1.1 \) in bounded polygonal domains \( \Omega \) whose sides have slope \( \pm 1, 0 \) or \( \infty \). We also require that the corners of \( \Omega \) belong to the coarsest mesh. The constant \( K \) is a constant depending only on \( p_0 \), and the \( \infty \) norm of the first and second derivatives of \( p(x,y) \). Moreover, if \( p(x,y) \equiv \text{const.} \) then \( K = 0 \). In general, throughout this paper \( K \) will denote such a constant.

In section 2 we formulate the problem and the basic three-grid multigrid iteration. In particular we introduce the coarse grid operators \( \mathcal{L}_H, \mathcal{L}_H^\ast, \mathcal{L}_{2h}, \mathcal{L}_{2h}^\ast \). In section 3 we develop more notation and recall some basic estimates.
of [3]. In this section the reader is introduced to a number of additional difference operators \( L_H^{(1)}, \tilde{L}_H^{(1)}, L_{2h}^{(1)}, \tilde{L}_{2h}^{(1)}, Q_x, M_x, \tilde{L}_x \). This plethora of operators gets a bit confusing. However if one first concentrates on the case \( p(x,y) \equiv 1 \) (i.e., the Poisson equation) the situation simplifies. In this case \( L_H = L_H^{(1)}, L_{2h} = L_{2h}^{(1)} \) and (we always have) \( \tilde{L}_H = \frac{1}{2} L_H^{(1)} + \frac{1}{2} \tilde{L}_H^{(1)}, \tilde{L}_{2h} = \frac{1}{2} L_{2h}^{(1)} + \frac{1}{2} \tilde{L}_{2h}^{(1)} \). Moreover, in this case

\[
\tilde{L}_x = Q_x = L_{2h} = \left. \tilde{L}_H^{(1)} \right|_{\Omega_{2h}},
\]

[\( \Omega_{2h} \) is the coarsest grid] and

\[
M_x = \left. \tilde{L}_H^{(1)} \right|_{\Omega_H/\Omega_{2h}}
\]

[\( \Omega_H \) is the intermediate grid]. Another observation which should be useful is the fact that, in this case \( \tilde{L}_H^{(1)} \) is the same difference operator as \( L_{2h} \) except for points in \( \Omega_H \) which are next to the boundary. Moreover, these exceptional points are in \( \Omega_H/\Omega_{2h} \) not in \( \Omega_{2h} \). This perturbation of \( \tilde{L}_H^{(1)} \) causes a technical difficulty in the proof of lemma 5.2 even in this simplest case. In all cases the introduction of the variable diffusion coefficient \( p(x,y) \) introduces perturbation of the basic operators. However, the essence of the proof of the main result [Theorem 5.1 or the estimate (1.2)] is contained in the constant coefficient case. The analysis of the algorithm is given in two parts, sections 4 and 5.

Remark: The purpose of this work is to study and develop methods of multigrid analysis that may lead to actual numerical estimates on convergence rates. We are not suggesting that this particular algorithm is the optimal MGR[v] algorithm.
2. The Problem

Given a (small) value $h > 0$ let \( \{(x_k, y_j) = (kh, jh); \ k, j = 0, \pm 1, \pm 2, \ldots \} \) be the associated mesh points in the \( x \)-\( y \) plane. Let

\[
(2.1) \quad R_0 := \{(x_k, y_j); \ k+j \equiv 1 \pmod{2}\}
\]

\[
(2.2) \quad R_B := \{(x_k, y_j); \ k \equiv j \equiv 0 \pmod{2}\}
\]

\[
(2.3) \quad R_G := \{(x_k, y_j); \ k \equiv j \equiv 1 \pmod{2}\}.
\]

Let \( \Omega \) be a bounded polygonal domain in the plane whose sides have slope \( \pm 1, 0, \) or \( \infty \), and every corner point \( (x, y) \) of \( \partial \Omega \) belongs to \( R_B \).

Define

\[
(2.4a) \quad \Omega_h = (R_0 \cup R_B \cup R_G) \cap \Omega
\]

\[
(2.4b) \quad \partial \Omega_h = (R_0 \cup R_B \cup R_G) \cap \partial \Omega
\]

\[
(2.5a) \quad \Omega_H = (R_B \cup R_G) \cap \Omega
\]

\[
(2.5b) \quad \partial \Omega_H = (R_B \cup R_G) \cap \partial \Omega
\]

\[
(2.6a) \quad \Omega_{2h} = R_B \cap \Omega
\]

\[
(2.6b) \quad \partial \Omega_{2h} = R_B \cap \partial \Omega.
\]

For any function \( F(x, y) \) defined on \( \overline{\Omega} \) we write:
(2.7a) \[ F_{k,j} = F(x_k, y_j), \]

(2.7b) \[ F_{k+\frac{h}{2}, j} = F((k+\frac{h}{2})h, y_j), \]

(2.7c) \[ F_{k, j+\frac{h}{2}} = F(x_k, (j+\frac{h}{2})h). \]

The algebraic problem to be solved is: Find a mesh function \( U = \{U_{k,j}\} \) defined on \( \Omega_h \cup \partial \Omega_h \) which satisfies

(2.8a) \[ [L_h U]_{k,j} = F_{k,j}, \quad (x_k, y_j) \in \Omega_h \]

(2.8b) \[ U_{k,j} = 0, \quad (x_k, y_j) \in \partial \Omega_h \]

where

(2.8c) \[ [L_h U]_{k,j} = \frac{1}{h^2} \{ p_{k-\frac{h}{2}, j} [U_{k,j} - U_{k-1,j}] - p_{k+\frac{h}{2}, j} [U_{k+1,j} - U_{k,j}] \} + \frac{1}{h^2} \{ p_{k, j-\frac{h}{2}} [U_{k,j} - U_{k,j-1}] - p_{k, j+\frac{h}{2}} [U_{k,j+1} - U_{k,j}] \}. \]

We turn to solution of these linear algebraic equations by a three-grid method.

Let \( S_h, S_H, S_{2h} \) be the linear spaces of mesh functions defined on \( \Omega_h \cup \partial \Omega_h, \Omega_H \cup \partial \Omega_H \) and \( \Omega_{2H} \cup \partial \Omega_{2h} \) respectively which vanish on the respective boundaries \( \partial \Omega_h, \partial \Omega_H, \partial \Omega_{2h} \). We set up communication between these spaces. Specifically we define the linear interpolation and projection operators \( I_h^H, I_{2H}^H, I_h^{2h}, I_H^{2h} \) as follows. The interpolation operator \( I_h^H \) (see the definition of \( I_E^h \) of [3]) is given by
\[(2.9a) \quad I^h_H: S_H \rightarrow S_h\]

where

\[(2.9b) \quad [I^h_H u]_{kj} = u_{kj}, \text{ if } (x_k, y_j) \in \Omega_H \cup \partial\Omega_H\]

and, if \((x_k, y_j) \in \Omega_h / \partial\Omega_H\), then

\[(2.9c) \quad [I^h_H u]_{kj} = \frac{1}{c_{kj}} \left\{ p_{k+1/2, j} u_{k-1, j} + p_{k-1/2, j} u_{k+1, j} + p_{k, j-1/2} u_{k, j-1} + p_{k, j+1/2} u_{k, j+1} \right\}\]

where

\[(2.9d) \quad c_{kj} = \left\{ p_{k+1/2, j} + p_{k-1/2, j} + p_{k, j-1/2} + p_{k, j+1/2} \right\}.

Of course, if \((x_k, y_j) \in \partial\Omega_h / \partial\Omega_H\) then

\[(2.9e) \quad [I^h_H u]_{kj} = 0.\]

The projection operator \(I^H_h\) is defined by

\[(2.10) \quad I^H_h = \frac{1}{2} (I^H_h)^T.\]

**Remark:** The factor \(\frac{1}{2}\) in (2.10) is included merely to keep the method consistent with the MGR[v] methods of [4].

The interpolation operator \(I^H_{2h}\) is defined in a similar manner by

\[(2.11a) \quad I^H_{2h}: S_{2h} \rightarrow S_H\]
with

\[(2.11b) \quad [I_{2h}^H]_{kj} = U_{kj}, \text{ if } (x_k, y_j) \in \Omega_{2h} \cup \partial \Omega_{2h},\]

and, if \((x_k, y_j) \in \Omega_H / \partial \Omega_{2h} \ldots\) then

\[(2.11c) \quad [I_{2h}^H]_{kj} = \frac{1}{\bar{c}_{kj}} \{ p_{k+\frac{1}{2}, j+\frac{1}{2}} U_{k+1, j+1} + p_{k+\frac{1}{2}, j-\frac{1}{2}} U_{k+1, j-1} +
\]

\[+ p_{k-\frac{1}{2}, j+\frac{1}{2}} U_{k-1, j+1} + p_{k-\frac{1}{2}, j-\frac{1}{2}} U_{k-1, j-1} \}\]

where

\[(2.11d) \quad \bar{c}_{kj} = \{ p_{k+\frac{1}{2}, j+\frac{1}{2}} + p_{k+\frac{1}{2}, j-\frac{1}{2}} + p_{k-\frac{1}{2}, j+\frac{1}{2}} + p_{k-\frac{1}{2}, j-\frac{1}{2}} \}\]

and, if \((x_k, y_j) \in \partial \Omega_H / \partial \Omega_{2h}\), then

\[(2.11e) \quad [I_{2h}^H]_{kj} = 0.\]

The projection operator \(I_{2h}^H\) is given by

\[(2.12) \quad I_{2h}^H = \frac{1}{2} (I_{2h}^H)^T.\]

Finally we define the "coarse grid" operators \(L_H, L_{2h}\). These are

\[(2.13a) \quad L_H: S_H \rightarrow S_H\]

where, if \((x_k, y_j) \in \Omega_H\)

\[(2.13b) \quad [L_H]_{kj} = \frac{1}{2h^2} \{ \bar{c}_{kj} U_{k,j} - p_{k+\frac{1}{2}, j+\frac{1}{2}} U_{k+1,j+1} - p_{k+\frac{1}{2}, j-\frac{1}{2}} U_{k+1,j-1} - p_{k-\frac{1}{2}, j+\frac{1}{2}} U_{k-1,j+1} - p_{k-\frac{1}{2}, j-\frac{1}{2}} U_{k-1,j-1} \}\]
and

\[(2.14a) \quad L_{2h} : S_{2h} \rightarrow S_{2h} \]

where, if \((x_h, y_j) \in \Omega_{2h}\) then

\[(2.14b) \quad [L_{2h}U]_{kj} = \frac{1}{4h^2} \{ p_{k-1,j}[U_{k,j} - U_{k-2,j}] - p_{k+1,j}[U_{k+2,j} - U_{k,j}] \}
+ \frac{1}{4h^2} \{ p_{k,j-1}[U_{k,j} - U_{k,j-2}] - p_{k,j+1}[U_{k,j+2} - U_{k,j}] \} .\]

We are now ready to describe the three grid methods. Let \(B_h\) be a non-singular linear operator defined on \(S_h\)

\[(2.15) \quad B_h : S_h \rightarrow S_h .\]

Let the smoothing operator \(G_h\) be defined by

\[(2.16a) \quad G_h = I_h - B_h^{-1} L_h \]

and assume that

\[(2.16b) \quad \frac{\langle L_h G_h u, G_h u \rangle}{\langle L_h u, u \rangle} \leq 1 , \quad \forall u \in S_h, \quad u \neq 0 ,\]

Algorithm

Step 1: Given \(u^0 \in S_h\), form

\[(2.17) \quad \bar{u} = G_h u^0 + B_h^{-1} F .\]
Step 2: Perform one odd relaxation step. That is, construct \( \hat{u} \) via

\[(2.18a) \quad \hat{u}_{kj} = \tilde{u}_{kj}, \quad (x_k,y_j) \in \Omega_H \]

\[(2.18b) \quad [L_h \hat{u}]_{kj} = F_{kj}, \quad (x_k,y_j) \in \Omega_h/\Omega_H \]

\[\hat{u}_{kj} = 0, \quad (x_k,y_j) \in \partial \Omega_H.\]

Step 3: Set \( r = F - L_h \hat{u}, \quad r_H = I^H_h r. \)

Step 4: Let \( \hat{\psi} \) be obtained as follows.

\[(2.19a) \quad \hat{\psi}_{ij} = 0, \quad (x_i,y_j) \in \Omega_{2h} \]

\[(2.19b) \quad [L_h \hat{\psi}]_{ij} = r_H, \quad (x_i,y_j) \in \Omega_h/\Omega_{2h}. \]

Step 5: Set \( \tilde{r}_H = r_H - L_h \hat{\psi}, \quad r_{2h} = I^{2h}_H \tilde{r}_H. \)

Step 6: Solve

\[L_{2h} \phi = r_{2h} \]

Step 7: Set \( u^1 = \hat{u} + I^h_H [\hat{\psi} + [2h] \phi]. \)

Step 8: Set \( u^1 \rightarrow u^0 \) and return to step 1.

Observe that the red-black or odd-even nature of the basic equations means that \((2.18b)\) and \((2.19b)\) are explicit equations for the determination of \( \hat{u}_{kj} \) and \( \hat{\psi}_{ij} \) respectively.
3. Some Notation and Facts

Let \( u, v \in S_h \) or \( S_H \) or \( S_{2h} \). Then

\[
\langle u, v \rangle = \sum_{k,j} u_{k,j}^* v_{k,j}
\]

where the sum is taken over all indices \((k,j)\) so that \((x_k, y_j) \in \Omega_h\), or \(\Omega_H\) or \(\Omega_{2h}\) respectively. Whenever it seems that further clarity is required we will indicate the space by writing

\[
\langle u, v \rangle_a, \quad a = h \text{ or } H \text{ or } 2h.
\]

Since \( L_h, L_H \) and \( L_{2h} \) are positive definite operators we have the inner products

\[
[u, v]_a = \langle L^a u, v \rangle_a, \quad a = h \text{ or } H \text{ or } 2h.
\]

Let

\[
N_h := \text{Nullspace } I_{h}^H \subseteq S_h
\]

\[
R_h := \text{Range } I_{h}^H \subseteq S_h
\]

\[
N_H := \text{Nullspace } I_{H}^{2h} \subseteq S_H
\]

\[
R_H := \text{Range } I_{2h}^H \subseteq S_H
\]

**Lemma 3.1:** We have

\[
S_h = N_h \oplus R_h, \quad S_H = N_H \oplus R_H.
\]
In fact, \( N_h \) and \( R_h \) are \( L^2 \) orthogonal; \( N_H \) and \( R_H \) are \( L^2 \) orthogonal. That is, if \( \eta \in N_a, \omega \in R_a \), \( a = h \) or \( H \), then

\[
(3.4b) \quad [\eta, \omega]_a = \langle L_a \eta, \omega \rangle_a = 0.
\]

A function \( u \in S_h \) is in \( R_h \) if and only if

\[
(3.5a) \quad [L_h u]_{k,j} = 0, \quad (x_k, y_j) \in \Omega_h / \Omega_H.
\]

A function \( v \in S_h \) is in \( N_h \) if and only if

\[
(3.5b) \quad v_{k,j} = 0, \quad (x_k, y_j) \in \Omega_H.
\]

A function \( u \in S_H \) is in \( R_H \) if and only if

\[
(3.6a) \quad [L_H u]_{k,j} = 0, \quad (x_k, y_j) \in \Omega_H / \Omega_{2h}.
\]

A function \( v \in S_H \) is in \( N_H \) if and only if

\[
(3.6b) \quad v_{k,j} = 0, \quad (x_k, y_j) \in \Omega_{2h}.
\]

\textbf{Proof:} The assertions (3.5a) and (3.6a) follow from the definition of \( I_{h}^H, I_{2h}^H \) etc. given by (2.9)-(2.12). The assertions (3.4a), (3.4b), (3.5b), (3.6b) now follow immediately. See [3].

Let

\[
(3.7a) \quad \tilde{L}_H^H = I_{h}^H L_H I_{h}^H,
\]

\[
(3.7b) \quad \tilde{L}_{2h}^H = I_{h}^H L_H I_{2h}^H.
\]

Using the basic relations (2.10), (2.12) we see that
(3.7c) \[ \| \mathcal{I}^{h}_{1} v \|_{L_{h}}^{2} = \langle L_{h}^{1} \mathcal{I}^{h}_{1} v, v \rangle_{H} = 2 \langle \hat{L}_{h}^{1} v, v \rangle_{H}, \]

(3.7d) \[ \| \mathcal{I}^{H}_{2h} u \|_{L_{2h}}^{2} = \langle L_{h}^{2} \mathcal{I}^{H}_{2h} u, u \rangle_{H} = 2 \langle \hat{L}_{2h}^{H} u, u \rangle_{2h}. \]

The formulae (2.9), (2.10), (2.11), and (2.12) together with (3.5a) and (3.6a) imply

(3.8a) \[ \hat{L}_{h}^{u} = \frac{1}{2} L_{h}^{1} \mathcal{I}^{h}_{u} \Big|_{\Omega_{H}} \]

(3.8b) \[ \hat{L}_{2h}^{v} = \frac{1}{2} L_{h}^{2} \mathcal{I}^{H}_{2h} v \Big|_{\Omega_{2h}} \]

The analysis of [3] is based on the following facts about \( \hat{L}_{h}, \hat{L}_{2h} \).

**Lemma 3.2:** There are operators \( L_{H}^{(1)}, \tilde{L}_{H}^{(1)}, L_{2h}^{(1)}, \tilde{L}_{2h}^{(1)} \) such that:

(3.9a) \[ \hat{L}_{h} = \frac{1}{2} L_{h}^{(1)} + \frac{1}{2} \tilde{L}_{h}^{(1)}, \]

(3.9b) \[ \hat{L}_{2h} = \frac{1}{2} L_{2h}^{(1)} + \frac{1}{2} \tilde{L}_{2h}^{(1)}, \]

The operator \( L_{H}^{(1)} \) is based on the five points \((x_{k}, y_{j}), (x_{k+1}, y_{j+1}), (x_{k-1}, y_{j+1}), (x_{k+1}, y_{j-1}), (x_{k}, y_{j-1})\). These are the same points on which \( L_{h} \) is based. The operator \( \tilde{L}_{H}^{(1)} \) is based on the five points \((x_{k}, y_{j}), (x_{k+2}, y_{j}), (x_{k-2}, y_{j}), (x_{k}, y_{j+2}), (x_{k}, y_{j-2})\). If \( k \equiv j \equiv 0 \pmod{2} \), these are the same points on which \( L_{2h} \) is based. Similarly, if \( k \equiv j \equiv 0 \pmod{2} \),
The five point star for $L_H, L_H^{(1)}$

The five point star for $\tilde{L}_H^{(1)}, L_{2h}, L_{2h}^{(1)}$

Figure 1
$L_{2h}^{(1)}$ is based on these same points. The operators $L_H^{(1)}$, $L_{2h}^{(1)}$ are "almost" the operators $L_H$, $L_{2h}$. To be precise, we have: let

\[(3.10a) \quad a_{k-\frac{1}{2},j-\frac{1}{2}} = \left[ \frac{p_{k-\frac{1}{2},j} p_{k-1,j-\frac{1}{2}} + p_{k,j} p_{k-\frac{1}{2},j-\frac{1}{2}}}{c_{k-1,j}} + \frac{p_{k,j} p_{k-\frac{1}{2},j-\frac{1}{2}}}{c_{k,j-1}} \right],\]

\[(3.10b) \quad b_{k+\frac{1}{2},j-\frac{1}{2}} = \left[ \frac{p_{k,j+\frac{1}{2}} p_{k+\frac{1}{2},j-\frac{1}{2}} + p_{k+\frac{1}{2},j} p_{k+\frac{1}{2},j-\frac{1}{2}}}{c_{k+1,j-1}} + \frac{p_{k+\frac{1}{2},j} p_{k+\frac{1}{2},j-\frac{1}{2}}}{c_{k+1,j}} \right],\]

\[(3.10c) \quad d_{kj} = [a_{k-\frac{1}{2},j-\frac{1}{2}} + a_{k+\frac{1}{2},j+\frac{1}{2}} + b_{k+\frac{1}{2},j-\frac{1}{2}} + b_{k-\frac{1}{2},j+\frac{1}{2}}].\]

If $(k+j) \equiv 0 \pmod{2}$, then

\[(3.11) \quad [L_H^{(1)} U]_{kj} = \frac{1}{\hbar^2} \{ -a_{k+\frac{1}{2},j+\frac{1}{2}} U_{k+1,j+1} - a_{k-\frac{1}{2},j-\frac{1}{2}} U_{k-1,j-1} - b_{k+\frac{1}{2},j+\frac{1}{2}} U_{k+1,j-1} - b_{k-\frac{1}{2},j+\frac{1}{2}} U_{k-1,j+1} + d_{kj} U_{kj} \}.\]

An easy computation shows that

\[|2a_{k-\frac{1}{2},j-\frac{1}{2}} - p_{k-\frac{1}{2},j-\frac{1}{2}}| \leq \hbar^2\]

\[|2b_{k+\frac{1}{2},j-\frac{1}{2}} - p_{k+\frac{1}{2},j-\frac{1}{2}}| \leq \hbar^2.\]

Hence, for every $U \in S_H$,

\[(3.12a) \quad |\langle L_H U, U \rangle - \langle L_H^{(1)} U, U \rangle| \leq \hbar^2 \langle L_H U, U \rangle,\]

\[(3.12b) \quad |\langle L_H U, U \rangle - \langle L_H^{(1)} U, U \rangle| \leq \hbar^2 \langle L_H^{(1)} U, U \rangle.\]
A basic estimate is: for every $U \in S_H^*$,

\begin{equation}
0 \leq \langle \tilde{L}^{(1)}_H U, U \rangle \leq (1+K) \langle L^{(1)}_H U, U \rangle .
\end{equation}

Hence, if we write

\begin{equation}
\hat{L}_H = \frac{1}{2} L_H + \frac{1}{2} \tilde{L}^{(2)}_H ,
\end{equation}

then

\begin{equation}
-Kh \langle \hat{L}_H U, U \rangle \leq \langle \tilde{L}^{(2)}_H U, U \rangle \leq (1+K) \langle L_H U, U \rangle .
\end{equation}

Similarly, let

\begin{equation}
A_{k+1,j} = \left[ \frac{P_{k+\frac{1}{2}, j+\frac{1}{2}} P_{k+\frac{3}{2}, j+\frac{1}{2}}}{\xi_{k+1,j+1}} \right] + \frac{P_{k+\frac{3}{2}, j+\frac{1}{2}} P_{k-\frac{3}{2}, j-\frac{1}{2}}}{\xi_{k+1,j-1}} ,
\end{equation}

\begin{equation}
B_{k,j+1} = \left[ \frac{P_{k+\frac{1}{2}, j+\frac{1}{2}} P_{k+\frac{3}{2}, j+\frac{3}{2}}}{\xi_{k+1,j+1}} \right] + \frac{P_{k-\frac{3}{2}, j+\frac{3}{2}} P_{k-\frac{3}{2}, j+\frac{1}{2}}}{\xi_{k-1,j+1}} ,
\end{equation}

\begin{equation}
D_{k,j} = [A_{k+1,j} + A_{k-1,j} + B_{k,j+1} + B_{k,j-1}] .
\end{equation}

If, $k \equiv j \equiv 0 \pmod{2}$,

\begin{equation}
L^{(1)}_{2h} = \frac{1}{2h^2} \left\{ -A_{k+1,j} U_{k+2,j} - A_{k-1,j} U_{k-2,j} - B_{k,j+1} U_{k,j+2} - B_{k,j-1} U_{k,j-2} + D_{k,j} U_{k,j} \right\} .
\end{equation}
An easy calculation shows that

\[(3.17a) \quad |2A_{k+1,j} - p_{k+1,j}| \leq Kh^2,\]

\[(3.17b) \quad |2B_{k,j+1} - p_{k,j+1}| \leq Kh^2.\]

Hence, for all \( U \in \mathcal{S}_{2h} \)

\[(3.17c) \quad |\langle L_{2h}^{(1)} U, U \rangle_{2h} - \langle L_{2h}^{(1)} U, U \rangle_{2h}| \leq Kh^2 \langle L_{2h}^{(1)} U, U \rangle_{2h}.\]

The analog of the basic estimate (3.13) holds. That is

\[(3.18) \quad 0 \leq \langle \hat{L}_{2h}^{(1)} U, U \rangle \leq (1 + Kh) \langle L_{2h}^{(1)} U, U \rangle.\]

Hence, if we write

\[(3.19a) \quad \hat{L}_{2h} = \frac{1}{2} L_{2h} + \frac{1}{2} \tilde{L}_{2h}^{(2)}\]

then

\[(3.19b) \quad (-Kh) \langle L_{2h}^{(1)} U, U \rangle \leq \langle \tilde{L}_{2h}^{(2)} U, U \rangle \leq (1 + Kh) \langle L_{2h}^{(1)} U, U \rangle.\]

Of course, if \( p(x,y) \equiv 1 \), then

\[(3.20) \quad L_H = L_H^{(1)} , \quad L_{2h} = L_{2h}^{(1)} .\]

**Proof:** The construction of \( L_H^{(1)} \) and the basic estimate (3.13) is found in [3]. The construction of \( L_{2h}^{(1)} \) and the estimate (3.18) then follows from the same arguments. The estimates (3.11), (3.17) are direct computations. \( \blacksquare \)
Our next result looks at the operator $\tilde{\mathcal{L}}_H^{(1)}$.

**Lemma 3.3:** The operator $\tilde{\mathcal{L}}_H^{(1)}$ is of the form

\[
(3.21) \quad \tilde{\mathcal{L}}_H^{(1)} U_{kj} = \frac{1}{\hbar^2} \left\{ -\tilde{A}_{k+1,j} U_{k+2,j} - \tilde{A}_{k-1,j} U_{k-2,j} - \tilde{B}_{k,j+1} U_{k,j+2} - \tilde{B}_{k,j-1} U_{k,j-2} + \tilde{D}_{k,j} U_{k,j} \right\}.
\]

The coefficients, $\tilde{A}, \tilde{B}, \tilde{D}$ are given by

\[
(3.22a) \quad \tilde{A}_{k+1,j} = \frac{p_{k+\frac{1}{2},j} p_{k+\frac{1}{2},j}}{c_{k+1,j}} + \frac{1}{2} \left( p_{k+\frac{1}{2},j} \right)^2 \theta_{k+1,j},
\]

\[
(3.22b) \quad \tilde{A}_{k-1,j} = \frac{p_{k-\frac{1}{2},j} p_{k-\frac{1}{2},j}}{c_{k-1,j}} + \frac{1}{2} \left( p_{k-\frac{1}{2},j} \right)^2 \theta_{k-1,j},
\]

\[
(3.22c) \quad \tilde{B}_{k,j+1} = \frac{p_{k,j+\frac{1}{2}} p_{k,j+\frac{1}{2}}}{c_{k,j+1}} + \frac{1}{2} \left( p_{k,j+\frac{1}{2}} \right)^2 \theta_{k,j+1},
\]

\[
(3.22d) \quad \tilde{B}_{k,j-1} = \frac{p_{k,j-\frac{1}{2}} p_{k,j-\frac{1}{2}}}{c_{k,j-1}} + \frac{1}{2} \left( p_{k,j-\frac{1}{2}} \right)^2 \theta_{k,j-1},
\]

\[
(3.22e) \quad \tilde{D}_{k,j} = \tilde{A}_{k+1,j} + \tilde{A}_{k-1,j} + \tilde{B}_{k,j+1} + \tilde{B}_{k,j-1}
\]
where

\[
\theta_{\mu,\sigma} = \begin{cases} 
1, & (x_{\mu}, y_{\sigma}) \in \partial \Omega_h \\
0, & (x_{\mu}, y_{\sigma}) \notin \partial \Omega_h
\end{cases}
\]

(3.23)

**Proof:** These coefficients were computed in [3].

**Remark:** If

\[ \theta_{k \pm 1, j} \neq 0, \text{ then } U_{k \pm 2, j} = 0, \]

\[ \theta_{k, j \pm 1} \neq 0, \text{ then } U_{k, j \pm 2} = 0. \]

**Lemma 3.4:** Let \((x_k, y_j) \in \Omega_{2h}\). Then all 4 of its \(h\) grid neighbors
\((x_{k \pm 1}, y_j), (x_k, y_{j \pm 1}) \in \Omega_h\). Hence

\[ \theta_{k \pm 1, j} = \theta_{k, j \pm 1} = 0. \]

**Proof:** (See Figure 2). This result follows immediately from the fact that all corner points of \(\partial \Omega\) lie in \(R_B\).

It is useful to write \(i_H^{(1)}\) as the sum of two operators, one essentially based on \(\Omega_{2h}\) and the other on \(\Omega_h / \Omega_{2h}\).
Reentrant Corner

○ denotes a point in $R_0$
■ denotes a point in $R_B$
□ denotes a point in $R_G$

Figure 2
**Definition:** Let \( M_x, Q_x : S_H \to S_H \) be defined by

\[(3.24a) \quad [Q_x u]_{k,j} = 0, \quad (x_k, y_j) \in \Omega_h / \Omega_{2h}, \]

\[(3.24b) \quad [Q_x u]_{k,j} = \tilde{[L_H u]}_{k,j}, \quad (x_k, y_j) \in \Omega_{2h}, \]

\[(3.25a) \quad [M_x u]_{k,j} = \tilde{[L_H u]}_{k,j}, \quad (x_k, y_j) \in \Omega_h / \Omega_{2h}, \]

\[(3.25b) \quad [M_x u]_{k,j} = 0, \quad (x_k, y_j) \in \Omega_{2h}. \]

**Lemma 3.5:** Let \( v \in S_{2h} \). Then

\[(3.26) \quad |\langle Q_x I_{2h}^H v, I_{2h}^H v \rangle_H - \langle L_{2h} v, v \rangle_{2h} | \leq K h^2 \langle L_{2h} v, v \rangle. \]

**Proof:** The lemma follows from Lemma 3.4 and the estimates

\[|4\mathcal{A}_{k+1,j} - p_{k+1,j}| \leq K h^2 p_{k+1,j}, \]

\[|4\mathcal{B}_{k,j+1} - p_{k,j+1}| \leq K h^2 p_{k,j+1}. \]

**Remark:** When \( p(x,y) \equiv 1 \), then \( k \equiv 0 \).

Finally, we "lift" \( L_{2h} \) (an operator defined on \( S_{2h} \)) as follows: let \( \tilde{L}_x : S_{2h} \to S_H \) be defined by

\[(3.27a) \quad [\tilde{L}_x (I_{2h}^H v)]_{k,j} = 0, \quad (x_k, y_j) \in \Omega_h / \Omega_{2h}, \]

\[(3.27b) \quad [\tilde{L}_x (I_{2h}^H v)]_{k,j} = [L_{2h} v]_{k,j}, \quad (x_k, y_j) \in \Omega_{2h}. \]

**Remark:** Using this definition we may rephrase (3.26) as

\[(3.28) \quad |\langle Q_x I_{2h}^H v, I_{2h}^H v \rangle_H - \langle \tilde{L}_x I_{2h}^H v, I_{2h}^H v \rangle_H | \leq K h^2 \langle \tilde{L}_x I_{2h}^H v, I_{2h}^H v \rangle. \]
4. **Analysis I.**

Let \( \varepsilon^0 = u - u^0 \) be the initial error. Then \( \tilde{\varepsilon} = u - \tilde{u} \) is the error after step 1, the smoothing step. Assumption (2.16) asserts that

\[
(4.1) \quad \| \tilde{\varepsilon} \|_{L^2_h}^2 = \langle L_h \tilde{\varepsilon}, \tilde{\varepsilon} \rangle_L \leq \langle L_h \varepsilon^0, \varepsilon^0 \rangle_L = \| \varepsilon^0 \|_{L^2_h}^2.
\]

Using the decomposition (3.4a) we have

\[
(4.2) \quad \tilde{\varepsilon} = \eta_h + I^h_H w, \quad \eta_h \in N_h, \quad w \in S_H.
\]

From step 2 [i.e. (2.18)] of the algorithm and Lemma 3.1 [i.e. (3.5b)] we see that

\[
(4.3) \quad \hat{\varepsilon} = u - \hat{u} = I^h_H w.
\]

Hence, using (3.4a) we see that

\[
(4.4) \quad \| \hat{\varepsilon} \|_{L^2_h}^2 = \| I^h_H w \|_{L^2_h}^2 \leq \| \eta_h \|_{L^2_h}^2 + \| I^h_H w \|_{L^2_h}^2 = \| \tilde{\varepsilon} \|_{L^2_h}^2 \leq \| \varepsilon^0 \|_{L^2_h}^2.
\]

Using (4.3) and (3.7a) and step 3 of the algorithm we see that

\[
(4.5) \quad \hat{L}_H w = (I^h_H I^h_H - I^h_H) w = r_H.
\]

See [3] for a more complete discussion of the significance of this fact.

**Lemma 4.1:** Let \( v \in S_H \) be the solution of

\[
(4.6) \quad L_H v = r_H = \hat{L}_H w.
\]

Let

\[
(4.7) \quad v = \eta_H + I^H_{2h} v, \quad \eta_H \in N_H, \quad v \in S_{2h}.
\]
Let \( \hat{\psi} \) be the function in \( S_H \) constructed in step 4 [i.e. (2.19)] of the algorithm. Then

\[
(4.8) \quad \hat{\psi} = \eta_H.
\]

**Proof:** Observe that (2.19a) and (3.5b) imply that \( \hat{\psi} \in N_H \). Also (2.19b) and (4.6) yield

\[
[L_H(v-\hat{\psi})]_{k,j} = 0, \quad (x_k, y_j) \in \Omega_H/\Omega_{2h}.
\]

That is

\[
(4.9a) \quad (v-\hat{\psi}) = [(\eta_H-\hat{\psi}) + i^N_{2h} V] \in \mathbb{R}_H
\]

while

\[
(4.9b) \quad (\eta_H-\hat{\psi}) \in N_H.
\]

Using (3.4a) and (3.4b) we see that (4.8) holds. \( \blacksquare \)

Consider the function \( \phi \) which is constructed in step 6 of the algorithm. We have

\[
(4.10) \quad L_{2h}\phi = i^{2h}_{LH}(v-\hat{\psi}) = i^{2h}_{LH}i^H_{2h} V.
\]

thus

\[
(4.11) \quad L_{2h}\phi = \hat{\omega}_{2h} V.
\]

From (4.3), (4.11) and step 7 of the algorithm we see that
\[(4.12) \quad \varepsilon^1 = u - u^1 = I^h_H[(w - \hat{\psi}) - I^H_{2h} \phi] \in \mathbb{R}_h.\]

Thus, if we seek an eigenfunction \( \varepsilon^0 \), it must have the form
\[\varepsilon^0 = I^h_H \tilde{v}.\]

As we shall see, the generality of \( G_h \) and the estimate (4.1) implies that it suffices to consider the case where \( G_h = I_h \). In that case
\[(4.13) \quad \tilde{v} = \varepsilon^0 = I^h_H[\tilde{\eta}_H + I^H_{2h} U]; \quad \tilde{\eta}_H \in \mathcal{N}_H, \ U \in S_{2h} .\]

If \( \varepsilon^1 = \mu \varepsilon^0 \) (4.12) becomes
\[\varepsilon^1 = I^h_H[(\tilde{\eta}_H - \hat{\psi}) + I^H_{2h} (U - \phi)] = \mu I^h_H[\tilde{\eta}_H + I^H_{2h} U].\]

Thus
\[(4.14) \quad \hat{\psi} = \lambda \tilde{\eta}_H, \quad \phi = \lambda U, \quad \lambda = (1 - \mu).\]

Returning to Lemma 4.1 we have
\[(4.15) \quad L_H(\hat{\psi} + I^H_{2h} V) = \hat{L}_H(\tilde{\eta}_H + I^H_{2h} U).\]

From (3.8b), (3.6a), (4.11) and (4.14) we see that
\[(4.16a) \quad L_h^H I^H_{2h} V \bigg|_{\Omega_{2h}} = 2\hat{L}_{2h} V = 2L_{2h} \phi = 2\lambda L_{2h} U,\]
\[(4.16b) \quad L_h^H I^H_{2h} V \bigg|_{\Omega_H / \Omega_{2h}} = 0 .\]
Thus, (4.16) and the definition of $\tilde{\sigma}_X$ [i.e. (3.27)] allows us to rewrite (4.15) as

$$\lambda [L_H\tilde{H} + 2L_X I_{2h}^H U] = \hat{\Lambda}_H [\tilde{H} + I_{2h}^H U].$$

To simplify the eigenvalue problem (4.17) we define

$$L^# : S_H \rightarrow S_H$$

as follows: let $v \in S_H$. Then there is a unique representation

$$(4.18a) \quad v = \zeta_H + I_{2h}^H W, \quad \zeta_H \in N_H, \quad W \in S_{2h}.$$

Then

$$(4.18b) \quad L^# v = L_H \zeta_H + 2L_X I_{2h}^H W.$$

The eigenvalue problem (4.17) now becomes

$$(4.19a) \quad \lambda L^# v = \hat{\Lambda}_H v,$$

$$(4.19b) \quad v = \tilde{H} + I_{2h}^H U.$$

Observe that both $L^#$ and $\hat{\Lambda}_H$ are symmetric positive definite operators. Therefore, there is a complete set of eigenfunctions $\{v_k\}$ which satisfy

$$(4.20) \quad \langle L^# v_k, v_j \rangle = \langle \hat{\Lambda}_H v_k, v_j \rangle = 0, \quad k \neq j.$$

Then (3.7c) implies that

$$(4.21) \quad \frac{\| \varepsilon_1^1 \|_{L_h}}{\| \varepsilon_0^0 \|_{L_h}} \leq \max |1-\lambda| = \max |\mu|.$$
Thus, in view of (4.1), the general three-grid iteration \((G_h \neq I_h)\)
also satisfies (4.21).
5. **Analysis II**

Consider the basic eigenvalue problem (4.19). Let us now focus our attention on the right-hand-side of (4.19a). Using (3.9a) and (4.19b) we have

\begin{align}
(5.1a) \quad \hat{\mathbf{L}}_H \mathbf{v} &= \frac{1}{2} \mathbf{L}^{(1)}_H \bar{\mathbf{n}}_H + \frac{1}{2} \mathbf{L}^{(1)}_H \bar{\mathbf{1}}_{2h} \mathbf{U} + \frac{1}{2} \mathbf{L}^{(1)}_H \bar{\mathbf{1}}_{2h} \mathbf{U},
\end{align}

and

\begin{align}
(5.1b) \quad \langle \mathbf{v}, \hat{\mathbf{L}}_H \mathbf{v} \rangle &= \frac{1}{2} \langle \bar{\mathbf{n}}_H, \mathbf{L}^{(1)}_H \bar{\mathbf{n}}_H \rangle + \frac{1}{2} \langle \bar{\mathbf{n}}_H, \mathbf{L}^{(1)}_H \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle + \frac{1}{2} \langle \bar{\mathbf{n}}_H, \mathbf{L}^{(1)}_H \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle \\
&+ \frac{1}{2} \langle \bar{\mathbf{1}}_{2h} \mathbf{U}, \mathbf{L}^{(1)}_H \bar{\mathbf{n}}_H \rangle + \frac{1}{2} \langle \bar{\mathbf{1}}_{2h} \mathbf{U}, \mathbf{L}^{(1)}_H \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle.
\end{align}

The basic estimate (3.12a) allows us to replace \( \mathbf{L}^{(1)}_H \) by \( \mathbf{L}_H \) provided we accept error terms of the form

\begin{align}
(5.2a) \quad \delta_1 &= K h^2 \left[ (\mathbf{L}_H \bar{\mathbf{1}}_{2h} \mathbf{U}, \bar{\mathbf{1}}_{2h} \mathbf{U}) \langle \mathbf{L}_H \bar{\mathbf{n}}_H, \bar{\mathbf{n}}_H \rangle \right]^\frac{1}{2}, \\
(5.2b) \quad \delta_2 &= K h^2 \langle \mathbf{L}_H \bar{\mathbf{1}}_{2h} \mathbf{U}, \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle, \\
(5.2c) \quad \delta_3 &= K h^2 \langle \mathbf{L}_H \bar{\mathbf{n}}_H, \bar{\mathbf{n}}_H \rangle.
\end{align}

Thus we may rewrite (5.1b) as

\begin{align}
\langle \mathbf{v}, \hat{\mathbf{L}}_H \mathbf{v} \rangle &= \frac{1}{2} \langle \bar{\mathbf{n}}_H, \mathbf{L}_H \bar{\mathbf{n}}_H \rangle + \frac{1}{2} \langle \bar{\mathbf{1}}_{2h} \mathbf{U}, \mathbf{L}_H \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle \\
&+ \frac{1}{2} \langle \bar{\mathbf{n}}_H, \mathbf{L}_H \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle + \frac{1}{2} \langle \bar{\mathbf{1}}_{2h} \mathbf{U}, \mathbf{L}_H \bar{\mathbf{1}}_{2h} \mathbf{U} \rangle + O(\delta)
\end{align}
where

\[ 0(\delta) = 0(\delta_1 + \delta_2 + \delta_3) . \]

From (3.6b) of Lemma (3.1) we see that

\[ (\tilde{n}_H)_{k,j} = 0 , \quad (x_k , y_j) \in \Omega_{2h} . \]

Hence

\[ \langle \tilde{n}_H , \tilde{(1)}_H \tilde{n}_H \rangle = \langle \tilde{n}_H , M_x \tilde{n}_H \rangle, \]

\[ \langle \tilde{n}_H , \tilde{(1)}_{2h} U \rangle = \langle \tilde{n}_H , M_{x_{2h}} U \rangle . \]

Thus, we may rewrite (5.3) as

\[ \langle v , \tilde{L}_H v \rangle = \frac{1}{2} \langle \tilde{n}_H , L_H \tilde{n}_H \rangle + \frac{1}{2} \langle \tilde{1}_{2h} U , L_H \tilde{1}_{2h} U \rangle + \frac{1}{2} \langle \tilde{n}_H , M_{x} \tilde{n}_H \rangle \]

\[ + \langle \tilde{n}_H , M_{x_{2h}} U \rangle + \frac{1}{2} \langle \tilde{1}_{2h} U , M_{x_{2h}} U \rangle + \]

\[ \frac{1}{2} \langle \tilde{1}_{2h} U , Q_{x_{2h}} \tilde{1}_{2h} U \rangle + O(\delta) . \]

Let us consider the term

\[ J := \frac{1}{2} \langle \tilde{1}_{2h} U , L_H \tilde{1}_{2h} U \rangle_H . \]

From (2.12), (3.7b) and (3.9a) we have

\[ J = \langle U , \tilde{L}_{2h} U \rangle_{2h} = \frac{1}{2} \langle U , L_{2h} U \rangle_{2h} + \frac{1}{2} \langle U , \tilde{L}_{2h} U \rangle_{2h} . \]
Thus, using the definition of $\tilde{L}_X$ and (3.17c) we obtain

\begin{equation}
J = \frac{1}{2} \langle I_{2h}^H U, \tilde{L}_X^H U \rangle_H + \frac{1}{2} \langle U, \tilde{L}_{2h}^{(2)} U \rangle_{2h}.
\end{equation}

The estimate (3.28) allows us to replace $Q_X$ by $\tilde{L}_X$ provided we accept errors of the form

\begin{equation}
\tilde{\delta} = Kh^2 \langle L_{X,2h}^H U, i_{2h}^H U \rangle = Kh^2 \langle L_{H,2h}^H U, i_{2h}^H U \rangle.
\end{equation}

Thus, we rewrite (5.6) as

\begin{equation}
\langle v, \hat{L}_H^j v \rangle = \frac{1}{2} \langle v, L^H v \rangle + \frac{1}{2} \langle U, \hat{L}_{2h}^{(2)} U \rangle_{2h}
+ \frac{1}{2} \langle v, M_X v \rangle + O(\delta) + O(\tilde{\delta}).
\end{equation}

The eigenvalue problem (4.19) becomes

\begin{equation}
(\lambda - \frac{1}{2}) \langle v, L^H v \rangle = \frac{1}{2} \langle U, \hat{L}_{2h}^{(2)} U \rangle_{2h}
+ \frac{1}{2} \langle v, M_X v \rangle + O(\delta + \tilde{\delta}).
\end{equation}

Hence

\begin{equation}
\lambda - \frac{1}{2} \geq -Kh
\end{equation}

and

\begin{equation}
\lambda \geq \frac{1}{2} (1-Kh).
\end{equation}
The complete proof of our basic estimate requires a further analysis of the terms which appear in (5.11). Let

\[ J = \langle U, L_2h^{(2)} \rangle_{2h} + \langle v, M_x v \rangle_H. \]  

Using the basic estimate (3.19), Lemma 3.5 and the estimate (3.28) we obtain

\[ \langle U, L_2h^{(2)} \rangle_{2h} \leq (1+K) \langle I_{2h}^H U, Q_x I_{2h}^H U \rangle_H + O(\delta). \]  

Expand the second term in \( J \). We now have

\[ J \leq (1+K) \langle I_{2h}^H U, Q_x I_{2h}^H U \rangle_H + O(\delta) + \]

\[ \langle \bar{\eta}_H, M_x \bar{\eta}_H \rangle_H + 2 \langle \bar{\eta}_H, M_x I_{2h}^H U \rangle_H + \langle I_{2h}^H U, M_x I_{2h}^H U \rangle_H. \]  

Using (5.5) we see that

\[ \langle \bar{\eta}_H, Q_x \bar{\eta}_H \rangle_H = \langle \bar{\eta}_H, Q_x I_{2h}^H U \rangle_H = 0. \]  

Since (3.24) and (3.25) yield

\[ \bar{L}_H^{(1)} = Q_x + M_x \]

we may rewrite (5.15) as

\[ J \leq (1+K) \langle \bar{L}_H^{(1)} \bar{I}_{2h}^H U, \bar{I}_{2h}^H U \rangle_H + \langle \bar{\eta}_H, \bar{L}_H^{(1)} \bar{\eta}_H \rangle_H + 2 \langle \bar{\eta}_H, \bar{L}_H^{(1)} I_{2h}^H U \rangle_H + O(\delta). \]

That is

\[ J \leq (1+K) \langle v, \bar{L}_H^{(1)} v \rangle_H + O(\delta). \]
Using the basic estimates (3.12), (3.13) and (5.9) - the definition of \( \tilde{\delta} \) - we now have

\[
(5.16a) \quad \tilde{J} \leq (1+K) \langle v, L_H^* v \rangle_H,
\]

or

\[
(5.16b) \quad \tilde{J} \leq (1+K) \left[ \langle \tilde{\eta}_H^+, L_H^* \eta_H^- \rangle_H + \langle L_H^1 U, L_H^1 U \rangle_H \right].
\]

From (3.8) (the representation of \( \hat{L}_{2h} \)), (3.9) and the basic estimate (3.19b) we have

\[
(5.17) \quad \tilde{J} \leq (1+K) \left[ \langle \tilde{\eta}_H^+, L_H^* \eta_H^- \rangle_H + 2 \langle U, L_{2h} U \rangle_{2h} \right].
\]

However, using the definition of \( \hat{L}_x \) (3.27) and the definition of \( L^\# \) (4.18) we may rewrite (5.17) as

\[
(5.18) \quad \tilde{J} \leq (1+K) \langle v, L^\# v \rangle_H.
\]

**Lemma 5.1**: Let \( v \in S_H \). Then

\[
(5.19) \quad \langle v, \hat{L}_H^* v \rangle_H \leq (1+K) \langle v, L^\# v \rangle_H.
\]

Further, let \( (\lambda, v) \) be an eigenpair for the eigenvalue problem (4.19). Then

\[
(5.20) \quad \frac{1}{2} (1-Kh^2) \leq \lambda \leq (1+K).
\]

Moreover, if \( p(x,y) = \text{const} > 0 \), then \( K = 0 \).
Proof: From (5.10), (5.18) and using the fact that both \( \delta \) and \( \bar{\delta} \) are dominated by \( \langle v, L_y^# v \rangle_H \), we obtain (5.19). The left hand inequality of (5.20) was established in (5.12). The right hand inequality of (5.20) follows immediately from (5.11) and (5.18).

This result and (4.21) yields

**Theorem 5.1:** Consider the three grid iterative scheme described in section 2: steps 1-8. Let

\[
\varepsilon^0 = u - u^0, \quad \varepsilon^1 = u - u^1.
\]

There is a constant \( K \geq 0 \), depending only on \( p(x,y) \) and its first and second derivatives, such that

\[
\| \varepsilon^1 \|_{L_h} \leq \frac{1}{2} (1 + Kh) \| \varepsilon^0 \|_{L_h}.
\]

Moreover, if \( p(x,y) \equiv \text{const} > 0 \), then \( K = 0 \).
REFERENCES


