COMPUTABLE NUMERICAL BOUNDS FOR LAGRANGE MULTIPLIERS OF STATIONARY POINTS OF NONCONVEX DIFFERENTIABLE NONLINEAR PROGRAMS

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ABSTRACT

It is shown that the satisfaction of a standard constraint qualification of mathematical programming [5] at a stationary point of a nonconvex differentiable nonlinear program provides explicit numerical bounds for the set of all Lagrange multipliers associated with the stationary point. Solution of a single linear program gives a sharper bound together with an achievable bound on the 1-norm of the multipliers associated with the inequality constraints. The simplicity of obtaining these bounds contrasts sharply with the intractable NP-complete problem of computing an achievable upper bound on the p-norm of the multipliers associated with the equality constraints for integer $p \geq 1$.

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Consider the constrained optimization problem

(1) $$\text{minimize } f(x) \text{ subject to } g(x) \leqq 0,\ h(x) = 0$$

where $f : R^n \to R$, $g : R^n \to R^m$ and $h : R^n \to R^k$. It is well known that if a standard constraint qualification [2, 5]

(2) $$\begin{cases} \nabla g_I(\bar{x})z \leqq -e,\ \nabla h(\bar{x})z = 0 \text{ for some } z \in R^n, \text{ and} \\ \text{rows of } \nabla h(\bar{x}) \text{ are linearly independent} \end{cases}$$

holds at a local solution $\bar{x}$ of (1) at which f, g and h are continuously differentiable, $I = \{i \mid g_i(\bar{x}) = 0\}$, $\nabla g(\bar{x})$, $\nabla g_I(\bar{x})$ and $\nabla h(\bar{x})$ are $m \times n$, $\bar{m} \times n$ and $k \times n$ Jacobian matrices respectively, e is a vector of ones and $\bar{m}$ is the number of elements in I, then $\bar{x}$ is a <u>stationary point</u> of (1), that is it satisfies the Karush-Kuhn-Tucker conditions [2]

(3) $$\nabla f(\bar{x}) + \bar{u}\nabla g(\bar{x}) + \bar{v}\nabla h(\bar{x}) = 0,\ \bar{u}g(\bar{x}) = 0,\ g(\bar{x}) \leqq 0,\ \bar{u} \geqq 0,\ h(\bar{x}) = 0$$

for some Lagrange multipliers $(\bar{u},\bar{v}) \in R^{m+k}$. Let $\bar{W}$ denote the set of all Lagrange multipliers which satisfy (3) for a fixed $\bar{x}$. It follows from Gauvin's theorem [1] that if $\bar{x}$ is a local solution of (1), then $\bar{W}$ is nonempty and bounded if and only if the constraint qualification (2) holds. What we would like to point out in this note is that <u>any</u> z in the set Z of points satisfying the constraint qualification (2) for a fixed $\bar{x}$ provides an explicit numerical bound for all $(\bar{u},\bar{v})$ in $\bar{W}$ as follows:

(4) $$\|\bar{u}\|_p \leqq \nabla f(\bar{x})z$$

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(5) $$\|\bar{v}\|_p \leqq \max_{j \in I}\{\|\nabla f(\bar{x})B\|_p \ , \ \|(\nabla f(\bar{x}) + (\nabla f(\bar{x})z)\nabla g_j(\bar{x}))B\|_p\}$$

where B is the $n \times k$ matrix defined by

(6) $$B := \nabla h(\bar{x})^T(\nabla h(\bar{x})\nabla h(\bar{x})^T)^{-1}$$

and $\|\bar{u}\|_p$ denotes the p-norm $\left(\sum_{j=1}^{m} |\bar{u}_j|^p\right)^{1/p}$ for $p \in [1,\infty)$ and $\|\bar{u}\|_\infty = \max_{1 \leqq j \leqq m} |\bar{u}_j|$. In particular we have the following.

1. <u>Theorem</u>. Let $\bar{x}$ be a stationary point of (1). The corresponding nonempty set of all Lagrange multipliers $\bar{W}$ satisfying the Karush-Kuhn-Tucker conditions (3) is bounded if and only if the constraint qualification (2) holds, in which case each $(\bar{u},\bar{v})$ in $\bar{W}$ is bounded by (4) - (5) for $p \in [1,\infty]$.

<u>Proof</u>. The nonempty set $\bar{W}$ is bounded if and only if there exists <u>no</u> (u_I,v) satisfying

(7) $$u_I\nabla g_I(\bar{x}) + v\nabla h(\bar{x}) = 0, \ u_I \geqq 0, \ (u_I,v) \neq 0$$

which by a theorem of the alternative [3, Theorem 1(i') & (iii)], is equivalent to the constraint qualification (2). Hence for such a case we have for $(\bar{u},\bar{v}) \in \bar{W}$ and $p \in [1,\infty]$ that

(8) $$\|\bar{u}\|_p \leqq \|\bar{u}\|_1 \leqq \max_{(u_I,v) \in R^{m+k}} \{eu_I \mid u_I\nabla g_I(\bar{x}) + v\nabla h(\bar{x}) + \nabla f(\bar{x}) = 0, \ u_I \geqq 0\}$$

(8a) $$= \min_{z \in R^n} \{\nabla f(\bar{x})z \mid \nabla g_I(\bar{x})z \leqq -e, \ \nabla h(\bar{x})z = 0\}$$

(By linear programming duality)

$$\leqq \nabla f(\bar{x})z \quad \text{for} \quad z \in Z$$

which establishes (4).

Now, for any $(\bar{u},\bar{v}) \in \bar{W}$, $z \in Z$ and $p \in [1,\infty]$ we have that

(9) $$\|\bar{v}\|_p \leqq \max_{v,u_I} \{\|v\|_p \mid -v\nabla h(\bar{x}) = \nabla f(\bar{x}) + u_I\nabla g_I(\bar{x}), \ u_I \geqq 0\}$$

$$\leqq \max_{v,u_I} \{\|v\|_p \mid v = -(\nabla f(\bar{x}) + u_I\nabla g_I(\bar{x}))B, \ u_I \geqq 0, \ eu_I \leqq \nabla f(\bar{x})z\}$$

$$= \max_{u_I}\{\|(\nabla f(\bar{x}) + u_I \nabla g_I(\bar{x}))B\|_p \mid u_I \geqq 0,\ eu_I \leqq \nabla f(\bar{x})z\}$$

$$= \max_{j \in I}\{\|\nabla f(\bar{x})B\|_p\ ,\ \|(\nabla f(\bar{x}) + (\nabla f(\bar{x})z)\nabla g_j(\bar{x}))B\|_p\}$$

where the last equality follows from the fact that the maximum of a continuous convex function on a bounded polyhedral set is attained at a vertex [7, Corollary 32.3.4]. This establishes the bound (5).

□

2. Corollary. The bounds (4) - (5) of Theorem 1 can be sharpened by replacing z by $\bar{z}$ where $\bar{z}$ is a solution of the solvable linear program (8a).

We note that the bound (4) with $p = 1$ and $z = \bar{z}$, where $\bar{z}$ is a solution of (8a) is implicitly given in the elegant proof of Gauvin [1] which characterizes the nonemptiness and boundedness of $\bar{W}$ for a local solution $\bar{x}$ of (1) by the satisfaction of the constraint qualification (2).

It is interesting to note that the first part of the constraint qualification (2) (existence of z) gives an achievable bound on $\|\bar{u}\|_1$, whereas the second part of (2) (linear independence of the rows of $\nabla h(\bar{x})$) gives a bound on $\|\bar{v}\|_p$, which is not necessarily achievable. It is however possible (but impractical for large k) to compute $\max_{(u,v)\in \bar{W}} \|v\|_\infty$ by solving $2k$ linear programs: $\max_{1\leqq i\leqq k} \max_{(u,v)\in \bar{W}} \pm v_i$. However to obtain $\max_{(u,v)\in \bar{W}} \|v\|_1$ one is faced with the essentially impossible task (even for a moderate-sized $k \geqq 15$) of solving 2^k linear programs: $\max_{c\in C} \max_{(u,v)\in \bar{W}} cv$, where C is the set of 2^k vertices of the cube $\{y \mid y \in R^k, -e \leqq y \leqq e\}$. In fact for integer $p \geqq 1$ the problem $\max_{(u,v)\in \bar{W}} \|v\|_p$ has been shown to be an intractable NP-complete problem [6]. We finally note that the methods of [4] could also be used to obtain the bounds of this work.

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