ON DIGRAPHS WITH THE
ODD CYCLE PROPERTY

by

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Abstract

We say that a digraph $D$ has the odd cycle property if there exists an
eledge subset $S$ such that every cycle of $D$ has an odd number of edges from
$S$. We give necessary and sufficient conditions for a digraph to have the odd
cycle property. We also consider the analogous problem for graphs.

1. Introduction

A signed digraph $(D, S)$, is a digraph $D$ whose edges are associated with
signs, positive or negative, where $S$ is the set of negative edges. Several
problems related to the parity of the number of negative edges in the cycles
of signed graphs and digraphs have been studied. For instance, balanced
signed digraphs and graphs have been studied extensively. See for example [2].
A signed digraph (graph) is balanced if all its cycles have an even (possibly
zero) number of negative edges. Another, more recent example is the study of
signed digraphs, all of whose cycles have an odd number of negative edges.

This problem has been considered in the study of qualitative matrices, see
for example [1, 4, 6, 7]. Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ii} < 0$.
We say that $A$ is sign-nonsingular (or an L-matrix) if $A$ is nonsingular
and every matrix $B$ whose sign-pattern is the same as the sign-pattern of $A$,
is nonsingular. We associate with \( A \) a signed digraph \((D(A), S)\) with vertex set \( \{1, 2, \ldots, n\} \) corresponding to the rows of \( A \) and with edge set \( \{(i, j): a_{ij} \neq 0\} \). The set \( S \) of negative edges consists of those edges \((i, j)\) for which \( a_{ij} < 0 \). It has been shown that \( A \) is sign-nonsingular if and only if every elementary cycle in \( D(A) \) has an odd number of negative edges \([1]\). The time complexity of recognizing digraphs with the above property is still open. However, the time-complexity of several related problems have been determined in \([4, 5, 8]\).

It seems natural to consider the following question: Given a digraph \( D \), is there a subset \( S \) of the edge set such that every cycle of \( D \) has an odd number of edges from \( S \)? If such a set \( S \) exists, we say that \( D \) has the odd cycle property. We study this question together with the analogous question for graphs. Harary, Lundgren and Maybee \([3]\) have independently considered this question and have presented three families of digraphs with the odd cycle property. They have however left open the problem of characterizing general digraphs with the odd cycle property. We give here a necessary and sufficient condition for a digraph to have the odd cycle property. We also consider the special class of symmetric digraphs, which is one of the families treated in \([3]\). Our characterization of symmetric digraph with the odd cycle property is essentially the same as that of \([3]\) but our method of proof is different.
2. Basic definitions.

As the term is used here, a **walk** in a digraph [resp. graph] is a sequence $W$ of vertices $(i_0, i_1, \ldots, i_k)$ such that $k \geq 1$ and $(i_{j-1}, i_j)$ (resp. $[i_{j-1}, i_j]$) is an edge, $j = 1, \ldots, k$. The walk is a **path** if no vertex is repeated. The sequence $W$ is a **closed walk** if $i_0 = i_k$ and it is a **cycle** if no other vertex is repeated. Let $P$ be the walk $(u = x_0, x_1, \ldots, x_t = v)$ and let $Q$ be the walk $(v = y_0, y_1, \ldots, y_s)$. We denote by $P \cup Q$ the walk $(x_0, x_1, \ldots, x_t, y_1, \ldots, y_s)$. We say that two paths $P$ and $Q$ from $u$ to $v$ are **vertex disjoint** if $V(P) \cap V(Q) = \{u, v\} = \emptyset$.

Here we use the common notation $V(P)$ and $E(P)$ to denote the vertex set and edge set of $P$ respectively. In a digraph, every closed walk can be viewed as a union of cycles. However, this is not the case for an undirected graph. For example, if $[u, v]$ is an edge of a graph, the closed walk $(u, v, u)$ is not a union of cycles. Let $D$ be a digraph. The multiset of edges of a closed walk $W$ is written as a union of the edge sets of several cycles, where the union is understood to be in the multiset sense; for example, $\{e\} \cup \{e\} = \{2 \cdot e\}$.

We say that a digraph has the **unique parity property** if for every closed walk $W$ in $D$, $E(W) = \bigcup_{i=1}^k E(C_i)$ and $E(W) = \bigcup_{j=1}^\ell E(D_j)$ imply that $k \equiv \ell (\text{mod } 2)$, where $C_i$ and $D_j$ are cycles in $D$ ($i = 1, \ldots, k$; $j = 1, \ldots, \ell$). In the remainder of the paper we use the notation $x \equiv y$ to abbreviate $x \equiv y (\text{mod } 2)$. Let $D$ be a digraph with the unique parity property. For every closed walk $W$ we define the **parity** of $W$, denoted by $N(W)$, as follows

$$
N(W) = \begin{cases} 
1 & \text{if } E(W) = \bigcup_{i=1}^k E(C_i) \text{ and } k \equiv 1 \\
0 & \text{if } E(W) = \bigcup_{j=1}^\ell E(D_j) \text{ and } \ell \equiv 0
\end{cases}
$$
Since \( W \) has the unique parity property, \( N(W) \) is well defined. We observe that \( W_1 \) and \( W_2 \) are two closed walks having a common vertex, then with \( W = W_1 \cup W_2 \),

\[
N(W) = N(W_1) + N(W_2).
\]

The digraph \( D_1 \) of Fig. 1 has the unique parity property and \( N(W) = 0 \) for every closed walk of \( D_1 \). However, the digraph \( D_2 \) does not have the unique parity property since with \( W = 3,1,2,4,3,4,1,2,3 \), \( E(W) \) is a multiset and \( E(W) = E(C_1) \cup E(C_2) = E(D_1) \cup E(D_2) \cup E(D_3) \) where

\[
C_1 = 3,1,2,4,3 ; \quad C_2 = 3,4,1,2,3 ;
\]

\[
D_1 = 1,2,3,1 ; \quad D_2 = 1,2,4,1 \quad \text{and} \quad D_3 = 3,4,3
\]

but \( 2 \neq 3 \). We also note that \( D_1 \) has the odd cycle property while \( D_2 \) does not.

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**Figure 1**
Denote by \( c(D) \) the number of strongly connected components of the digraph \( D \). The **condensation digraph** of \( D \) is a multi-digraph whose vertices are the strongly connected components of \( D \) and for every edge \((u,v)\) of \( D \) with \( u \in D_i \) and \( v \in D_j \) there is an edge in the condensation graph from \( D_i \) to \( D_j \). The condensation digraph is clearly acyclic.
3. Digraphs with the odd cycle property

Theorem 3.1: Let \( D = (V,E) \) be a strongly connected digraph. Then \( D \) has the odd cycle property if and only if it has the unique parity property.

Proof: Suppose \( D \) has the odd cycle property. Then there exists a subset \( S \subseteq E \) such that for every cycle \( C \), \( |E(C) \cap S| \equiv 1 \). Let \( W \) be a closed walk in \( D \) with \( E(W) = E(C_1) \cup \ldots \cup E(C_k) \) and \( E(W) = E(D_1) \cup \ldots \cup E(D_\ell) \)

where \( C_i \) and \( D_j \) are cycles of \( D \) \( (i=1,\ldots,k; j=1,\ldots,\ell) \). Then

\[
E(W) \cap S = (S \cap E(C_1)) \cup \ldots \cup (S \cap E(C_k)) =
\]

\[
= (S \cap E(D_1)) \cup \ldots \cup (S \cap E(D_\ell)).
\]

Hence

\[
|E(W) \cap S| = \sum_{i=1}^{k} |S \cap E(C_i)| \equiv k
\]

and

\[
|E(W) \cap S| = \sum_{j=1}^{\ell} |S \cap E(D_j)| \equiv \ell.
\]

Thus \( k \equiv \ell \) and \( D \) has the unique parity property.

Now let \( D \) have the unique parity property. Let \( e \in E \) and \( D_e = (V,E\setminus \{e\}) \). Harary et al [3, Theorem 7], have shown that if for every edge \( e \), \( D_e \) is not strongly connected then \( D \) has the odd cycle property. They refer to such digraphs as ministrong. Thus ministrong digraphs have the odd cycle property and this is independnet of the unique parity property. We may now assume there exists an edge \( e = (u,v) \) such that \( D' = D_e \) is strongly connected. We proceed by induction on the number of edges in \( D \). When \( |E| = 2 \), either \( D \) has no cycles or \( D \) consists of a single cycle of length 2 and with \( S \) being any one
any one edge of $D$, we see that $D$ has the odd cycle property. Since $D$
has the unique parity property, so does $D'$. Hence by induction there is a
subset $S' \subseteq E - \{e\}$ such that every cycle of $D'$ has an odd number of
edges from $S'$. If every two paths $P_1$ and $P_2$ from $v$ to $u$ satisfy
$\vert E(P_1) \cap S' \vert \equiv \vert E(P_2) \cap S' \vert$, then let

$$(3.2) \quad S = \begin{cases} S' & \text{if } \vert E(P_1) \cap S \vert \equiv 1 \\ S' \cup \{e\} & \text{if } \vert E(P_1) \cap S \vert \equiv 0. \end{cases}$$

It follows that every cycle of $D$ has an odd number of edges from $S$. We
show that indeed, if $P_1$ and $P_2$ are paths from $v$ to $u$ then
$\vert E(P_1) \cap S' \vert \equiv \vert E(P_2) \cap S' \vert$. Since $D'$ is strongly connected there exists a path
$P$ from $u$ to $v$ in $D'$. Consider the walk $W = P_1 \cup P \cup P_2 \cup (u,v)$ in $D$.
Then since both $P_1 \cup (u,v)$ and $P_2 \cup (u,v)$ are cycles, we have

$$N(W) \equiv N(P \cup P_1) + N(P_2 \cup \{e\}) \equiv N(P \cup P_2) + 1$$

and

$$N(W) \equiv N(P \cup P_2) + N(P_1 \cup \{e\}) \equiv N(P \cup P_1) + 1.$$ 

Hence $N(P \cup P_1) \equiv N(P \cup P_2)$. But since $D'$ has the odd cycle property,
$N(W) \equiv \vert E(W) \cap S' \vert$ for every closed walk $W$ in $D'$. Hence

$$N(P \cup P_1) \equiv \vert (E(P) \cup E(P_1)) \cap S' \vert \equiv \vert E(P) \cap S' \vert + \vert E(P_1) \cap S' \vert$$

and

$$N(P \cup P_2) \equiv \vert (E(P) \cup E(P_2)) \cap S' \vert \equiv \vert E(P) \cap S' \vert + \vert E(P_2) \cap S' \vert.$$ 

Thus $\vert E(P_1) \cap S' \vert \equiv \vert E(P_2) \cap S' \vert$ and hence $D$ has the odd cycle property.

Since every closed walk in a digraph is contained in one of its strongly
connected components, we get:
Corollary 3.3: A digraph $D$ has the odd cycle property if and only if $D$ has the unique parity property.

The unique parity property is not easy to check. However, for some special classes of digraphs, the odd cycle property can be more easily checked. For example the class of ministrong digraphs can be recognized in polynomial time. More information on the structure of ministrong digraph is given in the next proposition.

Proposition 3.4: Let $D = (V,E)$ be a ministrong digraph and let $e \in E$.

Then the condensation digraph of $D_e = (V,E\setminus\{e\})$ is a directed path.

Proof: First we show that the condensation digraph is simple, namely it has no multiple edges. For suppose $G_i$ and $G_j$ are two connected components of $D_e$ and $(x,y) \neq (x',y')$ are edges of $D_e$ with $x,x' \in G_i$ and $y,y' \in G_j$. Then $D(x,y)$ is strongly connected, a contradiction. Now choose $e = (u,v) \in E$ such that $m = c(D_e) \leq c(D_f)$ for every $f \in E$. Let $G_1, \ldots, G_m$ be the strongly connected components of $D_e$ where $G_1$ contains $v$ and $G_m$ contains $u$.

Then $G_1 \neq G_m$. Note that since $D$ is strongly connected, there is a path in the condensation digraph of $D_e$ from $G_1$ to every component $G_j$, and there is no path from $G_m$ to any other component. Let $P$ be a path in the condensation digraph from $G_1$ to $G_m$. Suppose that for some $G_i$ in $P$ ($1 \leq i < m$) and for $G_j$ not in $P$ there exists an edge $e' = (u',v') \in E - \{e\} - E(P)$ with $u' \in G_i$ and $v' \in G_j$. Since $e$ is an edge of $D_e'$, the vertices belonging to the same strongly connected components in $P$, all belong to the same connected component of $D_e'$. Since $G_1 \neq G_m$ it follows that $c(D_e') < c(D_e)$, a contradiction to the minimality of $c(D_e)$. Thus the vertices of $P$ are all the strongly connected components of $D_e$ and we may assume $P = (G_1,G_2,\ldots,G_m)$. Moreover, if there exists an edge $f \in E - \{e\}$ joining a vertex of $G_i$ to a vertex of $G_j$ with
j \neq i + 1 \text{ then } G_j^r \text{ is strongly connected, a contradiction. It follows that }
the condensation digraph of } D_e \text{ is a path } P = (G_1, G_2, \ldots, G_m).

We conclude this section with a more structural characterization of
another special class of digraphs. As noted before, Harary et al [3] have
independently considered this class of digraphs. A digraph } D = (V,E) \text{ is
symmetric if } (u,v) \in E \text{ implies } (v,u) \in E. \text{ The graph } G \text{ obtained by iden-
tifying the edges } (u,v) \text{ and } (v,u) \text{ to form an unordered pair } [u,v] \text{ is
the underlying graph of } D. \text{ We denote by } D_G \text{ the symmetric digraph whose
underlying graph is } G.

Let } G \text{ be a graph and } C \text{ a cycle in } G. \text{ A diagonal path of } C \text{ is an
(elementary) path } P = (x_1, \ldots, x_k) \text{ such that } x_1, x_k \in V(C) \text{ and } x_i \notin V(C), i \neq 1, k.\n\text{ We start with two Lemmas.}

\textbf{Lemma 3.5:} \text{ Let } G \text{ be a graph and suppose } D_G \text{ has the odd cycle property
with the edge subset } S. \text{ If } P_1 \text{ and } P_2 \text{ are two vertex disjoint directed
paths of odd length from } u \text{ to } v \text{ in } D_G. \text{ Then } |E(P_1) \cap S| = |E(P_2) \cap S|.

\textbf{Proof.} \text{ Let } P_i \text{ be the directed path } (u = x_{i1}, x_{i2}, \ldots, x_{ik_i} = v) \text{ and let } \overset{+}{P_i}
\text{ be the directed path } (v = x_{ik_i}, \ldots, x_{i1} = v), (i=1,2). \text{ Then } P_1 \cup \overset{+}{P_2}
\text{ is a
cycle in } D_G \text{ and hence } |E(P_1 \cup \overset{+}{P_2}) \cap S| = 1. \text{ Since } P_i \text{ has odd length
}|E(P_i \cup \overset{+}{P_i}) \cap S| = 1 \text{ for } i = 1,2. \text{ It follows that } |E(P_1) \cap S| = |E(P_2) \cap S|.

Let } P \text{ be the directed path } (x_1, x_2, \ldots, x_k). \text{ Let } x = x_i \in V(P) \text{ and } \text{y} = x_j \in V(P) \text{ with } i < j. \text{ We denote by } xPy \text{ the directed path
}(x = x_i, x_{i+1}, \ldots, x_j = y) \text{ and by } yPx \text{ the directed path } (y = x_j, x_{j-1}, \ldots, x_i = x).

\textbf{Lemma 3.6.} \text{ Let } G = (V,E) \text{ be a bipartite graph in which every diagonal
path of every cycle has odd length. Let } e = [u,v] \text{ be an edge of } G \text{ and}
\( G_e = (V,E\setminus \{e\}) \). Suppose \( P \) and \( Q \) are two directed paths from \( u \) to \( v \) in \( D_G \). Then there exist \( t > 0 \) and vertices \( x_{11}, x_{21}, \ldots, x_{t1} \) such that with \( x_{01} = u \) and \( x_{t+1,1} = v \) (3.7) holds.

\[
(3.7) \quad P = \bigcup_{i=0}^{t} x_{i1}P_{x_{i+1}(i+1)1}, \quad Q = \bigcup_{i=0}^{t} x_{i1}Q_{x_{i+1}(i+1)1}
\]

and \( x_{i1}P_{x_{i+1}(i+1)1} \) and \( x_{i1}Q_{x_{i+1}(i+1)1} \) are vertex disjoint directed paths of odd length from \( x_{i1} \) to \( x_{(i+1)1} \).

**Proof:** Let \( P \) be a directed path. We use the notation \( x <_P y \) if both \( x \) and \( y \) are vertices of \( P \) and \( x \) precedes \( y \) on \( P \). Let \( x_{11} \neq u \) be the first intersection of \( P \) and \( Q \) along \( P \), and let \( y \) be the next intersection along \( P \). That is \( y >_P x_{11} \). If \( y >_Q x_{11} \) as well, we denote \( y \) by \( x_{21} \) otherwise we denote \( y \) by \( x_{12} \). Using this scheme we label all vertices in \( V(P) \cap V(Q) \) to obtain the following

\[
\begin{align*}
x_{11} <_P x_{12} <_P \cdots <_P x_{1n_1} <_P x_{21} <_P \cdots <_P x_{2n_2} <_P \cdots \\
<_P x_{t1} <_P \cdots <_P x_{tn_t}, \text{ where } t > 0 \text{ and } n_i \geq 1.
\end{align*}
\]

Moreover, \( x_{i1} <_Q x_{(i+1)1} \) but \( x_{i1} >_Q x_{ij} \) for \( j = 2, \ldots, n_i \) (See Fig. 3)

---

**Figure 3**

(The path \( P \) is a straight line, and the other path is \( Q \)).
Note that with \( j > i, 2 \leq k \leq n_j \) and \( 1 \leq \ell \leq n_i \), it is possible that \( x_{jk} \leq^{Q} x_{il} \), but \( x_{jl} \geq^{Q} x_{il} \). We show that with \( x_{i1}, x_{21}, \ldots, x_{t1} \) is satisfied. Since \( x_{i1} \leq^{P} x_{(i+1)1} \) for \( i = 0, \ldots, t \), \( p = \sum_{i=0}^{t} x_{i1}^{P}(i+1)1 \). Similarly since \( x_{i1} \leq^{Q} x_{(i+1)1} \) for \( i = 0, \ldots, t \), \( p = \sum_{i=0}^{t} x_{i1}^{Q}(i+1)1 \).

Next we note that \( x_{i1}^{P}(i+1)1 = (x_{i1}^{P}x_{i2}) \cup \cdots \cup x_{in_i}^{P}(i+1)1 \) and \( x_{ij} \leq^{Q} x_{il} \leq^{Q} x_{(i+1)1} \). It follows that \( x_{i1}^{P}(i+1)1 \) and \( x_{i1}^{Q}(i+1)1 \) are vertex disjoint. It remains to show that both have odd length. For \( i = 1, \ldots, t \), let \( C_i \) be a cycle in the undirected graph \( G \) defined by

\[
C_i = (uP_{i1}) \cup (x_{i1}^{Q}v) \cup (v, u) \quad i = 0, \ldots, t.
\]

Clearly \( uP_{i1} \) and \( x_{i1}^{Q}v \) are paths in \( G \). Moreover, since for \( j \leq i - 1 \) and \( k \leq n_j \), \( x_{jk} \leq^{Q} x_{jl} \leq^{Q} x_{il} \), the vertex \( x_{jk} \) is not a vertex of any segment \( x_{i1}^{Q}(\ell+1)1 \) for \( \ell \geq i \). It follows that the above two paths are vertex disjoint and hence \( C_i \) is a cycle. It also follows that \( x_{i1}^{P}(i+1)1 \) is a diagonal path of \( C_i \) and hence is of odd length. But \( x_{i1}^{Q}(i+1)1 \) and \( x_{i1}^{P}(i+1)1 \) are vertex disjoint paths joining \( x_{i1} \) and \( x_{(i+1)1} \) in \( G \) and hence \( (x_{i1}^{P}(i+1)1) \cup (x_{(i+1)1}^{Q}x_{i1}) \) is a cycle in \( G \). Since \( G \) is bipartite, the path \( x_{i1}^{Q}(i+1)1 \) has odd length.

**Theorem 3.7**: Let \( G = (V, E) \) be a graph. Then \( D_G \) has the odd cycle property if and only if \( G \) is bipartite and every diagonal path of every cycle in \( G \) has odd length.

**Proof**: Suppose first that \( D_G \) has the odd cycle property. Let \( C = (v_1, \ldots, v_k, v_1) \) be a cycle in \( G \). Then the edge set of the closed walk \( D_C \) in \( D_G \) can be written both as a union of \( k \) cycles of length 2 and as
a union of 2 cycles of length $k$. It follows from Corollary 3.3 that $k \equiv 2$
and hence $G$ is bipartite. Now let $P = (u_1, \ldots, u_{s+1})$ be a diagonal path,
of $C$, that is $u_1 = v_1$ and $u_{s+1} = v_i$ for some $1 \leq i \leq k$. Denote by
$\vec{C}$ the directed cycle $(v_1, \ldots, v_k, v_1)$ and by $\vec{P}$ the directed path $(u_1, \ldots, u_{s+1})$.
Consider the following directed paths in $D_G$:

\[ Q_1 = v_1 \vec{C} v_i ; \quad Q_2 = v_i \vec{C} v_1 ; \quad P_1 = v_1 \vec{P} v_i \quad \text{and} \quad P_2 = v_i \vec{P} v_1 . \]

Then with $W = Q_1 \cup Q_2 \cup P_1 \cup P_2$,

\[ N(W) = N(Q_1 \cup Q_2) + N(P_1 \cup P_2) \equiv s + 1 \]

and

\[ N(W) = N(Q_1 \cup P_2) + N(Q_2 \cup P_1) \equiv 2 . \]

Hence $s \equiv 1$; that is, every diagonal path in $G$ has odd length.

Conversely, suppose $G$ is bipartite and every diagonal path in $G$ has
odd length. Then the same holds for every subgraph of $G$. We proceed by
induction on the number of edges in $G$. When $|E| = 2$ then $D_G$ has at most
one cycle which is of length 2 and hence $D_G$ has the odd cycle property.
Let $e = [u, v] \in E$ and let $G_{e} = (V, E - \{e\})$. By the inductive hypotheses,
$D_{G_{e}}$ has the odd cycle property with some $S' \subseteq E - \{e\}$. Let $P$ and $Q$
be two directed paths in $D_{G_{e}}$ from $u$ to $v$. Then by Lemma (3.6), there
exist vertices $x_{11}, x_{21}, \ldots, x_{t1}$ of $P$ such that (3.7) is satisfied. Moreover,
by Lemma (3.5), the two vertex disjoint paths $P_1 = x_{11} \vec{P} (i+1)1$ and
$Q_1 = x_{11} \vec{Q} (i+1)1$ satisfy $|E(P_1) \cap S'| = |E(Q_1) \cap S'|$. It now follows that
$|E(P \cap S')| = |E(Q \cap S')|$. Similarly, if $P$ and $Q$ are two directed paths
from $v$ to $u$ in $D_{G_{e}}$ then $|E(P \cap S')| = |E(Q \cap S')|$. Let $P$ be a directed
path from u to v in $D_{G_e}$ and Q a directed path from v to u in $D_{G_e}$.

Since G is bipartite and $[u,v]$ is an edge of G both P and Q have odd length. Hence $P \cup \bar{P}$ is the union of an odd number of cycles of length 2 in $D_{G_e}$. Moreover since $D_{G_e}$ has the odd cycle property $|E(P) \cap S'| \equiv |E(\bar{P}) \cap S'|$. But since as shown above $|E(\bar{P}) \cap S'| \equiv |E(Q) \cap S'|$ we get $E(P) \cap S' \not\equiv E(Q) \cap S'$.

Now let

$$S = \begin{cases} 
S' \cup (v,u) & \text{if } k \text{ is even } (\ell \text{ is odd}) \\
S' \cup (u,v) & \text{if } k \text{ is odd } (\ell \text{ is even})
\end{cases}$$

Then with S as above, $D_G$ has the odd cycle property and the proof is complete.
4. Graphs with the odd cycle property.

A graph $G$ has the \textit{odd cycle property} if there is an edge subset $S$ such that every cycle of $G$ has an odd number of edges in $S$. Harary et al. [2] have shown that if a signed-graph has a block with more than one cycle then this block contains a cycle with an even number of negative edges. This provides a structural characterization of graphs with the odd cycle property. We show here that an undirected graph with the odd cycle property can also be characterized by an analogue to the unique parity property of digraphs.

We have noted before that a closed cycle in an undirected graph is not always a union of cycles. However, it is true that the edge set of every closed walk $W$ in a graph can be written as

$$E(W) = E(C_1) \cup \ldots \cup E(C_k) \cup \{2n_1 \cdot e_1, \ldots, 2n_t \cdot e_t\},$$

where $C_1, \ldots, C_k$ are cycles, $e_1, \ldots, e_i$ are edges and $n_1, \ldots, n_t$ are non-negative integers. We say that a graph has the \textit{unique parity property} if for every closed walk $W$, $E(W) = E(C_1) \cup \ldots \cup E(C_k) \cup \{2n_1 \cdot e_1\} \cup \ldots \cup \{2n_t \cdot e_t\}$ and $E(W) = E(D_1) \cup \ldots \cup E(D_k) \cup \{2m_1 \cdot e'_1\} \cup \ldots \cup \{2m_s \cdot e'_s\}$ imply that $k \equiv l$.

Theorem 4.1: \textit{Let $G$ be a graph. Then the following are equivalent:}

(i) $G$ has the odd cycle property.

(ii) $G$ has the unique parity property.

(iii) No cycle of $G$ has a diagonal path.

(iv) Every block $G$ is either an edge or a single cycle.

Proof: (i) $\Rightarrow$ (ii) Suppose $S$ is an edge subset so that every cycle of $G$ has an odd number of edges from $S$. Let $W$ be a walk such that

$$E(W) = E(C_1) \cup \ldots \cup E(C_k) \cup \{2n_1 \cdot e_1\} \cup \ldots \cup \{2n_t \cdot e_t\}$$
and

\[ E(W) = E(D_1) \cup \ldots \cup E(D_{t}) \cup \{2m_1 \cdot e_1\} \cup \ldots \cup \{2m_s \cdot e_s\} . \]

Then \( |E(W) \cap S| = k \) and \( |E(W) \cap S| = \ell \) and hence \( k = \ell \).

(ii) \( \Rightarrow \) (iii). Let \( G \) be a graph with the unique parity property. Let \( C \) be a cycle in \( G \) and suppose \( P \) is a diagonal path of \( C \) joining the vertices \( u \) and \( v \) of \( C \). Let \( P_1 = uCV \), \( P_2 = vCU \) and \( E(P) = \{e_1, \ldots, e_t\} \). Then \( W = P_1 \cup P \cup P_2 \cup P \) is a closed walk in \( G \). Moreover \( E(W) = E(C) \cup \{2 \cdot e_1\} \cup \ldots \cup \{2 \cdot e_t\} \) and \( E(W) = E(P_1 \cup P) \cup E(P_2 \cup P) \) where \( C, P_1 \cup P \) and \( P_2 \cup P \) are cycles, but \( 1 \neq 2 \), a contradiction.

(iii) \( \Rightarrow \) (iv) Suppose no cycle of \( G \) has a diagonal path. Let \( B \) be a block of \( G \) with more than one edge. Then \( B \) has a cycle \( C \). If \( B \neq C \) then there exists an edge \( [u,v] = e \) in \( B \) but not in \( C \) with \( u \in V(C) \).

Let \( e' = [x,y] \) be any edge of \( C \). Since \( B \) is a block \( e \) and \( e' \) are contained in a cycle \( C' = (u,v,\ldots,u) \) in \( B \). Let \( z \neq u \) be the first vertex of \( C' \) which is in \( C \), then \( uC'z \) is a diagonal path of \( C \), a contradiction.

(iv) \( \Rightarrow \) (i) Since every cycle lies in some block and every block consists of at most one cycle, we construct \( S \) as follows. For every cycle \( C_i \) of \( G \) choose any edge \( e_i \) of \( C_i \) and let \( S = \bigcup_{i} e_i \).
References


