A CONJUGATE DECOMPOSITION OF THE EUCLIDEAN SPACE

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ABSTRACT

Given a closed convex cone \( K \) in the \( n \)-dimensional real Euclidean space \( \mathbb{R}^n \) and an \( n \times n \) real matrix \( A \) which is positive definite on \( K \), we show that each vector in \( \mathbb{R}^n \) can be decomposed into a component which lies in \( K \) and another which lies in the conjugate cone induced by \( A \) and such that the two vectors are conjugate to each other with respect to \( A + A^T \). As a consequence of this decomposition we establish the following characterization of positive definite matrices: An \( n \times n \) real matrix \( A \) is positive definite if and only if it is positive definite on some closed convex cone \( K \) in \( \mathbb{R}^n \) and \( (A+A^T)^{-1} \) exists and is positive semidefinite on the polar cone \( K^0 \). If \( K \) is a subspace of \( \mathbb{R}^n \) then \( K^0 \) is its orthogonal complement \( K^\perp \). Other applications include local duality results for nonlinear programs and other characterizations of positive definite and semidefinite matrices.

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Let $A$ be an $n \times n$ real matrix and $K$ be a closed convex cone in the $n$-dimensional real Euclidean space $\mathbb{R}^n$. A vector $a$ in $\mathbb{R}^n$ is said to have a conjugate decomposition with respect to $A$ and $K$ if there exists an $x$ in $K$ and a $y$ in the conjugate cone

$$K_A := \{y \mid y^T(A + A^T)x \leq 0, \forall x \in K\}$$

such that

$$a = x + y \quad \text{and} \quad x^T(A + A^T)y = 0.$$ 

When $K$ is a subspace and $A$ is the identity matrix such a decomposition becomes the classical orthogonal decomposition of a vector into its projections onto the subspace $K$ and its orthogonal complement $K^\perp$. It is well known that, in this case, such a decomposition exists and is unique for any given vector $a$. This result was generalized by Moreau [5] to the case where $K$ is any closed cone in a Hilbert space with any Hilbertian norm. Thus, if $A$ is positive definite on the entire space, it defines a norm $\|x\|_A^2 := x^T A x$, and our decomposition result, Theorem 1, follows directly from Moreau's theorem. The main point of this paper is to extend Moreau's result to the case where $A$ is not positive definite on the entire space but merely on the closed convex cone $K$, that is, $x^T A x > 0$ whenever $0 \neq x \in K$. Although the results of this paper are extendable to a Hilbert space, they are presented here only for a real Euclidean space. We begin with our first principal result.

**Theorem 1** Let $A$ be an $n \times n$ real matrix and $K$ be any closed convex cone in $\mathbb{R}^n$. If $A$ is positive definite on $K$ then any vector in $\mathbb{R}^n$ has a conjugate decomposition with respect to $A$ and $K$. Moreover, if $A$ is positive definite on the linear hull of $K$ then the decomposition is unique.
Proof (Existence) Let \( a \) be a given fixed vector in \( \mathbb{R}^n \), and let

\[
S := \{ x | x \in K \text{ and } \| x \| \leq \frac{\| a^T (A + A^T) \|}{\alpha} \}
\]

where \( \| \cdot \| \) denotes the Euclidean norm and

\[
\alpha := \min \{ x^T A x | \| x \| = 1, x \in K \} > 0.
\]

Let \( f(x) := (x-a)^T A (x-a) \) and consider the following problems

\((Q)\) \hspace{1cm} \min \{ f(x) | x \in K \},

\((Q')\) \hspace{1cm} \min \{ f(x) | x \in S \}.

Any solution of \((Q')\) also solves \((Q)\) because for any \( x \in K \setminus S \),

\[
f(x) = x^T A x - a^T (A + A^T) x + a^T A a \\
\geq (\alpha \| x \| - \| a^T (A + A^T) \|) \| x \| + a^T A a \\
> f(0).
\]

It follows from the compactness of \( S \) that \((Q)\) has a solution \( \bar{x} \), say. Then by the minimum principle [4, Theorem 9.3.3], we have that

\[
(x - \bar{x})^T (A + A^T) (x - \bar{x}) \geq 0 \quad \forall x \in K.
\]

By letting \( x = 2\bar{x} \) and \( x = 0 \) and letting \( \bar{y} = a - \bar{x} \), we have that

\[
a = \bar{x} + \bar{y}, \bar{x} \in K, \bar{y} \in K^A \text{ and } \bar{x}^T (A + A^T) \bar{y} = 0,
\]

which is a conjugate decomposition of \( a \) with respect to \( A \) and \( K \).

(Uniqueness) Let \( a = \hat{x} + \hat{y} = \bar{x} + \bar{y} \) be two conjugate decompositions of \( a \). Then it follows from \( \bar{x} - \hat{x} = \hat{y} - \bar{y} \) that
\[(\bar{x} - \hat{x})^T A (\bar{x} - \hat{x}) = \frac{1}{2} (\bar{x} - \hat{x})^T (A + A^T) (\bar{y} - \hat{y})
\]
\[= \frac{1}{2} \bar{x}^T (A + A^T) \bar{y} + \frac{1}{2} \hat{x}^T (A + A^T) \hat{y}
\]
\[\leq 0
\]

Since \(A\) is positive definite on the linear hull of \(K\), the last inequality can hold only when \(\bar{x} = \hat{x}\). Hence, the decomposition is unique. \(\Box\)

An important consequence of Theorem 1 is the following characterization of positive definite matrices.

**Theorem 2** Let \(A\) be an \(n \times n\) real matrix and let \(K\) be a closed convex cone in \(\mathbb{R}^n\). \(A\) is positive definite if and only if \(A\) is positive definite on \(K\) and \((A + A^T)^{-1}\) exists and is positive semidefinite on the polar cone \(K^0 := \{y | y^T x \leq 0, \forall x \in K\}\).

**Proof** The "only if" is trivially true. Let \(a\) be any given vector in \(\mathbb{R}^n\), then by Theorem 1, there exists a conjugate decomposition \(a = \bar{x} + \bar{y}\) with \(\bar{x} \in K\), \(\bar{y} \in K^0\) and \(\bar{x}^T (A + A^T) \bar{y} = 0\). Let \(\bar{z} = (A + A^T) \bar{y}\) then \(\bar{z} \in K^0\). Thus
\[a^T A a = (\bar{x} + \bar{y})^T A (\bar{x} + \bar{y})
\]
\[= \bar{x}^T A \bar{x} + \frac{1}{2} \bar{y}^T (A + A^T) \bar{y}
\]
\[= \bar{x}^T A \bar{x} + \frac{1}{2} \bar{z}^T (A + A^T)^{-1} \bar{z} \geq 0.
\]

Hence, \(A\) is positive semidefinite and so is \(A + A^T\). Since \(A + A^T\) is nonsingular, \(A + A^T\) is in fact positive definite and so is \(A\). \(\Box\)

A direct consequence of Theorem 2 is the following.
Corollary 3 Let $A$ be an $n \times n$ real matrix and let $K$ be a closed convex cone in $\mathbb{R}^n$ such that $-K^0 \subseteq K$. $A$ is positive definite if and only if $A$ is positive definite on $K$ and $(A+A^T)^{-1}$ exists and is positive semidefinite on $K$.

If we let $K = \{x | Bx \preceq 0\}$ in Corollary 3 where $B$ is some $m \times n$ real matrix, then $-K^0 = \{y | x^T y \geq 0, \forall x \in K\} = \{y | y = -B^T u, u \geq 0\}$. Hence $-K^0 \subseteq K$ if and only if $BB^T u \succeq 0$ for all $u \geq 0$ or equivalently if $BB^T \succeq 0$. Consequently we have the following.

Corollary 4 Let $A$ be $n \times n$ real matrix and let $B$ be an $m \times n$ real matrix such that $BB^T \succeq 0$. $A$ is positive definite if and only if $A$ is positive definite on $K = \{x | Bx \preceq 0\}$ and $(A+A^T)^{-1}$ exists and is positive semidefinite on $K$.

By letting $B$ be the negative of the identity matrix in Corollary 4 we obtain the following interesting characterization of positive definite matrices in terms of strictly copositive and copositive matrices.

Corollary 5 A necessary and sufficient condition for an $n \times n$ real matrix $A$ to be positive definite is that $A$ be strictly copositive (that is $x^T Ax > 0$ for $0 \neq x \succeq 0$) and $A + A^T$ has a copositive inverse (that is $x^T (A+A^T)^{-1} x \geq 0$ for all $x \succeq 0$).

By letting $K$ in Theorem 2 be a subspace of $\mathbb{R}^n$, we get the following result obtained in [1] by a different technique which does not extend to cones.

Corollary 6 Let $A$ be an $n \times n$ real symmetric matrix and $K$ be a subspace of $\mathbb{R}^n$. $A$ is positive definite if and only if $A$ is positive definite on $K$ and $A^{-1}$ exists and is positive semidefinite on the orthogonal complement $K^\perp$ of $K$. 

Applications of Corollary 6 and Theorem 2 to local duality results of nonlinear programming are given in [1,3]. Additional results pertaining to conjugate decomposition with respect to positive semidefinite matrices are given in [2]. Other possible applications are to the theory of penalty functions and augmented Lagrangians [6].
References


