COMPLETE CLOSEDNESS OF MAXIMAL MONOTONE OPERATORS

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Abstract

A maximal monotone operator in R^n is completely closed, i.e., not only closed for points, but also closed for directions. Such a completely closed operator is locally bounded at a point if and only if it is bounded at that point.

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We discuss some properties of maximal monotone operators in R^n. First we show that the recession cone of the function value set of a maximal monotone operator at each point of its domain is exactly the normal cone of the closure of its domain at that point. This implies that the recession directions of the function value are only determined by its domain. Then we show that a maximal monotone operator is not only closed for points but also closed for directions. We call such operators completely closed. A completely closed operator is locally bounded at a point if and only if it is bounded at that point. The local boundedness of a maximal monotone operator in the interior of its domain can be regarded as a corollary of this fact.

Suppose that $F:R^n \to R^n$ is a (set-valued) maximal monotone operator. Recall that a set-valued operator is called closed if its graph is closed. The abbreviations dom, int and cl denote domain, interior and closure respectively. Denote the recession cone of a set C by 0^+C. Recall that if C is a nonempty closed convex set,

$$0^+C = \{z \in R^n | \text{ for } \underline{\text{some}} \ \ y \in C \ \text{ and all } \ \lambda \geq 0,\ y + \lambda z \in C\}$$

[3] and that if $x \in \text{dom } F$, $F(x)$ is a nonempty closed convex set by the properties of a maximal monotone operator [2]. Denote the normal cone of a convex set C at $x \in C$ by $N_C(x)$. Recall [3] that

$$N_C(x) = \{z \in R^n | \langle u-x, z\rangle \leq 0,\ \forall u \in C\}.$$

We have

Theorem 1. If F is a maximal monotone operator in R^n, then for each $x \in \text{dom } F$,

$$0^+(F(x)) = N_C(x) ,$$

where $C = \text{cl dom } F$.

Proof. From [6], we know C is convex. Suppose $z \in N_C(x)$. For any $(u,v) \in \text{Graph } F$, we have $u \in \text{dom } F \subset C$,

$$\langle u-x, z\rangle \leq 0 .$$

Since $x \in \text{dom } F$, there is $y \in F(x)$. From the monotonicity,

$$<x-u,\ y-v> \geq 0 .$$

Therefore, for any $\lambda \geq 0$, and for each $(u,v) \in \text{Graph } F$,

$$<x-u,\ (y+\lambda z)-v> \geq 0 .$$

By maximality, $(x,y+\lambda z) \in \text{Graph } F$, i.e., $\{y+\lambda z | \lambda \geq 0\} \subset F(x)$ for every $y \in F(x)$. Therefore $z \in 0^+(F(x))$.

On the other hand, suppose $z \in 0^+(F(x))$. There is some $y \in F(x)$ with

$$\{y+\lambda z | \lambda \geq 0\} \subset F(x) .$$

From monotonicity, for any $(u,v) \in \text{Graph } F$,

$$<u-x,\ v-(y+\lambda z)> \geq 0 , \quad \forall \lambda \geq 0 .$$

i.e.,

$$<u-x,\ z+\frac{1}{\lambda}(y-v)> \leq 0 , \quad \forall \lambda > 0 .$$

Letting $\lambda \to +\infty$, we have

$$<u-x,\ z> \leq 0 , \quad \forall u \in \text{dom } F .$$

By closedness, this is also true for $u \in \text{cl dom } F$; therefore, $z \in N_C(x)$. This proves the theorem.

We now show that a maximal monotone operator has strong closedness.

Definition 1. A set valued operator F in R^n is called completely closed if

(a) it is closed;

(b) for any $x \in \text{dom } F$, $\{x_i,\ i=1,2,3,\ldots\} \subset \text{dom } F$, with $x_i \to x$, $y_i \in F(x_i)$, $i=1,2,3,\ldots$, $\|y_i\| \to +\infty$, $y_i/\|y_i\| \to z$, it holds that $z \in 0^+(F(x))$.

Theorem 2. A maximal monotone operator is completely closed.

Proof. It is known that a maximal monotone operator is closed. Suppose the conditions of (b) hold. For any $(u,v) \in \text{Graph } F$,

$$\langle x_i-u,\ y_i-v\rangle \geq 0\,, \qquad i=1,2,3,\ldots,$$

$$\langle x_i-u,\ (y_i-v)/\|y_i\|\rangle \geq 0\,, \quad i=1,2,3,\ldots.$$

Letting $i \to \infty$, we have for each $u \in \text{dom } F$,

$$\langle x-u,\ z\rangle \geq 0\,.$$

By closedness, this is true for any $u \in \text{cl dom } F$, i.e., $z \in N_C(x)$. Since $x \in \text{dom } F$, from Theorem 1, we know that $z \in 0^+(F(x))$. This completes the proof.

A completely closed operator has better properties than a merely closed operator.

Theorem 3. Suppose F is a completely closed operator in R^n. F is locally bounded at $x \in \text{dom } F$ iff $F(x)$ is bounded. In this case, F is upper semicontinuous at x.

Proof. The "only if" is obvious. If F is not locally bounded at x, there must be a sequence $\{(x_i,y_i),\ i=1,2,3,\ldots\} \subset \text{Graph } F$, $x_i \to x$, $\|y_i\| \to \infty$. Without loss of generality, suppose $y_i/\|y_i\| \to z$, $\|z\| = 1$.

From Definition 1, we know that $z \in 0^+(F(x))$, so $F(x)$ must be unbounded. Since closedness and local boundedness imply upper semicontinuity, the theorem is proved.

Since $N_C(x) = \{0\}$ iff $x \in \text{int } C$, from above three theorems, we easily get the following already known result [4]:

Corollary. If F is a maximal monotone operator in R^n, then F is locally bounded and upper semicontinuous at every $x \in \text{int dom } F$ and unbounded at boundary points of its domain.

A referee has kindly pointed out the first two of the following remarks:

Remark 1. The Theorem 1 can be readily extended to the infinite dimensional case. So is the Theorem 2, replacing the strong topology on the y variable by the weak topology. On the contrary, the Theorem 3 is no more true in the infinite dimensional case: One has only $\|z\| \leq 1$, hence z may be equal to zero and the conclusion fails.

Remark 2. The Theorem 1 gives us some information about the structure of the set $F(x)$. A closely related result is the following: Let F be a maximal monotone operator, M the closed subspace generated by the domain of F, and $M^\perp$ its orthogonal. Then the operator $\bar{F}(x) = \text{proj}_M F(x)$ is a maximal monotone operator in M, and $\forall x \in \text{dom } F$, $F(x) = \bar{F}(x) + M^\perp$. For the proof, see [10].

Remark 3. It is not difficult to see that:

(a) If F is completely closed, α is a real number, then αF is also a completely closed operator.

(b) If F and G are completely closed operators, then $F + G$ is also a completely operator.

Hence, If F and G are maximal monotone operators, then $-F$ and $G - F$ are completely closed, though they are not maximal monotone operators.

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References

1. Attouch, H., On the maximality of the sum of two maximal monotone operators, Nonlinear Analysis, Theory, Method & Applications 5, 143-147 (1981).

2. Brezis, H., Opérateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland/American Elsevier, Amsterdam/New York (1973).

3. Rockafellar, R. T., Convex Analysis, Princeton University Press, Princeton, N.J. (1970).

4. Rockafellar, R. T., Local boundedness of nonlinear, monotone operators, Michigan Math. J. 16, 397-407 (1969).

5. Rockafellar, R. T., On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149, 75-88 (1970).

6. Rockafellar, R. T., On the virtual convexity of the domain and range of a nonlinear maximal monotone operator, Math. Ann. 185, 81-90 (1970).

7. Attouch, H., Familles d'opérateurs maximaux monotones et mesurabilité, Ann. di math. IV, vol. CXX, 35-111 (1979).

8. Browder, F., Nonlinear maximal monotone operators in Banach spaces, Math. Annalen 175, 89-113 (1968).

9. Menou, S., Famille mesurable d'operateurs maximaux monotones, Thèse, Besançon (1979).

10. Robinson, S., Convex Programming, mimeographed notes, (1981).

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