# ON PACKING TWO-DIMENSIONAL BINS

by

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## Abstract

Suppose we are given a set L of rectangular items and wish to pack them into identical rectangular bins, so that no two items overlap and so that the number of bins used is minimized. This generalization of the standard one-dimensional bin packing problem models problems arising in a variety of applications, from truck loading to the design of VLSI chips. We propose a hybrid algorithm, based on algorithms for simpler bin packing problems, and show that proof techniques developed for the simpler cases can be combined to prove close bounds on the worst case behavior of the new hybrid. These are the first such close bounds obtained for this problem.

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## 1. Two-Dimensional Bin Packing

Let $L = \{r_1, r_2, \ldots, r_n\}$ be a set of rectangles, each rectangle $r$ having height $h(r)$ and width $w(r)$. A packing $P$ of $L$ into a collection $\{B_1, B_2, \ldots, B_m\}$ of $H \times W$ rectangular bins is an assignment of each rectangle to a bin and a position within that bin such that (a) each rectangle is contained entirely within its bin, with its sides parallel to the sides of the bin, and (b) no two rectangles in a bin overlap. See Figure 1 for an example of such a packing. In this paper we also assume that the orientations of the rectangles cannot be changed — the width of a rectangle must be aligned with the width of the bin. (The case when 90° rotations are allowed will be discussed in the conclusion.) In what follows, we shall assume that the bin dimensions $H$ and $W$ have been fixed and hence all packings are into bins of that size.

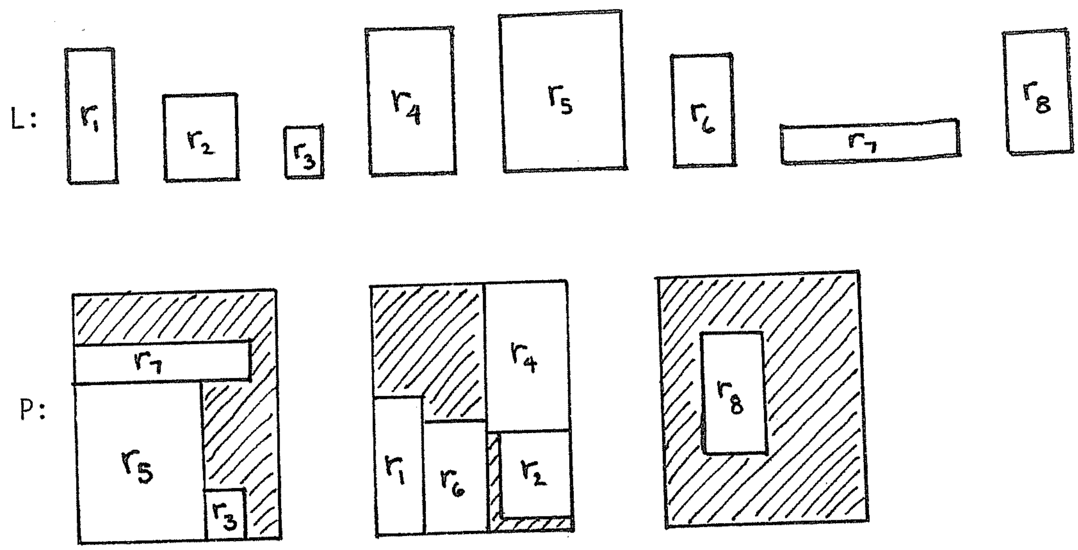



Figure 1. Example of a packing $P$ of a list $L$ of rectangles into 3 bins with $H = 20$, $W = 16$. Rectangle dimensions are 11×4, 7×6, 4×3, 12×7, 13×10, 9×5, 3×14, and 10×5.

If $P$ is a packing, let $|P|$ denote the number of non-empty bins in $P$. Given a list $L$, let $OPT(L)$ be defined to be $\min\{|P|: P$ is a packing of $L\}$. We are interested in finding packings $P$ with $|P|$ close to $OPT(L)$. (Determining $OPT(L)$, given $L$, is an NP-hard problem [1,7], and so it is unlikely that we can find optimal packings efficiently.)

This problem is related to two simpler and well-studied packing problems: one dimensional bin packing [9,10] and two dimensional strip packing [2,3,4]. The first is equivalent to the special case of our problem in which $w(r) = W$ for all $r \in L$. In the second we are once more given an arbitrary set of rectangles, but this time we are asked to pack them into a strip of width $W$ so as to minimize the height of the strip used. Although considerable progress has been made in analyzing the worst case behavior of algorithms for these two simpler problems, until now there has been little success in extending the results to the case of two dimensional bin packing. In this paper we make a start in this direction by proposing an appealing hybrid algorithm and obtaining close bounds on its asymptotic worst case behavior.

## 2. Asymptotic Worst Case Analysis

We measure the asymptotic worst case behavior of an algorithm A by the quantity $R_A^\infty$, defined as follows: Let A(L) be the value of the packing obtained by applying A to L. (A(L) would be either the number of bins or the strip height, depending on the problem.) Let OPT(L) be the corresponding optimal value. We then define $R_A(L) \equiv A(L)/OPT(L)$, $R_A^n \equiv \max \{R_A(L)$: L satisfies $OPT(L) = n\}$, and finally $R_A^\infty = \limsup_{n\to\infty} R_A^n$. The closer $R_A^\infty$ is to one, the better is the asymptotic worst case behavior of A.

Our hybrid algorithm is built from algorithms already developed for the simpler cases. The FIRST FIT algorithm (FF) for the one dimensional problem places the first item at the bottom of the first bin, and thereafter places each item in turn in the lowest indexed bin which has room for it. In [9,10] it is shown that $R_{FF}^\infty = \frac{17}{10}$. The FIRST FIT DECREASING algorithm (FFD) is the same as FF, except that the items to be packed are initially reordered so that $h(r_1) \geq h(r_2) \geq \ldots \geq h(r_n)$. For this algorithm we have [9,10] that $R_{FFD}^\infty = \frac{11}{9} = 1.222\ldots$

We shall be using FFD together with a strip packing algorithm based on FF, which we call FIRST FIT BY DECREASING HEIGHT (FFDH). The FFDH algorithm constructs a packing in which the strip is stratified into <u>blocks</u>, each block running the full width of the strip and resting on the top of the previous block (the first block rests on the bottom of the strip). Within the blocks, rectangles are packed linearly, each with its bottom edge resting on the bottom of the block. The height of a block is the height of the tallest rectangle it contains.

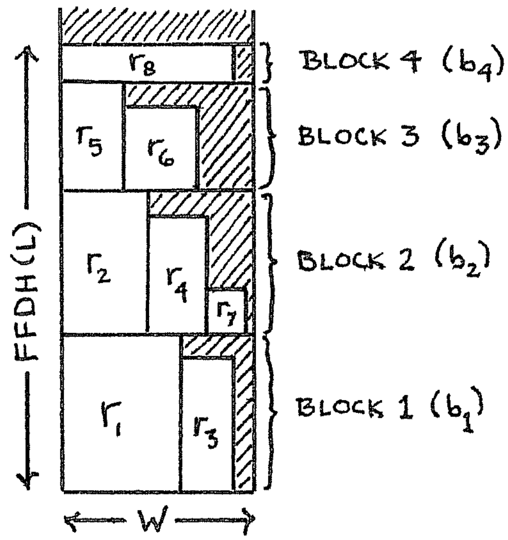



Figure 2. Example of an FFDH Packing of a list L of rectangles with dimensions 13×10, 12×7, 11×4, 10×5, 9×5, 7×6, 4×3, and 3×14.

Algorithm FFDH works by first reordering the set L of rectangles so that $h(r_1) \geq h(r_2) \geq \ldots \geq h(r_n)$ and then proceeding as follows: Place the first rectangle left-justified in the first block. Thereafter the rectangles are assigned in turn, each rectangle being placed as far to the left as possible in the lowest block which has room for it along its bottom edge. A new block is started on top of the current top block whenever the rectangle will not fit in any of the current blocks. See Figure 2 for the FFDH packing of the rectangles of Figure 1, appropriately re-indexed by height.

Note that if all the rectangles were the same height, FFDH would be equivalent to FF, with the blocks playing the role of bins. In [4] it is shown that the fact that rectangles may have differing heights is not as damaging as one might think, for $R^{\infty}_{FFDH} = R^{\infty}_{FF} = \frac{17}{10}$.

## 3. A Hybrid Algorithm

Our hybrid algorithm is now quite easily described. First create a strip packing for $L$ using FFDH and strip width $W$, thereby obtaining a collection $\{b_1,b_2,\ldots,b_k\}$ of blocks of non-increasing heights $h_1 \geq h_2 \geq \ldots \geq h_k$ each containing a subset of the rectangles. If we view these blocks as a new collection of rectangles $L' = \{b_1,b_2,\ldots,b_k\}$ with $h(b_i) = h_i$ and $w(b_i) = W$, $1 \leq i \leq k$, we have an instance of the one-dimensional problem and can apply FFD to pack the blocks (and hence the rectangles they contain) into $H \times W$ bins. See Figure 3, where FFD has been applied to the blocks of the strip packing in Figure 2. We call this hybrid algorithm HYBRID FIRST FIT (HFF). Our main result is

$$2.022 \leq R^{\infty}_{HFF} \leq 2.125$$

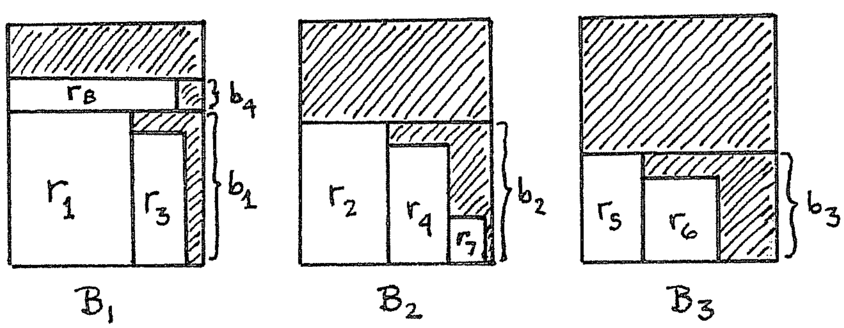



Figure 3. HFF Packing based on the FFDH packing of Figure 2.

In Figure 4 we present a schematic for instances $L$ of two-dimensional bin packing with arbitrarily large values of OPT(L) for which $HFF(L) = \frac{91}{45}(OPT(L) - 1)$. These instances will thus imply the lower bound $R^{\infty}_{HFF} \geq 2.0222\ldots$ The optimal packing is shown in 4(a) and consists of three types of bins: 42n bins containing items of types A, B, and E, packed as illustrated, followed by 48n bins containing items of types A, C, D, and E, packed as illustrated, followed by a single bin containing a single item of type A. The precise dimensions of the items are as follows ($\delta$ and $\varepsilon$ to be specified later):

$$\text{A-item in Bin } j\text{: Height} = \frac{1}{2} + \varepsilon,\ \text{Width} = \begin{cases} \frac{1}{6} + 4^j\delta & \text{if } j \text{ odd} \\ \frac{1}{6} - 4^j\delta & \text{if } j \text{ even} \end{cases}$$

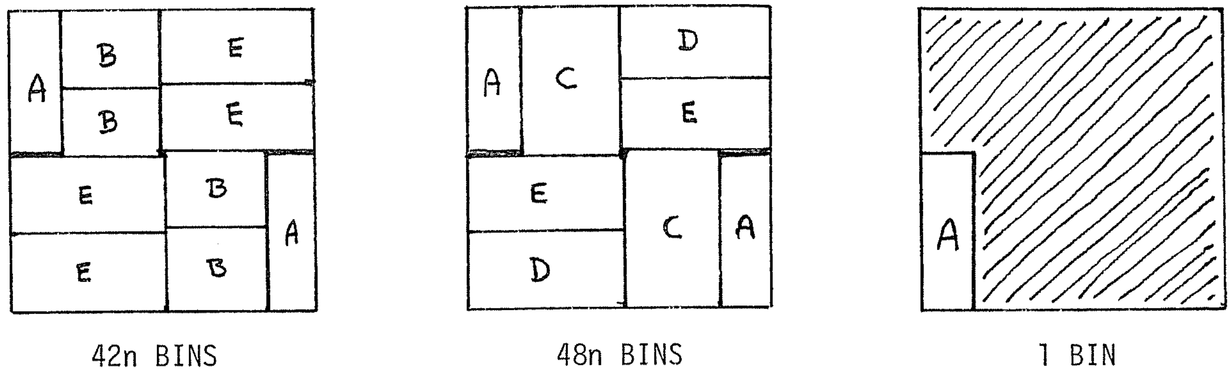
$$\text{B-item in Bin } j\text{: Height} = \frac{1}{4} + 2\varepsilon,\ \text{Width} = \begin{cases} \frac{1}{3} - (4^j+1)\delta & \text{if } j \text{ odd} \\ \frac{1}{3} + (4^j-1)\delta & \text{if } j \text{ even} \end{cases}$$

$$\text{C-item in Bin } j\text{: Height} = \frac{1}{2} + \varepsilon,\ \text{Width} = \begin{cases} \frac{1}{3} - (4^j+1)\delta & \text{if } j \text{ odd} \\ \frac{1}{3} + (4^j-1)\delta & \text{if } j \text{ even} \end{cases}$$

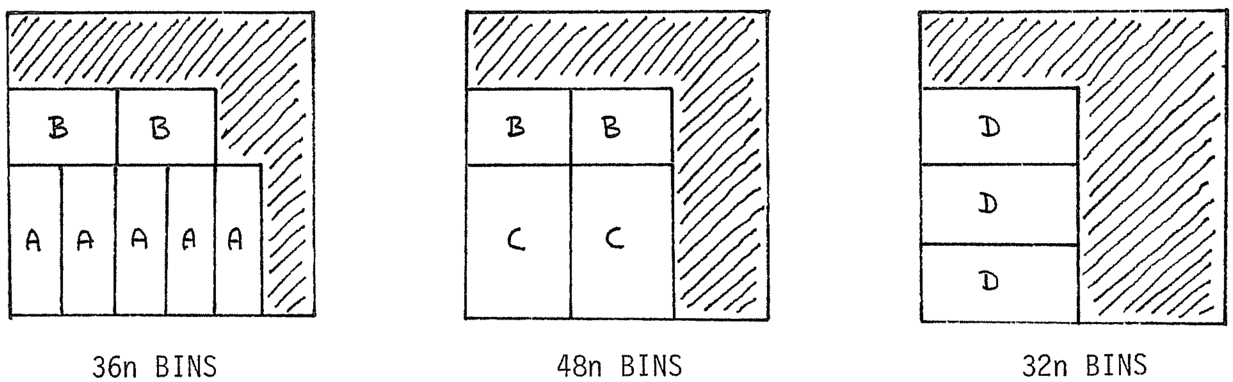
$$\text{All D-items have Height} = \frac{1}{4} + \varepsilon,\ \text{Width} = \frac{1}{2} + \delta$$

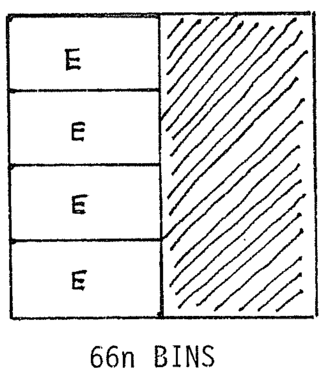
$$\text{All E-items have Height} = \frac{1}{4} - 2\varepsilon,\ \text{Width} = \frac{1}{2} + \delta$$

The reader may readily verify that if we choose $\varepsilon$ and $\delta$ so that $0 < \varepsilon < \frac{1}{16}$ and $0 < \delta < 4^{-50n}$, the items can be packed as claimed.




(a): OPTIMAL PACKING: OPT(L) = 90n + 1




(b) HFF PACKING: HFF(L) = 182n

Figure 4. Schematic of packings of lists L with $HFF(L) = \frac{91}{45}(OPT(L) - 1)$.

For the application of HFF, these items must be ordered by decreasing height. We assume that ties among items of the same height are broken so that the items are ordered as follows: First come the A-items, in reverse order, with the first A-item from Bin $2i + 1$ replaced by the first from Bin $2i + 3$, $0 \leq i \leq 45n - 1$. To illustrate this, here is a list of the values for the first 20 A-items of $w(r) - \frac{1}{6}$ (we let $m = 90n$).

$$- 4^m\delta,\ - 4^m\delta,\ + 4^{m+1}\delta,\ + 4^{m-1}\delta,\ - 4^{m-2}\delta$$
$$- 4^{m-2}\delta,\ + 4^{m-1}\delta,\ + 4^{m-3}\delta,\ - 4^{m-4}\delta,\ - 4^{m-4}\delta$$
$$+ 4^{m-3}\delta,\ + 4^{m-5}\delta,\ - 4^{m-6}\delta,\ - 4^{m-6}\delta,\ + 4^{m-5}\delta$$
$$+ 4^{m-7}\delta,\ - 4^{m-8}\delta,\ - 4^{m-8}\delta,\ + 4^{m-7}\delta,\ + 4^{m-9}\delta$$

Note that after each set of five items FFDH would start a new block: The sum of the first five exceeds $\frac{5}{6} + 4^m\delta$ and hence none of the remaining items will fit in the gap, and similar arguments hold for all remaining sets of five. Thus, since there are a total of $180n + 1$ type A-items, FFDH will create $36n$ "A-blocks" of 5 A-items each (the last A-item, having width $\frac{1}{6} + 4\delta$, will be postponed until after the C-items, and can be ignored since it will just fall in the first C-block).

The C-items follow the A-items, and are ordered so that they will go two per block. The values of $w(r) - \frac{1}{3}$ for the first eight are

$$+ (4^m-1)\delta,\ - (4^{m-1}+1)\delta,\ + (4^m-1)\delta,\ - (4^{m-1}+1)\delta,$$
$$+ (4^{m-2}-1)\delta,\ - (4^{m-3}+1)\delta,\ + (4^{m-2}-1)\delta,\ - (4^{m-3}+1)\delta$$

The reader should be able to see that this type of ordering will yield 48n blocks of 2 C-items each out of the total of 96n C-items. Similar tricks are played with the 168n B-items which follow next, yielding 84n blocks of two B-items each. (Note that sizes are arranged so that no B-item is narrow enough to fit in a block of C-items).

Finally, the list concludes with the 96n D-items, each going in a block by itself, followed by the 264n E-items, each going in a block by itself.

The reader may now verify that when HFF applies FFD to the blocks thus created, the packing of Figure 4(b) will result, using 182n bins or $\frac{91}{45}(\mathrm{OPT}(L)-1)$ as claimed. Note also that the bad behavior illustrated here is not dependent on our ability to order items of the same height in the worst possible way, since by appropriately shaving the height of the items we can insure that the given order is forced by the decreasing height rule, without changing the natures of the optimal and HFF packings.

The upper bound on $R^{\infty}_{\mathrm{HFF}}$ comes from the following Theorem:

Theorem: For any list $L$ of rectangles, $\mathrm{HFF}(L) < \frac{17}{8}\mathrm{OPT}(L) + 5$.

Proof: Suppose that $L$ is a counter-example. By normalizing widths and heights, we may assume without loss of generality that $W = H = 1$ and $0 \le w(r), h(r) \le 1$ for all $r \in L$. Let us further assume that $L$ is a counter-example containing the minimum possible number of rectangles.

We rely on three results about the one-dimensional bin packing and two-dimensional strip packing problems. Let $f: L \to [0, \frac{8}{5}]$ be a weighting function defined as follows:

$$f(r) = \begin{cases} (\frac{6}{5})\cdot w(r) & \text{if } 0 \leq w(r) \leq \frac{1}{6} \\ (\frac{9}{5})\cdot w(r) - \frac{1}{10} & \text{if } \frac{1}{6} < w(r) \leq \frac{1}{3} \\ (\frac{6}{5})\cdot w(r) + \frac{1}{10} & \text{if } \frac{1}{3} < w(r) \leq \frac{1}{2} \\ (\frac{6}{5})\cdot w(r) + \frac{4}{10} & \text{if } \frac{1}{2} < w(r) \leq 1 \end{cases}$$

Lemma 1 [6] If $R \subseteq L$ and $w(R) \equiv \sum_{r\in R} w(r) \leq 1$, then $f(R) \equiv \sum_{r\in R} f(r) \leq \frac{17}{10}$.

Lemma 2 [6] Suppose $R \subseteq L$ and $\{R_1, R_2, \ldots, R_m\}$ is a partition of $R$ into disjoint non-empty sets such that for all integers $i$ and $j$ with $1 \leq i < j \leq m$, $r \in R_j$ implies $w(r) > 1 - w(R_i)$. Then $f(R) \geq m - 1$.

Lemma 3 [4] Suppose $OPT_S(L)$ is the minimum possible strip height $H'$ such that $L$ can be packed into a strip of width 1 and height $H'$. Then $FFDH(L) \leq \frac{17}{10} OPT_S(L) + 1$.

Given $L = \{r_1, r_2, \ldots, r_n\}$, we now show that $HFF(L) < \frac{17}{8} OPT(L) + 5$, in contradiction to our assumption that $L$ was a counter-example. Let $P_{HFF}$ be the HFF packing of $L$ and $P_{OPT}$ be an optimal packing. Let $x$ denote the height of the tallest block in the last bin of $P_{HFF}$. Since $L$ is a minimum counter-example, we may assume that all rectangles $r_i \in L$ have height at least $x$: The number and heights of blocks of height $x$ or greater would not be affected by deleting all rectangles shorter than $x$, so that the number of bins required by HFF would not decrease, whereas the number of bins required by an optimal packing could not increase.

Thus if L contained any rectangle shorter than x, a counter example with fewer items would exist, contradicting the minimality of L.

Our proof divides into four cases, depending on the value of x. We shall treat the cases in order of difficulty.

Case 1. $x \leq \frac{1}{5}$.

In this case all but the last bin of $P_{HFF}$ must contain blocks whose total height is at least $\frac{4}{5}$. Thus, by Lemma 3,

$$\frac{4}{5}(\text{HFF}(L)-1) \leq \text{FFDH}(L) \leq \frac{17}{10}\text{OPT}_S(L)+1 \leq \frac{17}{10}\text{OPT}(L)+1,$$

where the last inequality results from the fact that one way to pack a strip of width 1 with L is to pack L into OPT(L) bins of width and height 1 and then pile them one on top of another. From this we conclude that

$$\text{HFF}(L) \leq \frac{5}{4}\cdot\frac{17}{10}\text{OPT}(L)+\frac{9}{4} < \frac{17}{8}\text{OPT}(L)+5,$$

as desired. □

In the remaining cases we assume that $x > \frac{1}{5}$ and so can divide the items of L into the following classes:

$$X_1 = \{r_i: h(r_i) > 1-x\}$$

$$X_2 = \{r_i: 1-x \geq h(r_i) > \tfrac{1}{2}\}$$

$$X_3 = \{r_i: \tfrac{1}{2} \geq h(r_i) > \tfrac{1-x}{2}\}$$

$$X_4 = \{r_i: \tfrac{1-x}{2} \geq h(r_i) > \tfrac{1}{4}\}$$

$$X_5 = \{r_i: \tfrac{1}{4} \geq h(r_i) \geq x\}$$

We shall say a block is of "type $X_i$" if its tallest item is from $X_i$.

Let $B_1, B_2, \ldots, B_\ell$ denote the bins of $P_{HFF}$ in order, where $\ell = HFF(L)$. For $1 \leq i \leq 5$, let $\beta_i$ denote the set of bins whose tallest block is of type $X_i$, and let $N_i = |\beta_i|$. Note that all bins from $\beta_i$ precede all bins from $\beta_{i+1}$, $1 \leq i \leq 4$.

Case 2. $x > \frac{1}{3}$.

If $x > \frac{1}{3}$ then $\frac{1-x}{2} < \frac{1}{3}$ and so $N_4 = N_5 = 0$. Let us look at an arbitrary bin B in $P_{OPT}$ and imagine lines drawn through it at heights $\frac{1}{3}$ and $\frac{2}{3}$. Let $S_1(B)$ be the set of items from $X_1$ in B. Let $S_{23}(B,1)$ be the set of items from $X_2$ and $X_3$ in B whose interiors are traversed by the line at height $\frac{1}{3}$, and let $S_{23}(B,2)$ be the set of items from $X_2$ and $X_3$ in B whose interiors are traversed by the line at height $\frac{2}{3}$ but not by the line at $\frac{1}{3}$. Note that since all items in L are of height exceeding $\frac{1}{3}$, every item in B must be in precisely one of these three sets.

Now observe that, since every item in $X_1$ has height exceeding $1-x$, no vertical line through B can traverse the interiors of both an item from $S_1(B)$ and one from $S_{23}(B,1) \cup S_{23}(B,2)$. We thus have

$$w(S_1(B)) + w(S_{23}(B,1)) \leq 1 \qquad (2.1)$$

and

$$w(S_1(B)) + w(S_{23}(B,2)) \leq 1. \qquad (2.2)$$

Using Lemma 1 and summing over all bins B of $P_{OPT}$ we conclude that

$$2f(X_1) + f(X_2) + f(X_3) \leq 2 \cdot \frac{17}{10} OPT(L) \leq \frac{17}{5} OPT(L). \qquad (2.3)$$

We now turn to the HFF packing. The bins of $\beta_1$ each contain one block, that block having height exceeding $1-x$, and these blocks induce a partition on $X_1$ which obeys the hypotheses of Lemma 2. Thus

$$f(X_1) \geq N_1 - 1. \tag{2.4}$$

None of the remaining bins contains a block of type $X_1$ and so the fact that a block of height $x$ went in the last bin means that all except that last bin must contain at least (and hence exactly) two blocks. Letting $X'_{23}$ denote the subset of $X_2 \cup X_3$ which is contained in these bins, and ordering the blocks in the same order as they were created by FFDH, we see that these $2(N_2+N_3) - 1$ blocks induce a partition of $X'_{23}$ which obeys the hypotheses of Lemma 2. Therefore

$$f(X_2) + f(X_3) \geq f(X'_{23}) \geq 2(N_2+N_3) - 2. \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3) we obtain

$$2(N_1-1) + 2(N_2+N_3-1) \leq \frac{17}{5}\,\mathrm{OPT}(L)$$

or $$\mathrm{HFF}(L) = N_1 + N_2 + N_3 \leq \frac{17}{10}\,\mathrm{OPT}(L) + 2 < \frac{17}{8}\,\mathrm{OPT}(L) + 5$$

as desired. □

<u>Case 3</u>. $\frac{1}{4} < x \leq \frac{1}{3}$.

In this case $N_5 = 0$. Let us once again consider a bin B in the optimal packing. This time we imagine 7 horizontal lines drawn through B: two (identical) lines at height $x$, one at height $\frac{1-x}{2}$, one at $\frac{1}{2}$, one at $\frac{1+x}{2}$, and two (identical) lines at $1-x$. It is easy to verify that, given these lines, each rectangle from $X_1$ in B will have its interior traversed by all 7 lines. Similarly, rectangles from $X_2$, $X_3$ and $X_4$ will have their interiors traversed by at least 4, 3, and 2 lines respectively. Let $S_i(B,j)$ be the set of rectangles from $X_i$ whose interiors are traversed by the jth line, $1 \le i \le 4$, $1 \le j \le 7$. We then have, for each $j$, $1 \le j \le 7$,

$$\sum_{i=1}^{4} w(S_i(B,j)) \le 1$$

Lemma 1 then yields for each $j$, $1 \le j \le 7$

$$\sum_{i=1}^{4} f(S_i(B,j)) \le \frac{17}{10}$$

Summing over all bins B of $P_{OPT}$ we thus conclude

$$7f(X_1) + 4f(X_2) + 3f(X_3) + 2f(X_4) \le 7 \cdot \frac{17}{10} OPT(L). \qquad (3.1)$$

Turning to the HFF packing, let $\beta_{2,3}$ be the set of bins from $\beta_2$ that, in addition to containing a block of type $X_2$, also contain a block of type $X_3$. Since a block of type $X_2$ has height at most $1-x$ and since the block of height $x$ in the last bin did not fit in any earlier bin, every bin in $\beta_{2,4} = \beta_2 - \beta_{2,3}$ must contain a block of type $X_4$. Let $N_{2,i} = |\beta_{2,i}|$ for $i \in \{3,4\}$.

Applying Lemma 2 to the partitions of $X_1$ and $X_2$ induced by the bins of $\beta_1$ and $\beta_2$ respectively, we obtain

$$f(X_1) \ge N_1 - 1. \qquad (3.2)$$

$$f(X_2) \ge N_2 - 1. \qquad (3.3)$$

There are at least $N_{2,3} + 2N_3 - 1$ blocks of type $X_3$: one in each bin of $\beta_{2,3}$ and two in all but possibly the last bin of $\beta_3$. If we let $X'_3$ be the subset of $X_3$ contained in these blocks and apply Lemma 2 to the partition of $X'_3$ induced by these blocks, we obtain

$$f(X_3) \geq f(X'_3) \geq N_{2,3} + 2N_3 - 2. \tag{3.4}$$

Finally, consider the blocks of type $X_4$. There are at least $N_{2,4} + 3N_4 - 2$ of these: one in each bin of $\beta_{2,4}$ and three in each bin of $\beta_4$ except the last. (A non-final bin from $\beta_4$ cannot have height less than $1 - x$, and since no block of type $X_4$ has height exceeding $\frac{1-x}{2}$, each such bin must contain at least three blocks). Lemma 2 thus yields

$$f(X_4) \geq N_{2,4} + 3N_4 - 3. \tag{3.5}$$

Substituting (3.2) through (3.5) in (3.1) yields

$$7N_1 + 6N_2 + 6N_3 + 6N_4 - 23 \leq 7 \cdot \frac{17}{10} OPT(L)$$

or $$HFF(L) = N_1 + N_2 + N_3 + N_4 \leq \frac{7}{6} \cdot \frac{17}{10} OPT(L) + \frac{23}{6} < \frac{17}{8} OPT(L) + 5$$

as desired. □

Case 4. $\frac{1}{5} < x \leq \frac{1}{4}$.

We divide this case into two subcases, depending on the value of $N_4$.

Subcase 4.1. $N_4 = 0$.

The total height of all blocks in $P_{HFF}$ is bounded by $(1-x)(N_1+N_2+N_3) + 4x(N_5-1)$ since all bins except the last must have total block height at least $1-x$, and all bins of $\beta_5$ except the last must contain four blocks. By Lemma 3 we thus have

$$(1-x)(N_1+N_2+N_3) + 4x(N_5-1) \leq \frac{17}{10}\mathrm{OPT}(L) + 1. \tag{4.1}$$

Furthermore, by the argument used in Case 2, we have

$$N_1 + N_2 + N_3 \leq \frac{17}{10}\mathrm{OPT}(L) + 2. \tag{4.2}$$

Using (4.1) and (4.2) we then can derive the following:

$$\begin{aligned} 4x\ell &= 4x(N_1+N_2+N_3+N_5) \\ &\leq 4x(N_1+N_2+N_3) + \frac{17}{10}\mathrm{OPT}(L) + 1 - (1-x)(N_1+N_2+N_3) + 4x \\ &\leq (5x-1)(\frac{17}{10}\mathrm{OPT}(L)+2) + \frac{17}{10}\mathrm{OPT}(L) + 1 + 4x \\ &\leq (5x)\frac{17}{10}\mathrm{OPT}(L) + 14x - 1 \end{aligned}$$

and hence $\ell = \mathrm{HFF}(L) \leq \frac{5}{4}\cdot\frac{17}{10}\mathrm{OPT}(L) + \frac{7}{2} < \frac{17}{8}\mathrm{OPT}(L) + 5$

as desired. □

Subcase 4.2. $N_4 > 0$.

Consider a bin B in $P_{OPT}$ and this time imagine seven horizontal lines drawn through it, at heights $\frac{j}{8}$, $1 \leq j \leq 7$. Then rectangles from classes $X_1$, $X_2$, $X_3$. $X_4$, and $X_5$ have their interiors traversed by at least 6, 4, 3, 2, and 1 lines respectively, since $1 - x \geq \frac{3}{4}$ and $\frac{1-x}{2} \geq \frac{3}{8}$.

Letting $S_i(B,j)$ be the set of rectangles from $X_i$ whose interiors are traversed by the jth line, $1 \le i \le 5$, $1 \le j \le 7$, we then have for each $j$, $1 \le j \le 7$

$$\sum_{i=1}^{5} w(S_i(B,j)) \le 1$$

Lemma 1 thus yields $\sum_{i=1}^{5} f(S_i(B,j)) \le \frac{17}{10}$, and summing over all bins B of $P_{OPT}$ we obtain

$$6f(X_1) + 4f(X_2) + 3f(X_3) + 2f(X_4) + f(X_5) \le 7\frac{17}{10}OPT(L). \tag{4.3}$$

We now turn to the HFF packing. Since $N_4 > 0$, there is a block of type $X_4$ which did not fit in any bin from $\beta_2$ or any bin from $\beta_4$ except the last. Thus any bin from class $\beta_2$ or any bin (except the last) from class $\beta_4$ that contains a block of type $X_5$ must contain blocks whose total height is at least $1 - (\frac{1-x}{2}) + x = \frac{1+3x}{2}$. Let us partition the bins in $\beta_2$ and $\beta_4$ as follows:

Any bin in $\beta_2$ must contain at least one block in addition to its block of type $X_2$. Let $\beta_{2,j}$, $3 \le j \le 5$, be the subset of bins from $\beta_2$ whose second block is of type $X_j$ (there may be a third block, but we ignore it in forming the partition). Similarly, any bin in $\beta_4$ must contain at least three blocks. Let $\beta_{4,5}$ be the set of bins in $\beta_4$, other than the last, for which the third block is of type $X_5$, and let $\beta_{4,4} = \beta_4 - \beta_{4,5}$. Letting $N_{i,j} = |\beta_{i,j}|$, we then have

$$(1 - x)(N_1 + N_{2,3} + N_{2,4} + N_3 + N_{4,4}) + (\frac{1+3x}{2})(N_{2,5} + N_{4,5}) + 4x(N_5 - 1)$$
$$\le FFDH(L) \le \frac{17}{10}OPT(L) + 1. \tag{4.4}$$

Our next inequalities are obtained by applying Lemma 1 to the blocks of type $X_i$, $1 \leq i \leq 5$, as in previous cases, using the facts that all but the last bin in $\beta_3$ contain 2 blocks of type $X_3$, all but the last bin in $\beta_4$ contain either 3 blocks of type $X_4$ (if in $\beta_{4,4}$) or two (if in $\beta_{4,5}$), and all but the last bin in $\beta_5$ contain 4 blocks of type $X_5$:

$$f(X_1) \geq N_1 - 1 \tag{4.5}$$

$$f(X_2) \geq N_2 - 1 \tag{4.6}$$

$$f(X_3) \geq N_{2,3} + 2N_3 - 2 \tag{4.7}$$

$$f(X_4) \geq N_{2,4} + 3N_{4,4} + 2N_{4,5} - 3 \tag{4.8}$$

$$f(X_5) \geq N_{2,5} + N_{4,5} + 4N_5 - 4 \tag{4.9}$$

Now a final dose of symbol manipulation yields the desired result. Combining (4.3) and (4.5) through (4.9) we obtain

$$6(N_1) + 6(N_{2,3} + N_{2,4}) + 5N_{2,5} + 6N_3 + 6N_{4,4} + 5N_{4,5} + 4N_5 \leq 7 \cdot \frac{17}{10} OPT(L) + 26. \tag{4.10}$$

Multiplying (4.4) by 2 and (4.10) by $(5x - 1)$ and then adding we obtain

$$(2(1-x) + 6(5x-1))(N_1 + N_{2,3} + N_{2,4} + N_3 + N_{4,4}) + ((1+3x) + 5(5x-1))(N_{2,5} + N_{4,5}) + (8x + 4(5x-1))(N_5) \leq \frac{17}{10} OPT(L)(7(5x-1)+2) + 26(5x-1) + 8x + 2$$

that is,

$$(28x - 4)(N_1 + N_{2,3} + N_{2,4} + N_{2,5} + N_3 + N_{4,4} + N_{4,5} + N_5)$$

$$\leq \frac{17}{10}OPT(L)(35x - 5) + 138x - 24$$

or $$HFF(L) < \frac{5}{4} \cdot \frac{17}{10}OPT(L) + 5$$

$$= \frac{17}{8}OPT(L) + 5$$

as desired. □

Thus in all cases $HFF(L) < \frac{17}{8}OPT(L) + 5$, in contradiction to our assumption that a counter-example exists. The theorem has been proved.

## 4. Directions for Further Research

By showing that close bounds can be obtained on the asymptotic worst case behavior of two-dimensional bin packing algorithms, we hope to encourage researchers to design other algorithms and investigate their behavior. Algorithms based on the "bottom-left" strip packing rule introduced in [3] are particularly attractive candidates for analysis. Although the bottom-left algorithms are all asymptotically worse than FFDH in the strip packing environment, they may well be more competitive for two-dimensional bin packing. There is also the possibility of constructing better hybrid algorithms. FFDH is not the best heuristic known for strip packing. An algorithm is presented in [2] with $R_A^\infty \leq \frac{5}{4}$ (although the structure of its packings is much more complicated than that for FFDH). Similarly, FFD has recently been improved on in the one-dimensional case by a modified algorithm [8] with $R_A^\infty = 1.18333...$

A second line of attack would be to design and analyze algorithms which could make use of the fact that, in some applications, 90° rotations of rectangles might be allowable. Algorithm HFF would still be applicable in such situations, assuming all rectangles were presented in such a way that they would fit in a bin without rotation. However, the performance guarantee of Theorem 1 would not necessarily hold. Algorithms which consider the possibility of rotations might well yield improvements. Can one prove worst case bounds that reflect these improvements?

Finally, there is of course the problem of further narrowing the gap between upper and lower bounds on $R_{HFF}^\infty$. We suspect that the upper bound can be lowered further, although we fear that a considerable blow-up in proof length might be necessary. As to the actual value of $R^{\infty}_{HFF}$, we hesitate to conjecture. It is amusing to note that one possibility still left open by our bounds is $(\frac{17}{10})(\frac{11}{9}) = 2.07777\ldots$, the product of the values of $R^{\infty}_{A}$ for the two algorithms whose combination yields the algorithm HFF, although we suspect that the actual value may be somewhat less than this.

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