CONTINUITY PROPERTIES OF LINEAR PROGRAMS

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Abstract

Continuity properties of linear programs that are subjected to data perturbations are characterized. It is shown that the optimal value is continuous relative to joint right-hand-side and objective coefficient perturbations that preserve finiteness of the optimal value, whereas continuity properties of the optimal solution sets of the problem and its dual are equivalent to certain properties of the corresponding solution sets of the unperturbed problem. Extensions of the results to perturbations of the coefficient matrix are then considered, along with applications to nonlinear programming.

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1. Introduction

Continuity properties of linear programs that are subjected to data perturbations are of computational interest in a number of areas, including linear programming sensitivity analysis (where uncertainties and variability of the data give rise to questions of stability of the optimal value and optimal solutions) and nonlinear programming (where continuity properties of linear programming approximations are crucial in establishing the convergence of certain algorithms). Bereanu [1] also discusses some applications in stochastic programming.

Initially, we will be concerned with linear programs of the form

$$\text{LP}(b,c) \qquad \begin{array}{l} \underset{x}{\text{minimize}} \;\; cx \\ \text{subject to: } A^*x = b,\; x \geq 0, \end{array}$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, c is an n-dimensional row vector (the juxtaposition of two vectors, as in cx, denotes their inner product), and A^* is $m\times n$. (In this context the notation A^* does not indicate the conjugate transpose, but merely some fixed matrix; perturbation of the constraint coefficients will be considered in Section 5.) The notation LP(b,c) is intended to emphasize that in Sections 2-4 only right-hand side and objective perturbations are allowed. (When constraint coefficient perturbation is considered, obvious modifications to the notation will be employed.) To avoid trivial cases, we assume throughout this paper that the unperturbed problem <u>LP(b*,c*) has an optimal solution.</u>

The dual of LP(b,c) is the problem

$$D(b,c) \qquad \begin{array}{l} \text{maximize } by \\ \quad y \\ \text{subject to: } yA^* \leq c, \end{array}$$

where y is an m-dimensional row vector. For notational convenience, the unperturbed problems $LP(b^*,c^*)$ and $D(b^*,c^*)$ are denoted by LP^* and D^* respectively.

Since we are assuming that LP^* has a finite optimal value, perturbations that lead to infeasibility or unboundedness clearly result in discontinuities of the optimal value function. Thus, we will concentrate attention on those perturbations that preserve the existence of an optimal solution. This is equivalent to considering only perturbations for which the constraints of both the primal and dual remain solvable. Accordingly, let

$$\mathcal{B} \equiv \{b \mid A^* x = b,\ x \geq 0 \text{ is solvable}\}$$

and

$$\mathcal{C} \equiv \{c \mid yA^* \leq c \text{ is solvable}\},$$

i.e., $\mathcal{B}$ is the set of RHS's such that LP(b,c) has a feasible solution, and $\mathcal{C}$ is the set of objectives for the primal (right-hand sides for the dual) such that D(b,c) has a feasible solution. Note that $(b^*,c^*) \in \mathcal{B}\times\mathcal{C}$, and that LP(b,c) has an optimal solution if and only if $(b,c) \in \mathcal{B}\times\mathcal{C}$. Moreover, $\mathcal{B}$ and $\mathcal{C}$ are polyhedral and therefore closed sets. The optimal value of LP(b,c) is denoted by

$\omega(b,c)$, and the optimal value of the unperturbed problem is denoted by ω^*. Note that by assumption $-\infty < \omega^* < +\infty$, and that $-\infty < \omega(b,c) < +\infty$ if and only if $(b,c) \in \mathcal{B}\times\mathcal{C}$. Finally,

$$\Omega_P(b,c) \equiv \{x|A^*x=b,\ x\geq 0,\ cx=\omega(b,c)\}$$

denotes the set of optimal solutions of LP(b,c) and

$$\Omega_D(b,c) \equiv \{y|yA^*\leq c,\ yb=\omega(b,c)\}$$

denotes the set of optimal solutions of D(b,c). Note that $\Omega_P(b,c) \neq \phi \neq \Omega_D(b,c)$ if and only if $(b,c) \in \mathcal{B}\times\mathcal{C}$. For notational convenience, we denote the optimal solution sets of the unperturbed problems by Ω_P^* and Ω_D^* respectively.

The main result of Section 2 is the continuity of the optimal value function with respect to perturbations in $\mathcal{B}\times\mathcal{C}$. An application of this result to a nonlinear programming algorithm is also briefly discussed. In Sections 3 and 4, perturbations of only the right-hand side and only the objective function are considered. The principal results are that the primal optimal solution set is always continuous with respect to RHS perturbations, but is continuous with respect to objective perturbations if and only if it is a singleton (i.e., the primal has a unique solution). (By duality, corresponding results hold for the optimal solutions set of the dual.)

Section 5 takes up the issue of constraint coefficient perturbation, and discusses conditions related to continuity with respect to optimality-preserving perturbations. Finally, the main results of the paper are summarized in the two tables of Section 6.

2. Continuity with respect to joint right-hand-side and objective perturbations

In this section we will consider continuity properties of the optimal value and optimal solution sets with respect to perturbations in $\mathcal{B}\times\mathcal{C}$, i.e., perturbations of the RHS and objective that preserve the existence of an optimal solution. The continuity of the optimal value function relative to such perturbations was established in [5], but, for the sake of completeness, we give a proof of this key result.

Theorem 1: The optimal value function ω is continuous on $\mathcal{B}\times\mathcal{C}$ relative to perturbations in $\mathcal{B}\times\mathcal{C}$.

Proof: Let $\{(b^k,c^k)\} \subset \mathcal{B}\times\mathcal{C}$ with $(b^k,c^k) \to (b^*,c^*)$. We will show that $\omega(b^k,c^k) \to \omega(b^*,c^*)$.

Without loss of generality we may assume that the matrix A has full row rank, since linearly dependent rows may be deleted. (The only case not covered by this assumption is the one in which A is the 0 matrix and $b = 0$, but the result is obvious in this instance since $\mathcal{B} = \{0\}$, $\mathcal{C} = \mathbb{R}^n_+$ and the optimal value is always 0.) Let K denote a sequence of indices such that there exists a fixed basis matrix B such that for $k \in K$ an optimal solution of $LP(b^k,c^k)$ is obtained by setting the corresponding basics to the values $x^k_B = B^{-1}b^k$ (and non-basics to 0) and an optimal solution of the dual of $LP(b^k,c^k)$ is given by $y^k = c^k_B B^{-1}$, where c^k_B are the components of c^k corresponding to B. Taking the limit over $k \in K$, it is easily seen that $x^*_B = B^{-1}b^*$ is feasible for $LP(b^*,c^*)$ and $y^* = c^*_B B^{-1}$ is feasible for its dual, and thus these solutions must be optimal for their respective problems. Clearly, for $k \in K$, $\omega(b^k,c^k) = c_B^k B^{-1} b^k$ converges to $c_B^* B^{-1} b^* = \omega(b^*,c^*)$. By considering subsequences K_I and K_S corresponding to lim inf and lim sup and extracting from them further subsequences corresponding to fixed bases, it is clear that $\liminf \omega(b^k,c^k) = \limsup \omega(b^k,c^k) = \omega(b^*,c^*)$. ▲

The key element in the proof of Theorem 1 is the existence of bounded sequences of optimal solutions of the primal and dual problems. Under the hypotheses of Theorem 1 it is possible to obtain such sequences because of the existence of optimal basic feasible solutions and the ability to work with a fixed basis. (This fixed basis property also leads to well-known linearity properties of primal and dual solutions when sensitivity analysis is performed with respect to given RHS and objective perturbation vectors — see, for example, [4].) In Section 5, where matrix perturbations are allowed, additional assumptions will be required in order to ensure boundedness.

Of course, the preceding result holds for general LP's (i.e., those not in standard form), since conversion to standard form has no effect on the optimal value of an LP. Note also that $\mathcal{B}\times\mathcal{C}$ contains a ball about (b^*,c^*) if and only if b^* is in the interior of $\mathcal{B}$ $(b^* \in \text{int}(\mathcal{B}))$ and $c^* \in \text{int}(\mathcal{C})$. In this case continuity relative to $\mathcal{B}\times\mathcal{C}$ is equivalent to continuity in the ordinary (i.e., unrestricted) sense.

Although no assumptions are required to ensure that the optimal value is continuous relative to $\mathcal{B}\times\mathcal{C}$, the same is not true with regard to continuity properties of the optimal solution sets. The next theorem, in fact, shows that the optimal solution set $\Omega_P(b,c)$ is continuous at (b^*,c^*) relative to $\mathcal{B}\times\mathcal{C}$ if and only if $LP(b^*,c^*)$ has a unique optimal solution, i.e. $\Omega_P(b^*,c^*)$ is a singleton. In order to define continuity of $\Omega_P(b,c)$, which is set-valued, we require the concepts of upper semi-continuity and lower semi-continuity of set valued mappings. A set-valued mapping Ω will be said to be <u>lower-semi-continuous</u> (l.s.c.) at (b^*,c^*) relative to a set S if $x^* \in \Omega(b^*,c^*)$ and $\{(b^k,c^k)\} \subset S$ with $(b^k,c^k) \to (b^*,c^*)$ imply the existence of $x^k \in \Omega(b^k,c^k)$ with $x^k \to x^*$. Ω will be said to be <u>upper semi-continuous</u> (u.s.c.) (or "closed" in the terminology of [6]) at (b^*,c^*) relative to S if $\{(b^k,c^k)\} \subset S$ and $x^k \in \Omega(b^k,c^k)$ $(k=1,2,\ldots)$ with $(b^k,c^k) \to (b^*,c^*)$ and $x^k \to x^*$ imply $x^* \in \Omega(b^*,c^*)$. Finally, Ω is said to be <u>continuous</u> at (b^*,c^*) relative to S if it is both l.s.c. and u.s.c. at (b^*,c^*) relative to S. If Ω is continuous at (b^*,c^*) with respect to a particular sequence $\{(b^k,c^k)\}$ we will write $\Omega(b^k,c^k) \to \Omega(b^*,c^*)$.

<u>Theorem 2</u>: The optimal solution set Ω_P is upper semi-continuous on $\mathcal{B}\times\mathcal{C}$ relative to $\mathcal{B}\times\mathcal{C}$. Ω_P is continuous at (b^*,c^*) relative to $\mathcal{B}\times\mathcal{C}$ if and only if Ω_P^* is a singleton.

<u>Proof</u>: The u.s.c. follows trivially from Theorem 1. Thus, we need only show that l.s.c. of Ω_P is equivalent to uniqueness of the optimal solution.

($\Leftarrow$) If Ω_P^* is a singleton $\{x^*\}$ then by the construction used in the proof of Theorem 1, every subsequence of the sets $\Omega_P(b^k,c^k)$ contains a corresponding subsequence of solutions x^k converging to x^*, so that Ω_P is also l.s.c. at (b^*,c^*).

($\Rightarrow$) Suppose that Ω_P^* is not a singleton. If A^* is the 0 matrix, a contradiction is obtained by choosing $c^k > 0$, $c^k \to 0$. Otherwise, let x^* be an optimal basic feasible solution of $LP(b^*,c^*)$. Now consider the family of problems $LP(b^*,c^k)$ where c^k is chosen so that $c_i^k = c_i^*$ for the i corresponding to variables basic with respect to x^* and $c_i^k = c_i^* + k^{-1}$ $(k=1,2,\ldots)$ otherwise. For these perturbed problems, x^* is the unique element of $\Omega_P(b^*,c^k)$, so that clearly $\Omega_P(b,c)$ cannot be continuous at (b^*,c^*). ▲

It turns out to be the case that Ω_P^* is bounded if and only if c^* is an interior point of $\mathcal{C}$ (see [15]). Thus, the following extension of Theorem 2 also holds.

Corollary 1: The optimal solution set $\Omega_P(b,c)$ is continuous at (b^*,c^*) relative to $\mathcal{B} \times \mathbb{R}^n$ if and only if Ω_P^* is a singleton.

Corollary 1 is related to a result of Mangasarian [8] who showed that a general linear program has a unique optimal solution x^* if and only if x^* is an optimal solution of all LP's whose objectives are are sufficiently close to the original objective (the coefficient matrix and RHS are assumed to be unchanged). In Corollary 1 RHS changes are allowed, so in general x^* is not in the optimal solution sets of the perturbed problems, but there must be optimal solutions of the perturbed problems "close" to x^*.

Analogous results hold for the optimal solution set of the dual as well as for general linear programs. Here we shall only state the result for the dual problem, and refer the reader to the Appendix for the proof for the general linear program.

Theorem 3: The optimal solution set Ω_D is upper semi-continuous on $\mathcal{B}\times\mathcal{C}$ relative to $\mathcal{B}\times\mathcal{C}$. Ω_D is continuous at (b^*,c^*) relative to $\mathbb{R}^m\times\mathcal{C}$ if and only if Ω_D^* is a singleton.

Proof: See Appendix. ▲

Note that a general LP may be converted to the form $D(b,c)$ by converting all of the constraints to inequalities in the usual manner. Since the optimal solution set of the original problem is unchanged by this conversion process, Theorem 3 may be used to establish continuity properties for general LP's. Lipschitz properties of ω and extensions to quadratic programs may be obtained by exploiting properties of polyhedral multifunctions (see Robinson [13]).

In a series of papers, Meyer [10, 11] and Kao and Meyer [7] describe iterative approximation methods for convex optimization problems of the form

$$\min_x \; f(x)$$
$$\text{subject to: } x \in S,$$

where $f(x)$ is real-valued on the nonempty, compact, convex set S. In certain cases, this method utilizes subproblems of the form

$$\min_x \; \tilde{f}(x)$$
$$\text{subject to: } x \in S, \; \tilde{\ell} \le x \le \tilde{u},$$

where $\tilde{f}$ is a convex piecewise-linear approximation of f and $\tilde{\ell}$ and $\tilde{u}$ are appropriately chosen bounds. When S is polyhedral, it may be shown that these approximating problems are equivalent to linear programs which differ only in their objective function coefficients and right-hand sides. In this situation, the convergence of the algorithm to an optimal solution of the original problem may be established by exploiting the continuity of the optimal value of the approximating LP's. The convergence proof is also easily extended to the case of non-polyhedral S.

3. Continuity of the optimal solution set with respect to right-hand-side perturbations

In this section we assume that the objective function is fixed at some vector in $\mathcal{C}$, but that the RHS is allowed to vary over $\mathcal{B}$. Continuity of the optimal value function relative to $\mathcal{B}$ follows as a special case of Theorem 1, but in this case the optimal solution set Ω_P is also continuous even in the absence of a uniqueness assumption.

Theorem 4: The optimal solution set $\Omega_P(b,c)$ is continuous on $\mathcal{B}\times\mathcal{C}$ relative to perturbations of b within $\mathcal{B}$.

Proof: By Theorem 1, u.s.c. holds, so we need only show l.s.c. Let $\{b^k\} \subset \mathcal{B}$ with $b^k \to b^*$ and let $\omega^k = \omega(b^k,c^*)$. Suppose that $x^* \in \Omega_P^*$ and consider the problem

$$(N^k) \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & \|x^*-x\|_1 \\ \text{subject to:} & A^*x = b^k,\ x \geq 0,\ c^*x = \omega^k. \end{array}$$

Since (N^k) may be written as an LP in which only the RHS depends on k, and since (N^k) has some optimal solution $\bar{x}^k$ for all k, it follows from the preceding theorem that $\|x^*-\bar{x}^k\|_1 \to 0$, since 0 is the optimal value of the limiting problem and thus $\bar{x}^k \to x^*$. ▲

Note that since this result depends only on the continuity of the optimal value, the extension to general LP's is straightforward. The result is an immediate consequence of a continuity theorem of Dantzig, Folkman, and Shapiro [3], and also strengthens a result of Böhm [2] who established (by a different argument) the continuity of Ω_p relative to perturbations within $\mathcal{B}$ under the assumption that Ω_p^* was compact. (For u.s.c. see also [6].)

Of course, perturbations of the RHS in the primal correspond to perturbations of the objective in the dual. This leads to the next Theorem.

<u>Theorem 5</u>: The optimal solution set $\Omega_D(b,c)$ is continuous on $\mathcal{B}\times\mathcal{C}$ relative to perturbations of c within $\mathcal{C}$.

Note that for $c^* = 0$, $\Omega_p(b,c^*)$ is simply the feasible set of $LP(b,c^*)$, so that Theorem 4 yields the continuity of the <u>feasible set</u> relative to perturbations of b within $\mathcal{B}$. In this fashion, results below that provide continuity properties of Ω_p can be used to establish continuity of feasible set mappings.

4. Continuity of the optimal solution set with respect to objective perturbations

Although the optimal solution set Ω_P is continuous relative to feasibility-preserving RHS perturbations, uniqueness is still required to guarantee continuity in the case of objective function perturbations. Since only perturbations within C were required in the proof of Theorem 2 to show that continuity implied uniqueness for problems in standard form, the next result follows directly from that proof.

Theorem 6: The optimal solution set $\Omega_P(b^*,c)$ is continuous at (b^*,c^*) relative to perturbations in C if and only if Ω_P^* is a singleton.

As previously noted, c^* is in $\text{int}(C)$ when Ω_P^* is bounded, so the following extension of Theorem 5 holds.

Corollary 2: The optimal solution set $\Omega_P(b^*,c)$ is continuous at (b^*,c^*) (relative to arbitrary objective perturbations) if and only if Ω_P^* is a singleton.

Actually, an even stronger result holds, namely, $\Omega_P(b^*,c) = \Omega_P(b^*,c^*)$ for all c sufficiently close to c^* if and only if $\Omega_P(b^*,c^*)$ is a singleton. This result is established in the Appendix, where its extension to general LP's is proved. Here we note only that the corresponding result holds for the dual problem.

Theorem 7: The optimal solution set $\Omega_D(b,c^*)$ is continuous at (b^*,c^*) relative to perturbations in $\mathbb{R}^m$ if and only if Ω_D^* is a singleton.

It should be noted that $\mathbb{R}^m$ may not be replaced by $\mathcal{B}$ in the preceding theorem. Although the singleton condition remains sufficient, it is not necessary for continuity if perturbations are restricted to $\mathcal{B}$. (For problems whose feasible sets contain extreme points, this difficulty does not arise, because, for any extreme point, objective functions may be constructed with the property that the extreme point is the unique optimal solution, as in the proof of Theorem 2.)

Example 1: For the problem

$$\min_x \; 0 \cdot x$$

$$\text{subject to: } 0 \cdot x = 0, \; x \geq 0,$$

we have $\mathcal{B} = \{0\}$. In this case continuity of the dual solution set relative to perturbations in $\mathcal{B}$ is trivial, but $\Omega_D^* = \mathbb{R}^1$. ▲

5. Constraint coefficient perturbations

There are well-known examples that illustrate that if all of the problem data, including the constraint coefficients, are perturbed, then the optimal value may behave discontinuously relative to perturbations that preserve the existence of an optimal solution.

Example 2: For $x \in \mathbb{R}^1$, consider the families of LP's:

$$\begin{aligned} \min_x \quad & x \\ \text{s.t.} \quad & tx = t \\ & x \geq 0. \end{aligned}$$

For $t = 0$, this LP has the unique optimal solution $x = 0$ and optimal value 0. For $t > 0$, the unique optimal solution is $x = 1$, with optimal value 1. Thus, both the optimal value and optimal solution set are discontinuous at $t = 0$. The optimal solution set of the dual is also discontinuous, being $\mathbb{R}^1$ for $t = 0$ and $\{t^{-1}\}$ for $t > 0$. ▲

To see why the proof used for Theorem 1 cannot be extended to the matrix perturbation case, note that if the columns associated with a fixed set of basic indices are substituted for the fixed basis matrix B in the proof, there is no guarantee that their limit will have an inverse. This is the case in the preceding example. However, a simple modification of the proof is valid if certain boundedness conditions hold. In a slight abuse of notation, $\omega(A,b,c)$, $\Omega_P(A,b,c)$, and $\Omega_D(A,b,c)$ will indicate the optimal value and optimal solution sets as functions of the data, where A is the constraint matrix.

Theorem 8: If the optimal solution sets $\Omega_P(A,b,c)$ and $\Omega_D(A,b,c)$ have non-empty intersections with fixed bounded sets for all (A,b,c) in a set S and sufficiently close to (A^*,b^*,c^*), then the optimal value function $\omega(A,b,c)$ is continuous at (A^*,b^*,c^*) relative to S.

Proof: If $\{(A^k,b^k,c^k)\} \subseteq S$ with $(A^k,b^k,c^k) \to (A^*,b^*,c^*)$, then by the boundedness assumption there exists a subsequence K such that $x^k \in \Omega_P(A^k,b^k,c^k)$ for $k \in K$, $y^k \in \Omega_D(A^k,b^k,c^k)$ for $k \in K$, and $x^k \overset{K}{\to} x^*$, $y^k \overset{K}{\to} y^*$. Since $c^k x^k = b^k y^k$ for $k \in K$ and x^* and y^* are feasible for LP^* and D^* respectively, it follows that

$$c^*x^* = \lim_{k \in K} c^k x^k = \lim_{k \in K} b^k y^k = b^*y^*,$$

so x^* and y^* are optimal for their respective problems. By considering sequences corresponding to lim inf and lim sup, continuity of ω may be established. ▲

Since this proof does not make use of the particular forms of the linear programs, the result is also valid for general LP's. The preceding theorem is an extension of a continuity theorem of Martin [9] who used the fact that $\Omega_P(A,b,c)$ and $\Omega_D(A,b,c)$ are uniformly bounded sets in some neighborhood of (A^*,b^*,c^*) provided that Ω_P^* and Ω_D^* are bounded. As we will see, there are interesting cases in which the perturbed sets are not uniformly bounded, but do nevertheless intersect fixed bounded sets for (A,b,c) in some neighborhood of (A^*,b^*,c^*).

In the case in which the coefficient matrix is fixed at A^*, the boundedness property required in the preceding Theorem is assured because of the form of the optimal basic feasible solutions and the corresponding optimal dual solutions. However, as illustrated by Example 2, this argument cannot be used when the constraint coefficients are perturbed, because a sequence of optimal bases may converge to a singular matrix and the corresponding sequence of optimal solutions may be unbounded. The sufficient boundedness property may be obtained by assuming the boundedness of Ω_P^* and Ω_D^*, which in turn is equivalent to the conditions $b^* \in \text{int}(\mathcal{B})$ and $c^* \in \text{int}(\mathcal{C})$. Moreover, the latter conditions are clearly necessary if arbitrary perturbations are allowed, so in that case they are equivalent to continuity of ω. With regard to the continuity of the optimal solution sets of the primal and dual, the results of the previous sections may be extended to allow constraint coefficient perturbation, provided that the optimal solution sets of the unperturbed problem are singletons.

Theorem 9: The optimal solution sets Ω_P^* and Ω_D^* are bounded if and only if the optimal value function ω is continuous at (A^*,b^*,c^*) relative to arbitrary data perturbations. The optimal primal solution set Ω_P is continuous at (A^*,b^*,c^*) relative to arbitrary perturbations if and only if Ω_P^* is a singleton and Ω_D^* is bounded. The optimal dual solution set Ω_D is continuous at (A^*,b^*,c^*) relative to arbitrary perturbations if and only if Ω_P^* is bounded and Ω_D^* is a singleton.

Proof: See Appendix. ▲

If attention is restricted to perturbations that are optimality-perserving, (i.e., an optimal solution exists for the perturbed problem) the conditions $b^* \in \text{int}(\mathcal{B})$ and $c^* \in \text{int}(\mathcal{C})$ are not necessary for continuity, as the next Theorem shows. In fact, the sufficient condition given in the next Theorem is a condition on the constraint coefficients rather than the objective and right-hand side.

Theorem 10: If every set of m columns of A^* is non-singular, then $\omega(A,b,c)$ is continuous at (A^*,b^*,c^*) relative to perturbations in Ψ, the set of (A,b,c) for which $LP(A,b,c)$ has an optimal solution. In addition, under this assumption, $\Omega_P(A,b,c)$ is continuous at (A^*,b^*,c^*) relative to perturbations in Ψ if and only if Ω_P^* is a singleton; and $\Omega_D(A,b,c)$ is continuous at (A^*,b^*,c^*) relative to perturbations in Ψ if Ω_D^* is a singleton.

Proof: The proof of Theorem 1 is easily modified to show the continuity of ω, since the fixed basis matrix may be replaced by a set of columns corresponding to fixed indices, and since all sufficiently small perturbations will preserve the non-singularity property. The proof of the continuity properties of Ω_P and Ω_D is given in the Appendix. ▲

When A^* has the property that every set of m columns of A^* forms a non-singular matrix, we say that A^* is a Haar matrix and write $A^* \in \mathcal{H}$. It is easy to see that $A^* \in \mathcal{H}$ is not a necessary condition for continuity of ω, since $b^* \in \text{int}(\mathcal{B})$ and $c^* \in \text{int}(\mathcal{C})$ may hold whether or not $A^* \in \mathcal{H}$. Conversely, if $A^* \in \mathcal{H}$, then ω

will be continuous relative to optimality-preserving perturbations even if $b^* \in \text{bdy}(\mathcal{B})$.

Example 3: With $m = n = 1$ and $c^* = A^* = 1$, $b^* = 0$, the primal problem becomes

$$\min_x \quad x$$
$$\text{subject to:} \quad x = 0,\ x \geq 0$$

and the dual problem is

$$\max_y \quad 0 \cdot y$$
$$\text{subject to:} \quad y \leq 1.$$

In this example, $A^* \in \mathcal{H}$, so the optimal value is continuous relative to perturbations that preserve the existence of an optimal solution. In this case, this allows all sufficiently small perturbations in c^* and A^*, but only non-negative perturbations of $b^* = 0$. Note that the conditions of Theorem 9 are not satisfied, since b^* is on the boundary of $\mathcal{B}$ (the optimal solution set of the dual is thus unbounded). However, the only discontinuities that can occur are those corresponding to primal infeasibility, and such perturbations are specifically excluded from consideration. ▲

Cases in which $A^* \in \mathcal{H}$ and $b^* \in \text{bdy}(\mathcal{B})$ have an interesting property. Robinson [12] showed that $b^* \in \text{bdy}(\mathcal{B})$ implies that for any $\bar{x}$ feasible for LP^*, there exist perturbations A and b arbitrarily close to A^* and b^* such that $\bar{x}$ is optimal for the perturbed problem LP(A,b,c^*). However, when $A^* \in H$, the optimal value function is continuous, so, when $b^* \in \text{bdy}(B)$, it must be the case that the objective function is constant over the entire feasible set, since Robinson's result implies that this is a necessary condition for continuity of ω. In fact, the apparent instability corresponding to the arbitrary nature of the optimal solutions of perturbed problems does not occur, because (as in the case of Example 3) the feasible set is a singleton whenever $A^* \in H$ and $b^* \in \text{bdy}(B)$ (see Theorem 14 of the Appendix for a proof). In such a case the optimal solution set Ω_p will also be continuous at (A^*,b^*,c^*) relative to optimality-preserving perturbations, since the next theorem establishes that uniqueness of the feasible solution of a general LP is sufficient for continuity of both the optimal value and optimal solution set. When $b^* \in \text{bdy}(B)$, the feasible solution will often be unique even if $A^* \notin H$.

Theorem 11: If the feasible set of a linear program is a singleton, then the optimal value and the optimal solution set are continuous relative to optimality preserving perturbations.

Proof: See Appendix. ▲

Finally, we provide an example that illustrates that for $b^* \in \text{bdy}(B)$, ω may be continuous even if the feasible set is not a singleton.

Example 4: Consider the LP

$$\min_x \; 0 \cdot x$$

$$\text{subject to:} \quad x_1 + 2x_2 = 2$$
$$x_3 = 0$$
$$x \geq 0 .$$

Because of the constraint $x_3 = 0$, $b^* \in \mathrm{bdy}(\mathcal{B})$. Clearly, the feasible set is not a singleton and every feasible solution is optimal. By considering the three possible basic feasible solutions that may arise as a result of perturbation, it is easily seen that the optimal value function must be continuous relative to optimality preserving perturbations. (Alternatively, boundedness of the feasible set implies that every feasible solution of the perturbed constraints will be close to some solution of the unperturbed constraints, so the constancy of the objective on the unperturbed feasible set implies continuity of the optimal value.) ▲

Theorem 8 may also be used to establish the continuity of the feasible set of a linear system under appropriate hypotheses. Let $F(A,b) \equiv \{x \mid Ax=b,\ x \geq 0\}$ and assume $F^* \equiv F(A^*,b^*)$ is non-empty. The behavior of the feasible set in Example 2 shows that the mapping F need not be continuous relative to feasibility-preserving perturbations. However, if the set $\Delta^* = \{y|yA^* \leqq 0,\ yb^*=0\}$ is <u>bounded</u> (which will occur if and only if $\Delta^* = \{0\}$), then continuity of F at (A^*,b^*) holds. (Observe that Δ^* may be thought of as the optimal solution set of the dual of the following problem,

$$\min_x \quad 0 \cdot x$$

$$\text{subject to} \quad A^*x = b^*,\ x \geq 0,$$

which has optimal solution set F^*. From this viewpoint, the continuity result to be obtained is related to Theorem 9, except that the objective function is fixed at 0 and the optimal primal solution set is not assumed to be a singleton.)

<u>Theorem 12</u>: If $F^* \neq \phi$ and $\Delta^* = \{0\}$, then F is continuous at (A^*,b^*).

<u>Proof</u>: We will first show that $F(A,b)$ has a non-empty intersection with some fixed bounded set for all (A,b) sufficiently close to (A^*,b^*).

Suppose that this is false, so that there exists a sequence $\{(A^i,b^i)\}$ converging to (A^*,b^*) with the property that the corresponding sequence $\{\omega^i\}$ of optimal values of the problems

$$(P^i) \qquad \min_x \quad ex \qquad \text{subject to:} \quad A^ix = b^i,\ x \geq 0$$

tends to $+\infty$ (where $e = (1,\ldots,1)$, and if (P^i) is infeasible, its optimal value ω^i is defined to be $+\infty$). Consider the corresponding sequence of dual problems

$$(D^i) \qquad \max_y \; b^i y \quad \text{subject to: } yA^i \leq e .$$

Since $y = 0$ is always feasible, each (D^i) has optimal value ω^i also, where $\omega^i = +\infty$ implies that the problem (D^i) is unbounded. Thus, there exists a sequence $\{y^i\}$ such that y^i is feasible for (D^i) and $b^i y^i \to +\infty$. Without loss of generality, we may assume that all $y^i \neq 0$ and that the sequence $\{y^i | \|y^i\|\}$ converges to a point $y^* \neq 0$. Since $b^i y^i \to +\infty$, $\|y^i\| \to +\infty$, and $\|y^i\|^{-1} y^i A^i \leq \|y^i\|^{-1} e$ implies $y^* A^* \leq 0$ and $y^* b^* \geqq 0$. However, $y^* b^* = 0$ contradicts $\Delta^* = \{0\}$ and $y^* b^* > 0$ contradicts $F^* \neq \phi$, since it would imply unboundedness of the dual of the limiting problem.

We now apply Theorem 8 to the optimal value function of the problem class

$$\begin{aligned} \min_{u,x} \quad & eu \\ \text{subject to:} \quad & Ax = b \\ & -x + u \geq -x^* \\ & x + u \geq x^* , \\ & x \geq 0 \end{aligned}$$

where A and b are parameters and $x^* \in F^*$. The optimal value of this problem is the distance from x^* to the closest point (in the ℓ_1-norm) in $F(A,b)$. By the property established for $F(A,b)$, the optimal solution of the above problem must also intersect some fixed bounded set for (A,b) sufficiently close to (A^*,b^*). The dual problem may be written as

$$\begin{aligned} \max_{y,v,w} \quad & by - x^* v + x^* w \\ \text{subject to:} \quad & yA - v + w \leq 0 \\ & v + w = e \\ & v , w \geq 0 \end{aligned}$$

The dual optimal solution sets must also intersect some fixed bounded set for (A,b) sufficiently close to (A^*,b^*), for, if they did not, there would be an optimal solution sequence $\{y^i\}$ with $\|y^i\| \to +\infty$ corresponding to a perturbation sequence $\{A^i,b^i\} \to \{A^*,b^*\}$, and a contradiction to $\Delta^* = \{0\}$ could be obtained along the lines of the first part of the proof. Thus, Theorem 8 implies the continuity of the optimal value of the minimum distance problem, which in turn yields the continuity of F. ▲

(Note that Theorem 12 does not assert that the sets $F(A,b)$ are bounded; in fact, if $\{y|yA^* \leq 0\} = \{0\}$, then $F(A,b)$ is <u>unbounded</u> for all (A,b) sufficiently close to (A^*,b^*). To see that F^* is unbounded under this assumption, consider the problem

$$\max_x \quad ex$$

$$\text{subject to:} \quad A^*x = 0, \; x \geq 0$$

and its dual

$$\min_y \quad 0y$$

$$\text{subject to:} \quad yA^* \geq e.$$

The dual is infeasible, but the primal is feasible and therefore unbounded. It is easily seen that these properties are invariant under perturbations of A^*.)

The continuity result of Theorem 12 may be extended in an obvious manner to linear systems of other forms. Lipschitz properties of F under the same assumptions will be described in a forthcoming paper.

6. Summary

The results of Sections 2-4 are summarized by Tables 1 and 2. For a given mapping and perturbation (or perturbation pair), the table entries give properties relevant to continuity. The interpretation of the tables is illustrated by the following examples.

Example 5: If arbitrary perturbations are allowed in b and c, the perturbation sets are designated as $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. In this case, ω is continuous at (b^*,c^*) if and only if Ω_D^* is bounded (Table 1) and Ω_P^* is bounded (Table 2); Ω_P is continuous at (b^*,c^*) if and only if Ω_D^* is bounded and Ω_D^* is a singleton. ▲

Example 6: If b is fixed at b^* and the perturbations of c are restricted to C, only Table 2 is required. In this case ω and Ω_D are continuous regardless of the nature of Ω_P^* and Ω_D^*. Ω_P is continuous if and only if Ω_P^* is a singleton. ▲

It should be noted that the continuity of a mapping in the tables is equivalent to the solution set property corresponding to the perturbation with one exception - in Table 1 the property that Ω_D^* is a singleton is sufficient but not necessary for Ω_D to be continuous relative to perturbation within B.

Note also that the results of Section 5 pertaining to arbitrary perturbations of all of the data (including the constraint coefficients) are the same as those corresponding to arbitrary perturbations of just the right-hand side and objective coefficients (the $\mathbb{R}^m$ and $\mathbb{R}^n$ rows

Limitations on Perturbation Set for b	Continuous Mapping ω	Ω_p	Ω_D
$\mathbb{R}^m$ (unrestricted)	Ω_D^* bounded	Ω_D^* bounded	Ω_D^* a singleton
$\mathcal{B}$	unconditional	unconditional	Ω_D^* a singleton†

†sufficient but not necessary for continuity

Table 1. The relationship between continuity conditions and perturbation sets for b

Limitations on Perturbation Set for c	Continuous Mapping ω	Ω_p	Ω_D
$\mathbb{R}^n$ (unrestricted)	Ω_p^* bounded	Ω_p^* a singleton	Ω_p^* bounded
C	unconditional	Ω_p^* a singleton	unconditional

Table 2. The relationship between continuity conditions and perturbation sets for c

of Tables 1 and 2). Moreover, the results of Section 5 based upon $A^* \in H$ correspond to the $\mathcal{B}$ and $\mathcal{C}$ rows of the tables, keeping in mind that when the constraint coefficients are also perturbed, $\mathcal{B}$ and $\mathcal{C}$ may be dependent upon the perturbed matrix A. It should be observed that for $A^* \notin H$, the conditions in the $\mathcal{B}$ and $\mathcal{C}$ rows are <u>not</u> in general valid when constraint coefficient perturbation is allowed. That is, ω is not necessarily continuous relative to optimality-perserving perturbations, and Ω_P is not necessarily continuous when Ω_P^* is a singleton (refer to Example 2).

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Appendix

The sufficient conditions stated previously for the continuity of optimal solution set mappings are a consequence of the following Lemma.

Lemma 1: Let Ω be a point-to-set mapping from one finite-dimensional space into the subsets of another such that $\Omega(t)$ is defined and non-empty on a set T. Let Ω be u.s.c. relative to T at t^*, and assume that $\Omega(t^*)$ is a singleton. If there exists an open set N containing t^* and a constant K such that for each $t \in N \cap T$, there is a $\bar{t} \in \Omega(t)$ with $\|\bar{t}\| \leq K$, then Ω is continuous at t^* relative to T.

Proof: The l.s.c. of Ω at t^* follows from the fact that if $\{t^k\} \subseteq T$ with $t^k \to t^*$, then there exists a convergent subsequence $\{\bar{t}^k\}$ with $\bar{t}^k \in \Omega(t^k)$. If $t' = \lim \bar{t}^k$, then, by u.s.c., $t' \in \Omega(t^*)$. so that $t' = t^*$. This establishes l.s.c., so Ω is continuous at t^*. ▲

To apply the Lemma to prove Theorem 3, note that the u.s.c. of the optimal solution set is implied by the continuity of the optimal value function. When Ω_D^* is a singleton, the boundedness property of the Lemma is implied by the uniform boundedness of Ω_D^* for t close to t^*. (If this uniform boundedness did not hold, then there would be sequences $\{y^K\}$, $\{b^K\}$, $\{c^K\}$ with $\|y^K\| \to \infty$, $b^K \to b^*$, $c^K \to c^*$ and such that $y^k A^* \leq c^k$, $y^k b^k = \omega(b^k, c^k)$. By dividing these relations by $\|y^K\|$ and taking the limit of an appropriate subsequence, we obtain a vector y^* satisfying $\|y^*\| = 1$, $y^* A^* \leq 0$, $y^* b^* = 0$, which contradicts the assumption that Ω_D^* is a singleton.)

In the cases of Theorems 9 and 10, in which perturbations of the constraint coefficients are allowed, it is easily seen that the analogous uniform boundedness properties may be proved in a similar manner. Thus, the sufficiency of the conditions assumed for the continuity of the optimal solution sets is established. As in the case of the optimal value function, it is easily seen via consideration of the appropriate objective function that boundedness of both optimal solution sets is a necessary condition for their continuity if arbitrary perturbations are allowed (Theorem 9). To show that continuity also imples uniqueness (Theorem 9) we introduce the family of problems GLP(q), where S is a fixed polyhedral set:

$$\text{GLP}(q) \qquad \begin{aligned} &\min_{z} \; qz \\ &\text{subject to:} \quad z \in S\,. \end{aligned}$$

Theorem 13: The following are equivalent:

(1) $\text{GLP}(q^*)$ has a unique optimal solution

(2) the optimal solution set of GLP(q) is continuous at q^* (relative to arbitrary perturbations of the objective)

(3) the optimal solution set of GLP(q) is the same singleton for all q sufficiently close to q^*

Proof: (1) $\Rightarrow$ (2) The feasible set S may be expressed as

$S = \{z \mid z = \sum_{i=1}^{r} \lambda_i z^i + \sum_{j=1}^{s} \mu_j d^j,\ \lambda \geq 0,\ \mu \geq 0,\ \sum_{i=1}^{r} \lambda_i = 1\}$, where the z^i are distinct elements of S and the d^j are non-zero vectors with the property that if $\bar{z}$ is any element of S, then $\bar{z} + \mu d^j \in S$ for all

d^j and all $\mu \geq 0$ (see Rockafellar [14]; in the case of problems in standard form, such as LP(b,c), the z^i may be taken to be the extreme points and the d^j, the extreme rays). Uniqueness of the optimal solution is equivalent to the existence of a z^t such that $q^*z^t < q^*z^i$ for $i \neq t$ and $q^*d^j > 0$ for all d^j. Clearly, if S is unchanged, then for all $\tilde{q}$ sufficiently close to q^* these inequalities must still hold, so that z^t remains the unique optimal solution for all perturbed problems with $\tilde{q}$ sufficiently close to q^*.

(2) ⇒ (1) Conversely, suppose that GLP(q^*) has more than one optimal solution. There are two cases to consider: (i) $q^*d^j > 0$ for all j, but $|I| \geq 2$, where $I = \{i | z^i$ is an optimal solution of GPL(q^*)$\}$ and (ii) there exists a u such $q^*d^u = 0$ (note that if $q^*d^j < 0$ for some j, then the problem would be unbounded). In case (i), there exists a $t \in I$ and a δ such that $\delta z^t < \delta z^u$ for all $u \in I$, $u \neq t$, and δ may be assumed to be scaled so that $(q^*+\delta)\, d^j > 0$ for all j. Thus, for all $\theta \in (0,1)$ it follows that the linear program GLP($q^*+\theta\delta$) has a unique optimal solution z^t, so that the set of optimal solutions is discontinuous at q^*. In case (ii), the linear problem GLP($q^*-\theta d^u$) is unbounded for all $\theta > 0$, so again the optimal solution set is discontinuous at q^*.

(1) ⇒ (3) Follows from the proof of (1) ⇒ (2).

(3) ⇒ (1) Trivial. ▲

In the presence of non-negativity constraints in GLP(Q), Theorem 13 and its proof may be modified so as to involve only optimality-preserving perturbations (thereby completing the proof of Theorem 10) by noting that in case (ii) of the proof that (2) implies (1), discontinuity may be demonstrated by means of an objective perturbation of the form θe, where e is the vector of 1's. Such a perturbation (for arbitrary $\theta > 0$) transforms the optimal solution set from an unbounded set into a subset of the convex hull of the (fixed) extreme points of GLP (q^*) (observe that this perturbation is optimality-preserving, since the problem remains feasible and cannot become unbounded).

It should be noted that similar arguments may be used to show that boundedness of the optimal solution set of GLP(q^*) is equivalent to q^* being in the interior of the set of feasible RHS's for the dual family.

We will now establish the uniqueness of the feasible solution under the conditions described in Section 5.

<u>Theorem 14</u>: If $A^* \in \mathcal{H}$ and $b^* \in \text{bdy}(\mathcal{B})$, then $F^* \equiv \{x | A^*x=b, x \geq 0\}$ is a singleton.

<u>Proof</u>: The proof is by induction on the number n of columns of A^*. Because $A^* \in \mathcal{H}$, it is the case that $n \geq m$. If $n = m$, the result is trivial because A^* is non-singular. Suppose the result is true for $n = m,\ldots,k$, and let A^* be $m \times (k+1)$. Assume that F^* contains two distinct elements, $\bar{x}$ and x'. Consider the vector $\tilde{x} = \frac{1}{2}(\bar{x}+x')$. Clearly $\tilde{x} \in F^*$, and if $\tilde{x} > 0$, it is easy to see that $b^* \in \text{bdy}(\mathcal{B})$ is contradicted. On the other hand, if $\tilde{x}_i = 0$ for some i, then $\bar{x}_i = x'_i = 0$, so that the $m \times k$ system obtained by deleting column i from A^* has two distinct solutions, contradicting the induction hypothesis. ▲

Finally, to prove Theorem 11, we first apply the preceding Lemma 1 to establish continuity of the feasible set by observing that u.s.c. is trivial and that uniform boundedness property of the feasible set may be established in the usual manner. Since the optimal solution set of a perturbed problem must be a subset of the feasible set, uniform boundedness of the optimal sets is obvious, and u.s.c. follows from the observation that a sequence of optimal solutions must converge to the limiting singleton. Continuity of the mapping follows from the Lemma, and continuity of the optimal value is then trivial.