ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS, ASYMPTOTIC BEHAVIOR I.

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ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS, ASYMPTOTIC BEHAVIOR I.*

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ABSTRACT

Consider solutions $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ of the von Kármán equations for the swirling flow between two rotating coaxial disks

1.1) $$\varepsilon H^{iv} + HH''' + GG' = 0 ,$$

and

1.2) $$\varepsilon G'' + HG' - H'G = 0 .$$

We also assume that $|H(x,\varepsilon)| \leq B\sqrt{\varepsilon}$ while $|G(x,\varepsilon)| \leq B$. This work considers the shapes and asymptotic behavior as $\varepsilon \to 0+$. We consider the kind of limit functions that are permissible. The only possible limits (interior) for $G(x,\varepsilon)$ are constants. If that limit constant is not zero, then $\frac{1}{\sqrt{\varepsilon}} H(x,\varepsilon)$ will also tend to a constant.

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SIGNIFICANCE AND EXPLANATION

Under appropriate conditions the steady-state flow of fluid between two planes rotating about a common axis perpendicular to them may be described by two functions $H(x,\varepsilon)$, $G(x,\varepsilon)$ which satisfy the coupled system of ordinary differential equations

$$\varepsilon H^{iv} + HH''' + GG' = 0$$

$$\varepsilon G'' + HG' - H'G = 0 .$$

The quantity $\varepsilon > 0$ is related to the kinematic viscosity and $\frac{1}{\varepsilon} = R$ is usually called the Reynolds number.

These equations have received quite a bit of attention. First of all, people who are truly interested in the phenomena modeled by these equations, e.g. fluid dynamicists, are interested in this problem. However, as these equations have been studied by a variety of mathematical methods, they have taken on an independent interest. The major methods employed have been (i) Matched Asymptotic Expansions and (ii) Numerical Computations. In both approaches technical problems have appeared. There may be "turning points," i.e. points at which $H(x,\varepsilon) = 0$. Such points require special and delicate analysis within the theory of (i). As numerical problems, these equations are "stiff" - precisely because ε is small. The occurrence of "turning points" only makes computation more difficult.

For these reasons, these equations have become "test" problems for methods of "matching in the presence of turning points" and "stiff O.D.E. solvers." However, when one has "test problems," one needs to know the answers. Unfortunately here the answers are largely unknown.

In this report we study the asymptotic behavior as ε becomes small. A wealth of qualitative information is obtained which will enable one to further the various "test" programs.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS, ASYMPTOTIC BEHAVIOR I.*

Heinz Otto Kreiss (1) and Seymour V. Parter (2)

1. Introduction

In 1921 T. von Kármán [5] developed the similarity equations for incompressible axi-symmetric fluid flows. In 1951 G. K. Batchelor [1] used the von Karman approach to study the fluid motion between two rotating planes, rotating about a common axis perpendicular to them. Despite the passage of time and the work of many people, this problem is far from being completely understood.

Following Batchelor, K. Stewartson [20] made a further study of the problem and disagreed with several of Batchelor's basic conclusions. In the ensuing years many people have attacked this problem. Numerical calculations have been carried out by Lance and Rogers [7], C. E. Pearson [15], D. Greenspan [3], D. Schultz and D. Greenspan [19], L. O. Wilson and N. L. Schryer [23], G. L. Mellor, P. J. Chapple and V. K. Stokes [13], N. D. Nguyen, J. P. Ribault and P. Florent [14], S. M. Roberts and J. S. Shipman [17]. Formal matched asymptotic expansion methods have been applied by A. M. Watts [22] (who also did numerical calculations) K. K. Tam [21], H. Rasmussen [16], B. J. Matkowsky and W. L. Siegman [12]. Undoubtably many others have also worked on this problem and we are unaware of their efforts.

Rigorous mathematical results are a bit sparse. There are (to our knowledge) exactly three papers concerned with the existence question, S. P. Hastings [4], A. R. Elcrat [2] and J. B. McLeod and S. V. Parter [10]. The first two obtained existence and uniqueness

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results for small values of the Reynolds number $(R = \frac{1}{\varepsilon})$. The third concerns itself only with the case of counter-rotating planes. An existence theorem is obtained for all $\varepsilon > 0$ and a complete asymptotic description (as $\varepsilon \to 0+$) is given for the solutions obtained. Not a word is said about unicity. A later paper by McLeod and Parter [11] gives a negative result (in the limit as $\varepsilon \to 0$) of the existence of solutions which are monotone in the angular velocity. That results contradicts a conjecture of Batchelor.

The interplay between all of these approachs has been extremely profitable and interesting. The conjectures and remarks of Batchelor and Stewartson, the "shapes" obtained in numerical calculations and the general qualitative results of the formal asymptotic expansions have all led to "target" questions. In their turn these target questions have been studied analytically, numerically and by formal expansion methods. For example, the results of [10] cast doubt on the calculations of [3] and a refined method was proposed in [19]. One of the goals of [12] was to obtain - via formal expansion techniques - the solutions of [10].

Let us now describe the problem. Let the planes be placed at $x = 0$ and $x = 1$ and rotate about the x-axis with constant angular velocities Ω_0, Ω_1 respectively. Let q_r, q_θ, q_x denote the velocities in cylindrical coordinates (r,θ,x). Following von Kármán [5] and Batchelor [1] we make the ansatz that q_x is a function of x alone, i.e. there is a function $H(x)$ such that

$$q_x = -H(x) .$$

Then, as a consequence of the steady state Navier stokes equations we find that

$$q_r = \frac{r}{2} H'(x)$$

and, there is a function $G(x)$ such that

$$q_\theta = \frac{r}{2} G(x) .$$

These functions $\langle H(x), G(x) \rangle$ satisfy the ordinary differential equations

1.1) $$\varepsilon H^{iv} + HH''' + GG' = 0$$

1.2) $$\varepsilon G'' + HG' - H'G = 0$$

where ε is the kinematic viscosity. The associated boundary conditions are

1.3) $$H(0) = H(1) = 0, \text{ (no penetration)}$$

1.4) $$H'(0) = H'(1) = 0, \text{ (no slip)}$$

1.5) $$G(0) = 2\Omega_0, \quad G(1) = 2\Omega_1 .$$

However, our results are independent of these boundary conditions. Hence they apply to the cases where one has "suction" or "blowing" on the planes.

In this work we are concerned with the asymptotic behavior of solutions $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ as $\varepsilon_n \to 0+$ under the basic hypothesis:

H.1) There is a constant B such that

1.6a) $$|H(x,\varepsilon_n)| \le B\sqrt{\varepsilon_n} ,$$

1.6b) $$|G(x,\varepsilon_n)| \le B .$$

This hypothesis has been used, implicitly or explicity, in many of the studies connected with this problem. There are good reasons for this. For example, if we set

1.7) $$\xi = \frac{x}{\sqrt{\varepsilon}}, \quad h(\xi,\varepsilon) = \frac{1}{\sqrt{\varepsilon}} H(x,\varepsilon), \quad g(\xi,\varepsilon) = G(x,\varepsilon) ,$$

then equations (1.1), (1.2) become

1.8a) $$\left(\frac{d}{d\xi}\right)^4 h + h\left(\frac{d}{d\xi}\right)^3 h + g \frac{d}{d\xi} g = 0 ,$$

1.8b) $$\left(\frac{d}{d\xi}\right)^2 g + h\left(\frac{d}{d\xi}\right) g - \left(\frac{dh}{d\xi}\right) g = 0 ,$$

on the larger interval $[0, \frac{1}{\sqrt{\varepsilon}}]$. Many of the formal asymptotic expansion studies [21], [16], [22] have used this "stretching" and then "matched" with the numerical results of Rogers and Lance [18] for the semi-infinite region, i.e. <u>the</u> von Karman problem. Numerical studies based on "shooting" methods [13], [17] have found it convenient to make this change and then actually compute $\langle h(\xi,\varepsilon), g(\xi,\varepsilon)\rangle$. In fact, Wilson and Schryer [23] who did not use a shooting method also found these variables convenient for calculation. Moreover, the solutions found by McLeod and Parter [12] do, in fact, satisfy these estimates.

One may integrate equation (1.1) to obtain

1.9) $$\varepsilon H''' + HH'' + \frac{1}{2} G^2 - \frac{1}{2} (H')^2 = \mu(\varepsilon)$$

where $\mu(\varepsilon)$ is a constant. This constant is of some independent interest. For example, in the semi-infinite problem i.e. single disk or von Karman problem, one sets

1.10) $$\mu = \frac{1}{2} \lim_{x\to\infty} [G(x)]^2 \geq 0 ,$$

and μ is a known quantity. On the other hand, in [10] it was found that $\mu(\varepsilon) \sim -\varepsilon$. And, of course, in the two disk problem, $\mu(\varepsilon)$ is unknown.

In section 2 we set $\varepsilon = 1$ and study solutions of (1.1), (1.2) on large intervals $[0,E]$ with $E >> 1$. Here we discover several points of interest for the single disk problem. In section 3 we return to the finite interval and values of $\varepsilon < 1$. Under the assumption H.1 we are able to make the change of variables (1.7) and apply the results of section 2. The main results of this section are (i) If $\mu(\varepsilon_n) \to \bar{\mu}$ as $\varepsilon_n \to 0+$, then $\bar{\mu} \geq 0$, (ii) one may select a subsequence $n_k \to 0$ such that, there is a constant g_∞ and for every δ, $0 < \delta < \frac{1}{10}$

1.11) $$\operatorname*{Max}_{x}\{|G(x,\varepsilon_{n_k}) - g_\infty|;\ 0 < \delta \leq x \leq 1 - \delta\} \to 0 \quad \text{as} \quad \varepsilon_{n_k} \to 0 .$$

Moreover,

1.12) $$\mu(\varepsilon_{m_k}) \to \frac{1}{2} g_\infty^2 .$$

These results are consistent with the suggestions of both Batchelor (who emphasized the case $g_\infty \neq 0$) and Stewartson (who emphasized the case $g_\infty = 0$).

In section 4 we consider the case $\bar{\mu} > 0$. In this case we find that there is a constant h_∞ such that

1.13) $$\operatorname{Max}\left\{\left|\frac{H(x,\varepsilon_{n_k})}{\sqrt{\varepsilon_{n_k}}} - h_\infty\right|;\ \delta \leq x \leq 1 - \delta\right\} \to 0 .$$

Much of our work described in this paper is based on the properties of the function

1.14) $$\Phi(x,\varepsilon) = [G'(x,\varepsilon)]^2 + [H''(x,\varepsilon)]^2 .$$

The basic result, due to McLeod [9], [10], is

Lemma Φ: The function $\Phi(x,\varepsilon)$ satisfies the differential equation

1.15) $$\varepsilon\Phi'' + H\Phi' = 2\varepsilon[(G'')^2 + (H''')^2] ,$$

and, the function

$$\Phi'(x,\varepsilon)\exp\{\frac{1}{\varepsilon}\int_{x_0}^{x} H(t,\varepsilon)dt\}$$

is nondecreasing. Since it is also holomorphic, it has at most one zero. Thus the behavior of $\Phi(x,\varepsilon)$ is described in one of the following three ways:

(α) Φ is monotone decreasing on its interval of definition,

(β) Φ is monotone increasing on its interval of definition,

(γ) there is an interior point γ such that $\Phi' < 0$ for $x < \gamma$ and $\Phi' > 0$ for $x > \gamma$.

2. Some Basic Estimates

In this section we are concerned with obtaining estimates on functions $\langle h(\xi), g(\xi)\rangle$ which satisfy the differential equations

2.1a) $$h^{iv} + hh''' + gg' = 0, \qquad 0 \le \xi \le E$$

2.1b) $$g'' + hg' - h'g = 0, \qquad 0 \le \xi \le E$$

where

2.2) $$0 \le 1 \le E \le \infty .$$

Moreover, these functions satisfy the a-priori estimate

2.3) $$|h(\xi)| \le B, \quad |g(\xi)| \le B .$$

Throughout this section the letters E, B will denote these constants.

We recall a basic estimate due to Landau [6].

Lemma 2.1: Let $f(\xi) \in C^N[0,E]$ and let $\eta > 0$ be a given positive number. There is a constant $C(\eta,N)$ depending only on η and N and not on the length E, such that: for $1 \le j \le N-1$

2.4) $$\left\|\left(\frac{d}{d\xi}\right)^j f\right\|_\infty \le C(\eta,N)\|f\|_\infty + \eta\left\|\left(\frac{d}{d\xi}\right)^N f\right\|_\infty .$$

Moreover, if $\eta \le \frac{1}{2}E$ then

2.5) $$\|f'\|_\infty \le \frac{\eta}{2}\|f''\|_\infty + \frac{2}{\eta}\|f\|_\infty .$$

Proof: See [6].

Remark: In most instances this lemma is applied when η is small, however we shall also use (2.5) when η is large.

Lemma 2.2: Let $h(\xi), g(\xi)$ satisfy (2.1a), (2.1b) and (2.3). There are constants B_j, $j = 1,2,\ldots$ (depending only on B and not on h,g) such that

2.6) $$\left\|\left(\frac{d}{d\xi}\right)^j h\right\|_\infty + \left\|\left(\frac{d}{d\xi}\right)^j g\right\|_\infty \le B_j, \qquad 0 \le \xi \le E .$$

Proof: Let $\eta = \frac{1}{(4B)}$. From (2.1a), (2.1b) and lemma 2.1 we obtain

$$\|h^{iv}\|_\infty \le B[BC(\eta,4) + \eta\|h^{iv}\|_\infty] + B[BC(\eta,2) + \eta\|g''\|_\infty],$$

$$\|g''\|_\infty \le B[BC(\eta,4) + \eta\|h^{iv}\|_\infty] + B[BC(\eta,2) + \eta\|g''\|_\infty].$$

Collecting terms and adding the inequalities we obtain

$$(1 - 2B\eta)[\|h^{iv}\|_\infty + \|g''\|_\infty] \leq 2B^2[C(\eta,4) + C(\eta,2)] \ .$$

That is

$$\|h^{iv}\|_\infty + \|g''\|_\infty \leq 4B^2[C(\eta,4) + C(\eta,2)] \ .$$

Thus, (2.6) follows from (2.4) and repeated differentiations of the basic equations.

In the remainder of this section we use these estimates and lemma ϕ to obtain even stronger estimates.

Let

2.7) $$\phi(\xi) = [g'(\xi)]^2 + [h''(\xi)]^2 \ .$$

Then lemma ϕ (with $\varepsilon = 1$) applies. Suppose $\phi'(\xi) > 0$ on a "large" subinterval of $[0,E]$. Since

$$0 \leq \phi(\xi) \leq B_1^2 + B_2^2 \ ,$$

then $\phi'(\xi)$ must be "small" on "relatively large" sets. Our next result makes this statement precise. The details of the proof are left for an appendix.

<u>Lemma 2.3</u>: <u>Let</u>

2.8a) $$K_0' = \mathrm{Max}\{1, \|\phi\|_\infty, \|\phi'\|_\infty, \|\phi''\|_\infty\} \ , \qquad K_0 = K_0' + \frac{1}{16} \ ,$$

2.8b) $$K_1 = \|\phi'''\|_\infty \ .$$

<u>Let</u>

2.9) $$16K_0^2 \leq L \leq E \ .$$

<u>Then for every interval</u> $[\alpha,\beta] \subset [0,E]$ <u>of length</u> L, i.e.,

2.10) $$\beta - \alpha = L \ ,$$

<u>such that</u>

2.11) $$\phi'(\xi) \geq 0, \quad \xi \in [\alpha,\beta] \ ,$$

<u>there is a subinterval</u> $[\alpha',\beta'] \subset [\alpha,\beta]$ <u>such that</u>

2.12a) $$\beta' - \alpha' \geq \frac{1}{16K_0^2} L^{1/2}$$

<u>and</u>

2.12b) $$0 \leq \phi'(\xi) \leq \left(\frac{1}{L}\right)^{1/4} \ .$$

Moreover, on this interval,

$$|\phi''(\xi)| \le (K_1 + 1)\left(\frac{1}{L}\right)^{1/8} . \tag{2.12c}$$

<u>Proof:</u> The estimates (2.12a), (2.12b) follow immediately from Theorem A of the Appendix while (2.12c) follows from (2.5) applied to ϕ' with

$$\eta = \left(\frac{1}{L}\right)^{1/8} .$$

<u>Corollary 2.3:</u> On this same interval $[\alpha',\beta']$ we have

$$(h''')^2 + (g'')^2 \le BL^{-1/4} + (1 + K_1)L^{-1/8} \le K_2L^{-1/8} . \tag{2.13}$$

<u>Proof:</u> Apply (2.3), (2.12b) and (2.12c) to the differential equation (1.16) (with $\varepsilon = 1$).

<u>Lemma 2.4:</u> Suppose $[\alpha',\beta'] \subset [0,E]$ is a large interval, i.e.

$$\beta' - \alpha' \ge \frac{1}{16K_0^2} L^{1/2} \tag{2.14}$$

on which

$$|h'''| + |g''| \le K_3L^{-1/16} \tag{2.15}$$

where L is so big that

$$L^{1/64} \le L^{1/32} \le \frac{1}{32K_0^2} L^{1/2} \le \frac{1}{2}(\beta' - \alpha') . \tag{2.16}$$

Then there is a constant K_4 such that

$$|h''(\xi)| \le K_4L^{-1/32}, \quad |g'(\xi)| \le K_4L^{-1/32} , \tag{2.17}$$

$$|h'(\xi)| \le K_4L^{-1/64} . \tag{2.18}$$

<u>Proof:</u> Let $\eta = L^{1/32}$. Applying (2.5) to the function $h'(\xi)$ we have

$$\|h''\|_\infty \le \frac{1}{2} L^{1/32}\|h'''\|_\infty + 2L^{-1/32}\|h'\|_\infty$$

$$\le \frac{1}{2} K_3L^{-1/32} + 2B_1L^{-1/32} \le K_4L^{-1/32} .$$

Then, with $\eta = L^{1/64}$ we obtain

$$\|h'\|_\infty \le \frac{1}{2} L^{1/64}\|h''\|_\infty + 2L^{-1/64}\|h\|_\infty \le K_4L^{-1/64} .$$

A similar argument gives the result for $\|g'\|_\infty$.

Further applications of lemma 2.1 give the following additional estimates.

Lemma 2.5: Suppose $[\alpha,\beta] \subset [0,E]$ is an interval on which

2.19) $$0 \leq \phi(\xi) \leq KL^{-1/16}$$

and

2.20) $$\beta - \alpha \geq \frac{1}{2} L^{1/32} .$$

Then, there is a constant M such that

2.21a) $$|\phi'(\xi)| \leq ML^{-1/32} ,$$

2.21b) $$|\phi''(\xi)| \leq ML^{-1/64} ,$$

2.21c) $$|h'''(\xi)| \leq ML^{-1/64} ,$$

2.21d) $$|g''(\xi)| \leq ML^{-1/64} ,$$

2.21e) $$|h'(\xi)| \leq ML^{-1/64} .$$

Theorem 2.1: Suppose $E = +\infty$. Then, either $\phi(\xi) \equiv 0$, or

2.22) $$\phi'(\xi) < 0, \quad 0 \leq \xi < \infty .$$

Proof: Suppose there is a point ξ_0, $0 \leq \xi_0 < \infty$ at which (2.22) is violated. Then

$$\phi'(\xi) > 0, \quad \xi_0 < \xi < +\infty .$$

Let L be so large that we may apply Lemma 2.3 in the interval $[\xi_0 + L, \xi_0 + 2L]$. Thus we find an interval $[\alpha,\beta] \subset [\xi_0 + L, \xi_0 + 2L]$ such that

$$|\phi''(\xi)| \leq KL^{-1/8}$$

and

$$|h'''(\xi)| \leq KL^{-1/16}, \qquad |g''(\xi)| \leq KL^{-1/16} .$$

If L is sufficiently large we may also apply Lemma 2.4 to discover that

2.23) $$|\phi(\xi)| \leq KL^{-1/16} .$$

By the nature of $\phi(\xi)$, this last estimate holds on the entire interval $[\xi_0, \xi_0 + L]$. However, since L is arbitrary we have

$$\phi(\xi) = 0, \quad \xi_0 < \xi .$$

However, since $\phi(\xi)$ is a holomorphic function, $\phi(\xi) \equiv 0$.

Remark: This result is similar to a result of McLeod [11]. However, the proof is quite different, as are the hypothesis.

Theorem 2.2: Suppose $E = +\infty$. Then $\lim_{\xi=\infty} g(\xi)$ exists, call it g_∞. The constant of integration μ is given by

$$\mu = \frac{1}{2} g_\infty^2 .$$

Proof: Choose a large number L and let

$$\tilde{h}(\xi) = -h(2L - \xi), \qquad 0 \le \xi \le 2L$$

$$\tilde{g}(\xi) = g(2L - \xi), \qquad 0 \le \xi \le 2L$$

$$\tilde{\phi}(\xi) = (\tilde{h}'')^2 + (\tilde{g}')^2, \qquad 0 \le \xi \le 2L .$$

Then, $\langle \tilde{h}(\xi), \tilde{g}(\xi) \rangle$ satisfy (2.1a), (2.1b) and (2.3). Moreover, $\tilde{\phi}(\xi)$ satisfies (with ~ appropriately placed) (1.16). Finally,

$$\tilde{\phi}'(\xi) = -\phi'(2L - \xi) \ge 0, \quad 0 \le \xi \le 2L .$$

As in the proof of theorem 2.1 we apply lemma 2.3 on the interval $[L, 2L]$. Applying lemma 2.4 we find that

$$0 \le \tilde{\phi}(\xi) \le ML^{-1/16}, \quad 0 \le \xi \le L .$$

Applying lemma 2.5 we find that, as $L \to +\infty$

$$\tilde{h}'(\xi) = O(L^{-1/64}), \quad 0 \le \xi \le L ,$$

$$\tilde{h}''(\xi) = O(L^{-1/32}), \quad 0 \le \xi \le L ,$$

$$\tilde{h}'''(\xi) = O(L^{-1/64}), \quad 0 \le \xi \le L .$$

That is

$$h'(\xi) = O(L^{-1/64}), \quad L \le \xi \le 2L ,$$

$$h''(\xi) = O(L^{-1/32}), \quad L \le \xi \le 2L ,$$

$$h'''(\xi) = O(L^{-1/64}), \quad L \le \xi \le 2L .$$

Inserting these estimates into (1.9) gives

$$\frac{1}{2} g^2(\xi) \to \mu \quad \text{as} \quad \xi \to \infty . \tag{2.24}$$

Thus $\mu \ge 0$. If $\mu = 0$ then $g_\infty = 0$. If $\mu > 0$, $|g(\xi)|$ is bounded away from zero for ξ sufficiently large. Let $\sigma = \operatorname{sgn} g(\xi)$, ξ large. Then the theorem follows with

$$g_\infty = (\sqrt{2\mu})\sigma .$$

3. The Asymptotic Behavior of $G(x,\varepsilon)$

Returning to the functions $\langle H(x,\varepsilon),G(x,\varepsilon)\rangle$ which satisfy (1.1), (1.2) on $[0,1]$ we consider their behavior as $\varepsilon \to 0+$. Of course, we also assume H.1, i.e. (1.6a), (1.6b). We make the change of variables (1.7) and consider the "stretched" functions $h(\xi,\varepsilon)$, $g(\xi,\varepsilon)$ on the interval $[0, \frac{1}{\sqrt{\varepsilon}_n}]$. We observe that

3.1)
$$\begin{cases} \left(\dfrac{d}{d\xi}\right)^r h(\xi,\varepsilon) = (\varepsilon)^{\frac{r-1}{2}} \left(\dfrac{d}{dx}\right)^r H(x,\varepsilon) \ , \\ \left(\dfrac{d}{d\xi}\right)^r g(\xi,\varepsilon) = (\varepsilon)^{\frac{r}{2}} \left(\dfrac{d}{dx}\right)^r G(x,\varepsilon) \ . \end{cases}$$

Lemma 3.1: Let $\langle H(x,\varepsilon),G(x,\varepsilon)\rangle$ be a solution of (1.1), (1.2) which satisfies (1.6a), (1.6b). Let B_j be the constants of lemma 2.2. Let

$$C_0 = \frac{1}{2} B^2 + \frac{1}{2} B_1^2 + BB_2 + B_3 \ .$$

Then

3.2)
$$|\mu(\varepsilon)| \le C_0 \ .$$

Proof: Using (3.1) we see that

$$\mu(\varepsilon) = \left(\frac{d}{d\varepsilon}\right)^3 h + h\left(\frac{d}{d\varepsilon}\right)^2 h + \frac{1}{2} g^2 - \frac{1}{2}\left[\left(\frac{d}{d\varepsilon}\right)h\right]^2 .$$

Lemma 3.2: Let δ, $0 < \delta < \frac{1}{4}$ be given. There exists an $\varepsilon(\delta) > 0$ and an $M(\delta) > 0$ depending only on δ and B such that: for $0 < \varepsilon \le \varepsilon(\delta)$ and $\delta \le \sqrt{\varepsilon}\,\xi \le 1 - \delta$ we have

3.3a)
$$|h''(\xi,\varepsilon)| \le M(\delta)\varepsilon^{1/64}, \quad |g'(\xi,\varepsilon)| \le M(\delta)\varepsilon^{1/64}$$

3.3b)
$$|h'(\xi,\varepsilon)| \le M(\delta)\varepsilon^{1/128}, \quad |h'''(\xi,\varepsilon)| \le M(\delta)\varepsilon^{1/128} \ .$$

Proof: Let

$$\phi(\xi) = [g'(\xi,\varepsilon)]^2 + [h''(\xi,\varepsilon)]^2 \ .$$

Then $\phi(\xi)$ satisfies (1.16) with $\varepsilon = 1$ and lemma Φ applies. Let K_0, K_1 be as in lemma 2.3. Let

$$\varepsilon(\delta) = \delta^2/(32)^2 K_0^4 \ .$$

Then, if $0 < \varepsilon \le \varepsilon(\delta)$

3.4)
$$16K_0^2 \le \frac{\delta}{2\sqrt{\varepsilon}} = L \le \frac{1}{\sqrt{\varepsilon}} = E \ .$$

Let $\gamma(\varepsilon)$ be the unique point at which $\phi(\xi)$ assumes its minimum.

Case 1: $\gamma(\varepsilon) \in \left[\frac{\delta}{\sqrt{\varepsilon}}, \frac{1-\delta}{\sqrt{\varepsilon}}\right]$. In this case

$$\phi'(\xi) \geq 0, \quad 1 - \delta \leq \sqrt{\varepsilon}\,\xi \leq 1 - \tfrac{1}{2}\delta ,$$

$$\phi'(\xi) \leq 0, \quad \tfrac{1}{2}\delta \leq \sqrt{\varepsilon}\,\xi \leq \delta .$$

The estimate (3.4) implies that (2.9) holds for the two intervals $\left[\frac{1}{2\sqrt{\varepsilon}}\delta, \frac{\delta}{\sqrt{\varepsilon}}\right]$, $\left[\frac{1-\delta}{\sqrt{\varepsilon}}, \frac{1-\frac{1}{2}\delta}{\sqrt{\varepsilon}}\right]$. Hence we may apply lemma 2.3 and lemma 2.4 to obtain subintervals outside $\left[\frac{\delta}{\sqrt{\varepsilon}}, \frac{1-\delta}{\sqrt{\varepsilon}}\right]$ and a constant $M(\delta)$ so that (3.3a) holds. Since $\phi(\xi)$ assumes its maximum on the boundary of any interval, (3.3a) holds on all of $\left[\frac{\delta}{\sqrt{\varepsilon}}, \frac{1-\delta}{\sqrt{\varepsilon}}\right]$. The estimates (3.3b) now follow from lemma 2.2.

Case 2: $\gamma(\varepsilon) \notin \left[\frac{\delta}{\sqrt{\varepsilon}}, \frac{1-\delta}{\sqrt{\varepsilon}}\right]$. For definiteness suppose $\gamma(\varepsilon) \leq \frac{\delta}{\sqrt{\varepsilon}}$. Then we argue as above to obtain the estimate (3.3a) on a subinterval of $\left[\frac{1-\delta}{\sqrt{\varepsilon}}, \frac{1-\frac{1}{2}\delta}{\sqrt{\varepsilon}}\right]$. Then since $\phi' > 0$ for $\xi > \gamma(\varepsilon)$ we see that (3.3a) holds on all of $\left[\frac{\delta}{\sqrt{\varepsilon}}, \frac{1-\delta}{\sqrt{\varepsilon}}\right]$. As before, (3.3b) follows from lemma 2.2.

Lemma 3.3: Let δ, $0 < \delta < \frac{1}{4}$ be given. Let $\varepsilon \leq \varepsilon(\delta)$ and let $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ be a solution of (1.1), (1.2) satisfying (1.6a), (1.6b). Then, for $\delta \leq x \leq 1-\delta$ we have

3.5) $$\left|\tfrac{1}{2}G^2(x,\varepsilon) - \mu(\varepsilon)\right| \leq M(\delta)[\varepsilon^{1/128} + B\varepsilon^{1/64} + \tfrac{1}{2}M(\delta)\varepsilon^{1/64}] .$$

Proof: We make the change of variables (1.7), using (3.1) and (3.3a), (3.3b) we obtain (3.5).

Note: While the analysis given in this paper is primarily concerned with "limit" behavior and families (i.e. sequences) of solutions $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ which satisfy H.1 - (1.6a), (1.6b) the estimate (3.5) provides a "check" which may be applied to any calculated pair $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$. We simply must carry out some messy computation. That is, (i) find a B for (1.6a), (1.6b); (ii) carefully follow the steps of section 2 and compute $M(\delta)$; (iii) check (3.5).

Theorem 3.1: Let $\varepsilon_n \downarrow 0+$ and let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a corresponding sequence of solutions of (1.1), (1.2) which satisfies H.1, i.e. (1.6a), (1.6b). Suppose

3.6) $$\mu(\varepsilon_n) \to \bar{\mu} \quad \text{as} \quad \varepsilon_n \to 0+ .$$

(Note: In view of lemma 3.1 one can always extract a subsequence so that (3.6) holds.)

Then

3.7a) $$\bar{\mu} \geq 0$$

and; for every δ, $0 < \delta < \frac{1}{10}$

3.7b) $$\text{Max}\{|G^2(x,\varepsilon_n) - 2\bar{\mu}|;\ \ \delta \leq x \leq 1-\delta\} \to 0 \quad \text{as} \quad \varepsilon_n \to 0+ .$$

Moreover, if $\bar{\mu} = 0$ then

3.8) $$\text{Max}\{|G(x,\varepsilon_n)|;\ \ \delta \leq x \leq 1-\delta\} \to 0 \quad \text{as} \quad \varepsilon_n \to 0+ .$$

If $\bar{\mu} > 0$ then there is a subsequence $n_k \to \infty$ and a square root, say A, of $2\bar{\mu}$ such that

3.9) $$\text{Max}\{|G(x,\varepsilon_{n_k}) - A|;\ \ \delta \leq x \leq 1-\delta\} \to 0 \quad \text{as} \quad \varepsilon_n \to 0+ .$$

In fact, if

3.10a) $$|\mu(\varepsilon_n) - \bar{\mu}| + M(\delta)[\varepsilon_n^{1/128} + B\varepsilon_n^{1/64} + \frac{1}{2}M(\delta)\varepsilon_n^{1/64}] \leq \sigma \leq \frac{1}{10}\bar{\mu}$$

then

3.10b) $$\frac{1}{2}G^2(x,\varepsilon_n) \geq \frac{9}{10}\bar{\mu}, \qquad \delta \leq x \leq 1-\delta .$$

Thus $G(x,\varepsilon_n)$ is of one sign and

3.11) $$|G(x,\varepsilon_n) - \sqrt{2\bar{\mu}}\ \text{sgn}\ G(x,\varepsilon_n)| \leq \frac{\sigma}{\sqrt{2\bar{\mu}}} .$$

Proof: The estimates (3.7b) follows immediately from (3.5). The inequality (3.7a) follows from (3.7b). Then (3.8) is apparent. When $\bar{\mu} > 0$ and (3.10a) holds we have from (3.5) and the triangle inequality

$$\left|\frac{1}{2}G^2(x,\varepsilon_n) - \bar{\mu}\right| \leq \sigma \leq \frac{1}{10}\bar{\mu} .$$

Then, (3.10b) follows at once. We then have

$$|G(x,\varepsilon_n) - \sqrt{2\bar{\mu}}| \cdot |G(x,\varepsilon_n) + \sqrt{2\bar{\mu}}| \leq \sigma$$

and (3.11) follows at once. Thus, choosing "signs" at a fixed point, say $x_0 = \frac{1}{2}$, we obtain (3.9).

4. The Asymptotic Behavior of $H(x,\varepsilon)$: $\bar{\mu} > 0$

Let δ, $0 < \delta < \frac{1}{10}$ be given. Let $\varepsilon(\delta)$ be the value determined in section 3 and suppose $0 < \varepsilon \leq \varepsilon(\delta)$. Let $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ be a solution of (1.1), (1.2) on $[0,1]$ which also satisfies H.1, i.e., (1.6a), (1.6b). We also suppose that there is a constant $\bar{\mu} > 0$ such that

4.1a) $$\mu(\varepsilon) \geq \bar{\mu}/2 > 0 ,$$

4.1b) $$\frac{1}{2} G^2(x,\varepsilon) \geq \bar{\mu}/4 > 0, \quad \frac{1}{2}\delta \leq x \leq 1 - \frac{1}{2}\delta .$$

The main result is

Theorem 4.1: There are positive values $\bar{\varepsilon} = \varepsilon(\delta,\bar{\mu})$, $K = K(\delta)$, $\sigma = \sigma(\bar{\mu})$ where $\bar{\varepsilon}$ depends only on $\delta, \bar{\mu}$ and B, K depends only on δ and B, while $\sigma(\bar{\mu})$ depends only on $\bar{\mu}$ and B, such that, for $\delta \leq x \leq 1 - \delta$

4.2a) $$\left|\frac{1}{2} G^2(x,\varepsilon) - \mu(\varepsilon)\right| \leq K(\delta)\exp\{-\sigma\varepsilon^{-1/384}\} ,$$

4.2b) $$\left|\left(\frac{d}{dx}\right)^r H(x,\varepsilon)\right| \leq \varepsilon^{\frac{1-r}{2}} K(\delta)\exp\{-\sigma\varepsilon^{-1/384}\}, \quad r = 1,2,3 ,$$

4.2c) $$\left|\frac{d}{dx} G(x,\varepsilon)\right| \leq \varepsilon^{-1/2}K(\delta)\exp\{-\sigma\varepsilon^{-1/384}\} .$$

Finally, there are constants a, b such that

4.3a) $$\left|\frac{1}{\sqrt{\varepsilon}} H(x,\varepsilon) - a\right| \leq \varepsilon^{-1/2}K(\delta)\exp\{-\sigma\varepsilon^{-1/384}\} ,$$

4.3b) $$|G(x,\varepsilon) - b| \leq \varepsilon^{-1/2}K(\delta)\exp\{-\sigma\varepsilon^{-1/384}\} .$$

4.3c) $$|b| = \sqrt{2\mu(\varepsilon)} .$$

Of course this theorem immediately implies certain limit theorems for subsequences of solutions.

The proof is relatively straight-forward and follows the general pattern McLeod's work in [8]. Unfortunately there are many details to check out. We outline our approach.

Step 1: We make the change of variables (1.7) and consider the functions $h(\xi,\varepsilon)$, $g(\xi,\varepsilon)$ on an "interior" interval $[\alpha,\beta]$ which satisfies

4.4a) $$\delta/\sqrt{\varepsilon} \leq \alpha < \beta < (1-\delta)/\sqrt{\varepsilon} ,$$

4.4b) $$\beta - \alpha = L(\varepsilon) = \varepsilon^{-1/384} .$$

Step 2: We find an equivalent integral equation.

Step 3: We prove local uniqueness for solutions of the integral equation.

Step 4: We prove that the desired solution of the integral equation can be obtained via Picard iteration.

Step 5: We see that the limit of the Picard iterates satisfies the appropriate estimates.

Step 6: We return to the original variables x, $H(x,\varepsilon)$, $G(x,\varepsilon)$.

We can imagine step 1 has been done.

Step 2: The integral equation. Let

4.5a)
$$h_0 = h(\alpha,\varepsilon), \qquad g_0 = G(\alpha,\varepsilon) \; ,$$

4.5b)
$$\begin{cases} u_1(\xi) = g(\xi,\varepsilon) - g_0, & u_2(\xi) = h(\xi,\varepsilon) - h_0 \\ u_3(\xi) = g'(\xi,\varepsilon), & u_4(\xi) = h'(\xi,\varepsilon) \; , \\ u_5(\xi) = h''(\xi,\varepsilon), & u_6(\xi) = h'''(\xi,\varepsilon) \; . \end{cases}$$

Let

4.6a)
$$U = (u_1,u_2,u_3,u_4,u_5,u_6)^T \; ,$$

4.6b)
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -h_0 & g_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g_0 & 0 & 0 & -h_0 \end{bmatrix} ,$$

4.6c)
$$b = b(U) = (0,0,u_1u_4 - u_2u_3,0,0, - u_1u_3 - u_2u_6)^T \; .$$

The equations (2.1a), (2.1b) now take the form

4.7)
$$\frac{dU}{d\xi} = AU + b(U) \; .$$

A direct calculation shows that the eigenvalues of A are the roots of

4.8)
$$\lambda^2[g_0^2 + \lambda^2(\lambda + h_0)^2] = 0 \; .$$

Essentially this same eigenvalue problem arises in [8] and we can easily check the following formulae for the eigenvalues λ_k, $k = 1,2,3,4,5,6$. Let

$$\rho_1 = -\frac{1}{2} h_0 + \frac{1}{2\sqrt{2}} \{(h_0^4 + 16g_0^2)^{1/2} + h_0^2\}^{1/2} > 0 ,$$

$$\rho_2 = -\frac{1}{2} h_0 - \frac{1}{2\sqrt{2}} \{(h_0^4 + 16g_0^2)^{1/2} + h_0^2\}^{1/2} < 0 ,$$

$$\tau = \frac{1}{2\sqrt{2}} \{(h_0^4 + 16g_0^2)^{1/2} - h_0^2\}^{1/2} .$$

Then, the eigenvalues of A are

4.9a) $$\lambda_1 = \lambda_2 = 0 ,$$

4.9b) $$\lambda_3 = \rho_1 + i\tau, \quad \lambda_4 = \bar{\lambda}_3 = \rho_1 - i\tau ,$$

4.9c) $$\lambda_5 = \rho_2 + i\tau, \quad \lambda_6 = \bar{\lambda}_5 = \rho_2 - i\tau .$$

It is easy to see that $\rho_1 > 0$ and $\rho_2 < 0$ provided that $g_0 \neq 0$. However, (4.1b) gives

4.10a) $$B^2 \geq g_0^2 \geq \bar{\mu}/2 .$$

And, for all ε_n, we have

4.10b) $$|h_0| \leq B .$$

Thus a simple compactness argument shows that there is a constant $\rho > 0$ such that

4.10c) $$\rho_2 \leq -\rho < 0 < \rho \leq \rho_1 .$$

Let us diagonalize the matrix A. We construct the matrix of eigenvectors. Let

4.11a) $$m(\lambda) = \frac{g_0}{\lambda + h_0} ,$$

and let

4.11b) $$T = \begin{bmatrix} 1 & 0 & m(\lambda_3) & m(\lambda_4) & m(\lambda_5) & m(\lambda_6) \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \lambda_3 m(\lambda_3) & \lambda_4 m(\lambda_4) & \lambda_5 m(\lambda_5) & \lambda_6 m(\lambda_6) \\ 0 & 0 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ 0 & 0 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & \lambda_3^3 & \lambda_4^3 & \lambda_5^3 & \lambda_6^3 \end{bmatrix} .$$

Using the fact that

4.11c) $$m(\lambda_k) = \frac{g_0}{\lambda_k + h_0} = -\frac{\lambda_k^2(\lambda_k + h_0)}{g_0}, \qquad 3 \le k \le 6$$

one can easily verify that the columns of T are eigenvectors of A and

4.11d) $$T^{-1}AT = \text{diagonal}(0,0,\lambda_3,\lambda_4,\lambda_5,\lambda_6) \equiv \Lambda .$$

Moreover

4.12) $$T^{-1} = \begin{bmatrix} 1 & 0 & x & x & x & x \\ 0 & 1 & x & x & x & x \\ 0 & 0 & x & x & x & x \\ 0 & 0 & x & x & x & x \\ 0 & 0 & x & x & x & x \\ 0 & 0 & x & x & x & x \end{bmatrix}$$

where "x" marks an element we do not need to compute. (See [8] where an analogous calculation is carried out.)

Let

4.13a) $$U = TV, \qquad V = T^{-1}U .$$

Then (4.7) takes the form

4.13b) $$\frac{dV}{d\xi} = \Lambda V + d(V) ,$$

where

4.13c) $$d(V) = T^{-1}b(TV) .$$

Thus, we have essentially found our integral equation.

<u>Lemma 4.1:</u> Let $V(\xi)$ be any solution of (4.13b). Then

4.14a) $$v_j(\xi) = v_j(\alpha) + \int_\alpha^\xi d_j(V(t))dt, \qquad j = 1,2 ,$$

4.14b) $$v_j(\xi) = v_j(\beta)e^{\lambda_j(\xi-\beta)} + e^{\lambda_j(\xi-\beta)} \int_\beta^\xi e^{-\lambda_j(t-\beta)} d_j(V(t))dt, \qquad j = 3,4 ,$$

4.14c) $$v_j(\xi) = v_j(\alpha)e^{\lambda_j(\xi-\alpha)} + e^{\lambda_j(\xi-\alpha)} \int_\alpha^\xi e^{-\lambda_j(t-\alpha)} d_j(V(t))dt, \qquad j = 5,6 .$$

<u>Proof:</u> Integrate (4.13b).

Step 3: Local uniqueness.

It is essential that we distinguish between the components $v_j(\xi)$ of $V(\xi)$. Let

4.15a) $$\|V(\xi)\| = \max_j\{|v_j(\xi)|; \quad j = 1,2,3,4,5,6.\}$$

4.15b) $$N(V(\xi)) = \max_j\{|v_j(\xi)|; \quad j = 3,4,5,6.\}$$

4.15c) $$M(V(\xi)) = \max\{|v_1(\xi)|,|v_2(\xi)|\} .$$

Lemma 4.2: There are positive constants a_1,a_2,a_3 such that; if V, Y are each 6-vectors,

4.16) $$\begin{cases} \|d(V) - d(Y)\| \le a_1M(V - Y) \cdot N(V) + \\ \quad a_2M(Y)N(V - Y) + a_3N(V - Y)[N(Y) + N(V)] . \end{cases}$$

These constants a_1,a_2,a_3 are uniformly bounded.

Proof: The coefficients of T are bounded functions of g_0,h_0. A compactness argument shows they are uniformly bounded. The form of T shows that: for $k = 3,4,5,6$ and $j = 1,2$

$$\ddot{u}_j\ddot{u}_k = v_jL_{jk}(v_3,v_4,v_5,v_6) + Q_{jk}(v_3,v_4,v_5,v_6)$$

where L_{jk} is linear and homogeneous while Q_{jk} is quadratic and homogeneous. Thus, (4.16) follows from the form of $b(U)$.

Lemma 4.3: Let $M(\delta)$ be the constant of lemma 3.2. Let K_1 be a uniform bound on $\|T^{-1}\|_\infty$. Let

4.17a) $$K_1M(\delta)\varepsilon^{1/192} = \theta ,$$

4.17b) $$100\ \theta = \bar{\theta} ,$$

and assume that

4.17c) $$\bar{\theta} \le \frac{1}{2} .$$

Assume that

4.18) $$(a_1 + a_2 + 2a_3)L(\varepsilon)\bar{\theta} \le \frac{1}{2} .$$

Let $v_1(\alpha)$, $v_2(\alpha)$, $v_3(\beta)$, $v_4(\beta)$, $v_5(\alpha)$, $v_6(\alpha)$ be specified so that

$$\max\{|v_1(\alpha)|,|v_2(\alpha)|,|v_3(\beta)|,|v_4(\beta)|,|v_5(\alpha)|,|v_6(\alpha)|\} \le \bar{\theta} .$$

Then, there is at most one solution $V(\xi)$ of the integral equation (4.14a), (4.14b), (4.14c) with these boundary values and satisfying

4.19) $$\|V(\xi)\| \le \bar{\theta} .$$

<u>Proof</u>: Suppose $Y(\xi), W(\xi)$ are two such solutions. Let

4.20a) $$D \equiv \max\{\|d(Y(\xi)) - d(W(\xi))\|; \quad \alpha \le \xi \le \beta\} ,$$

4.20b) $$E = \max\{\|Y(\xi) - W(\xi)\|; \quad \alpha \le \xi \le \beta\} .$$

From lemma 4.2 and (4.19) we have

4.21a) $$D \le (a_1 + a_2 + 2a_3)E\bar{\theta} .$$

From the integral equation we see that

4.21b) $$E \le L(\varepsilon)D \le L(\varepsilon)(a_1 + a_2 + 2a_3)E\bar{\theta} .$$

Thus, either $E = 0$ or

$$1 \le L(\varepsilon)(a_1 + a_2 + 2a_3)\bar{\theta}$$

which contradicts (4.18).

<u>Step 4</u>: <u>The Iteration</u>.

Let $v_1(\alpha)$, $v_2(\alpha)$, $v_3(\beta)$, $v_4(\beta)$, $v_5(\alpha)$, $v_6(\alpha)$ be determined from $h(\xi,\varepsilon)$, $g(\xi,\varepsilon)$ via the transformations (4.5b), (4.13a). We seek to recover the appropriate $V(\xi)$ via Picard iterations. That is, let $V^0(\xi) = 0$. Assuming that $V^r(\xi)$ has been computed we determine $V^{r+1}(\xi)$ from the equations

4.22a) $$v_j^{r+1}(\xi) = v_j(\alpha) + \int_\alpha^\xi d_j(V^r(t))dt, \qquad j = 1,2 ,$$

4.22b) $$v_j^{r+1}(\xi) = v_j(\beta)e^{\lambda_j(\xi-\beta)} + e^{\lambda_j(\xi-\beta)} \int_\beta^\xi e^{-\lambda_j(t-\beta)} d_j(V^r(t))dt, \qquad j = 3,4 ,$$

4.22c) $$v_j^{r+1}(\xi) = v_j(\alpha)e^{\lambda_j(\xi-\alpha)} + e^{\lambda_j(\xi-\alpha)} \int_\alpha^\xi e^{-\lambda_j(t-\alpha)} d_j(V^r(t))dt, \qquad j = 5,6 .$$

<u>Lemma 4.4</u>: Let $V^r(\xi)$ be computed as above. Assume that

4.23a) $$4(a_1 + a_2 + 4a_3)\theta^{1/4} \le 1 ,$$

and

4.23b) $$2L(\varepsilon)\theta^{3/4} \le \frac{1}{4} ,$$

then

4.24a) $$M(V^r(\xi) - V^{r-1}(\xi)) \le \frac{\theta}{2^{r-1}} ,$$

4.24b) $$N(V^r(\xi) - V^{r-1}(\xi)) \le \frac{\theta}{2^{r-1}} \left[e^{\rho_1(\xi-\beta)} + e^{\rho_2(\xi-\alpha)}\right] ,$$

4.24c) $$M(V^r(\xi)) \le 2\theta ,$$

4.24d) $$N(V^r(\xi)) \le 2\theta\left[e^{\rho_1(\xi-\beta)} + e^{\rho_2(\xi-\alpha)}\right] .$$

Moreover, the functions $V^{(r)}(\xi)$ converge uniformly to a function $V(\xi)$ which satisfies the integral equation (4.14a), (4.14b), (4.14c) and

4.25a) $$M(V(\xi)) \le 2\theta ,$$

4.25b) $$N(V(\xi)) \le 2\theta\left[e^{\rho_1(\xi-\beta)} + e^{\rho_2(\xi-\alpha)}\right] .$$

<u>Proof:</u> We observe that (3.2) together with the choice of $L(\varepsilon)$ implies that the solution $V(\xi)$ determined by $\langle h(\xi,\varepsilon), g(\xi,\varepsilon)\rangle$ satisfies

4.26) $$\|V(\xi)\| \le \theta ,$$

therefore, a-fortiori, the boundary conditions satisfy the same estimate. Thus, (4.24a), (4.24b) are satisfied for $r = 1$. We proceed by induction. Assume that (4.24a), (4.24b) are satisfied for $r = 1,2,\ldots,r_0$. Then, (4.24c), (4.24d) are also satisfied for $r = 1,2,\ldots,r_0$. Applying lemma 4.2 we have

4.27) $$\begin{cases} \|d(V^{r_0}(\xi)) - d(V^{r_0-1}(\xi))\| \le \dfrac{2\theta^2}{2^{r_0-1}}(a_1 + a_2 + 4a_3)\left[e^{\rho_1(\xi-\beta)} + e^{\rho_2(\xi-\alpha)}\right] \\ \qquad\qquad \le \dfrac{\theta^{7/4}}{2^{r_0}}\left[e^{\rho_1(\xi-\beta)} + e^{\rho_2(\xi-\alpha)}\right] . \end{cases}$$

Substitution into (4.22a) gives

$$\left[v_j^{r_0+1}(\xi) - v_j^{r_0}(\xi)\right] \le \frac{2L(\varepsilon)\theta^{7/4}}{2^{r_0}}, \qquad j = 1,2 .$$

However, using (4.23b) we have (4.24a) with $r = r_0 + 1$. Substitution of (4.27) into (4.22b) gives

$$\left|v_j^{r_0+1}(\xi) - v_j^{r_0}(\xi)\right| \le \frac{\theta^{7/4}}{2^{r_0}} e^{\rho_1(\xi-\beta)} \int_\xi^\beta e^{\rho_1(\xi-t)} dt$$

$$+ \frac{\theta^{7/4}}{2^{r_0}} e^{\rho_2(\xi-\alpha)} \int_\xi^\beta e^{\rho_1(\xi-t)} dt, \qquad j = 3,4 .$$

Since $\xi \le t \le \beta$ over the interval of integration, we have

$$\left|v_j^{r_0+1}(\xi) - v_j^{r_0}(\xi)\right| \le \frac{L(\varepsilon)\theta^{7/4}}{2^{r_0}} \left[e^{\rho_1(\xi-\beta)} + e^{\rho_2(\xi-\alpha)}\right], \qquad j = 3,4 .$$

A similar computation applies for $j = 5,6$. Thus, using (4.23b) we obtain (4.24b) for $r = r_0$. Thus, the lemma is proven.

Proof of Theorem 4.1: Let δ be replaced by $\frac{1}{2}\delta$. That is, replace $\varepsilon(\delta)$ by $\varepsilon(\frac{1}{2}\delta)$ and replace $M(\delta)$ by $M(\frac{1}{2}\delta)$ and consider $[\alpha,\beta]$ which satisfy

4.28a) $$\frac{1}{2}\frac{\delta}{\sqrt{\varepsilon}} \le \alpha < \beta \le (1 - \frac{1}{2}\delta)/\sqrt{\varepsilon} .$$

Let $\bar{\varepsilon}(\delta)$ be the largest ε so that (4.17c), (4.18), (4.23a), (4.23b) are satisfied, and

4.28b) $$\varepsilon^{1/4} \le \frac{\delta}{2} .$$

Then if $0 < \varepsilon \le \bar{\varepsilon}(\delta)$, every point $\xi \in \left(\frac{\delta}{\sqrt{\varepsilon}}, \frac{(1-\delta)}{\sqrt{\varepsilon}}\right)$ can be placed at the center of an interval $[\alpha,\beta]$ which satisfies (4.28a) and (4.4b).

On this interval we construct the function $V(\xi)$ of lemma 4.4. However, the local uniqueness result of lemma 4.3 assures us that this $V(\xi)$ is precisely the function $V(\xi)$ obtained from $\langle h(\xi,\varepsilon), g(\xi,\varepsilon)\rangle$ via the transformations (4.5a), (4.5b), (4.13a). Let K_2 be a bound on $\|T\|$ as h_0, g_0 range over the values allowed by (4.10a), (4.10b). Then, due to the form of T,

4.29) $$|u_j(\xi)| \le 4K_2\theta\exp\{-\frac{\rho}{2}\varepsilon^{-1/384}\}, \qquad j = 3,4,5,6.$$

where ρ is the constant of (4.10c). Let

4.30a) $$K(\delta) = 4K_2K_1M(\frac{1}{2}\delta) \le 4K_2\theta ,$$

4.30b) $$\sigma = \frac{\rho}{2} .$$

Then (4.2b), (4.2c) follow from (4.29) and (3.1). The estimates (4.2a), (4.3a), (4.3b), (4.3c) follow from these.

The next theorem is an immediate consequence of theorem 4.1.

Theorem 4.2: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2) which satiafy H.1. Suppose these are constants $\bar{\mu}$, g_∞ with $g_\infty^2 = 2\bar{\mu}$ such that

4.31) $$\mu(\varepsilon_n) \to \bar{\mu} ,$$

and, for every δ, $0 < \delta < \frac{1}{10}$ we have

4.32) $$\max\{|G(x,\varepsilon_n) - g_\infty|;\ \delta \le x \le 1-\delta\} \to 0 \quad \text{as} \quad \varepsilon_n \to 0+ .$$

Then there is a subsequence $\varepsilon_{n_k} \to 0+$ and a constant h_0 so that

4.33) $$\max\{|\frac{1}{\sqrt{\varepsilon_{n_k}}} H(x,\varepsilon_{n_k}) - h_0|;\ \delta \le x \le 1-\delta\} \to 0 \quad \text{as} \quad \varepsilon_{n_k} \to 0+ .$$

Moreover, if $\varepsilon_{n_k} \to 0+$ and there is a function $h(x)$ such that

4.34a) $$\max\{|\frac{1}{\sqrt{\varepsilon_{n_k}}} H(x,\varepsilon_{n_k}) - h(x)|;\ \delta \le x \le 1-\delta\} \to 0 \quad \text{as} \quad \varepsilon_{n_k} \to 0 ,$$

then

4.34b) $$h(x) \equiv \text{const} .$$

Appendix

In this appendix we prove the following very plausible result: if $\phi(\xi)$ is a smooth function defined on a very large interval and if $\phi(\xi)$ is both positive and monotone, then there are relatively large intervals on which $\phi'(\xi)$ is small. Unfortunately the complete proof is technically complicated.

Theorem A: Let $\phi(\xi)$ satisfy

A.1) $$0 \leq \phi(\xi), \qquad 0 \leq \xi \leq L ,$$

A.2) $$\phi'(\xi) \geq 0, \qquad 0 \leq \xi \leq L .$$

Let

A.3) $$K_0 = \max\{1, \|\phi\|_\infty, \|\phi'\|_\infty, \|\phi''\|_\infty\}$$

and suppose that

A.4) $$16K_0^2 \leq L .$$

Then, there is a subinterval $[\alpha', \beta'] \subset [0,L]$ such that

A.5) $$\beta' - \alpha' \geq \frac{1}{16K_0^2 + 1} L^{1/2} ,$$

and

A.6) $$0 \leq \phi'(\xi) \leq \left(\frac{1}{L}\right)^{1/4} .$$

We require a basic estimate based on the mean value theorem.

Lemma A.1: Let $f \in C^2[0,L]$. Suppose

A.7) $$\left\|\frac{d^2}{d\xi^2} f\right\|_\infty \leq M ,$$

and

A.8) $$\left|\frac{df}{d\xi}(\xi_0)\right| \geq A > 0 .$$

Let

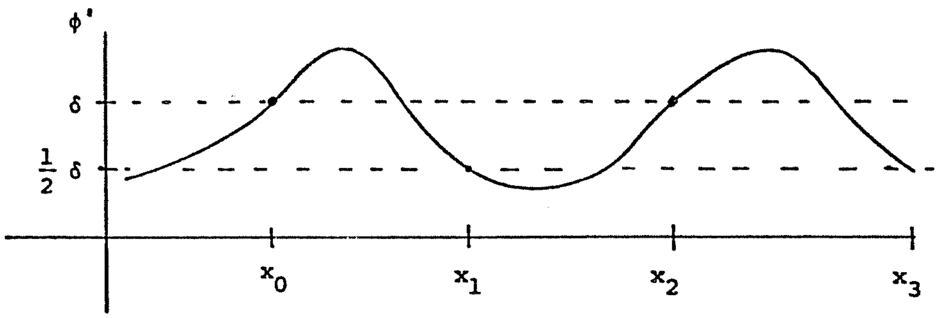
A.9) $$b = \min\{\frac{A}{2M}, \xi_0, L - \xi_0\} .$$

Then

A.10) $$\left|\frac{df}{d\xi}(\xi)\right| \geq \frac{A}{2}, \qquad \xi_0 - b \leq \xi \leq \xi_0 + b .$$

Proof of Theorem A: Let $\left(\frac{1}{L}\right)^{1/4} = \delta \geq \frac{8K_0}{L}$. We construct a sequence of points $0 \leq x_0 < x_1 < \cdots < \leq L$ with the following properties

$$\begin{array}{lll} \phi'(x_0) \geq \delta, & & \\ \phi'(x_1) = \frac{1}{2}\delta_1, & \phi'(x) > \frac{\delta}{2}, & \text{for } x_0 \leq x < x_1, \\ \phi'(x_2) = \delta, & \phi'(x) < \delta, & \text{for } x_1 \leq x < x_2, \\ \phi'(x_{2j+1}) = \frac{1}{2}\delta, & \phi'(x) > \frac{\delta}{2}, & \text{for } x_{2j} \leq x < x_{2j+1}, \\ \phi'(x_{2j+2}) = \delta, & \phi'(x) < \delta, & \text{for } x_{2j+1} \leq x \leq x_{2j+2}. \end{array}$$



To accomplish this we proceed as follows. If $\phi'(0) \geq \delta$ then $x_0 = 0$, if not x_0 is the first point at which $\phi(x_0) = \delta$. Let x_1 be the first point larger than x_0 such that $\phi(x_1) = \frac{1}{2}\delta$ and so on. If $x_0 \geq \frac{1}{4}L$ then the theorem is true. Assume $x_0 < \frac{1}{4}L$. By Lemma 1 the number of intervals is finite. Let N be the last index. Then $x_N \leq L$. If N even, then $\phi'(x) \geq \frac{\delta}{2}$ for $x_N \leq x \leq L$. Thus

$$K_0 \geq \phi(x_L) \geq \frac{\delta}{2}(L - x_N).$$

That is

$$(L - x_N) \leq \frac{2}{\delta}K_0 \leq \frac{L}{4}.$$

If N is odd, then $\phi'(x) < \delta$ for $x_N \leq x \leq L$. Thus, we can assume $|L - x_N| \leq \frac{L}{4}$. Therefore

$$x_N - x_0 \geq \frac{1}{2}L.$$

Let R be the number of interval (x_{2j}, x_{2j+1}) - on which $\phi'(x) > \frac{\delta}{2}$ -. We first seek a bound on R. By Lemma A.1

$$|x_{2j+1} - x_{2j}| \geq \frac{\delta}{2K_0}.$$

Thus

$$R\,\frac{\delta}{2}\,\frac{\delta}{2K_0} \le \sum_j \int_{x_{2j}}^{x_{2j+1}} \phi' dt \le K_0$$

and

A.11) $$R \le 4K_0^2/\delta^2 \ .$$

Similarly, the total length L' of these intervals satisfies

$$L'\,\frac{\delta}{2} \le \sum_j \int_{x_{2j}}^{x_{2j+1}} \phi' dt \le K_0$$

and

A.12) $$L' \le 2K_0/\delta \le \frac{L}{4} \ .$$

The number of intervals (x_{2j-1}, x_{2j}) - on which $\phi'(x) < \delta$ - is $(R \pm 1)$ and their total length L'' satisfies

A.13) $$L'' \ge \frac{L}{2} - L' \ge \frac{L}{4} \ .$$

Thus

$$\max_j (x_{2j} - x_{2j-1}) \ge \frac{L}{4(R+1)} \ge \frac{L\delta^2}{4(4K_0^2 + \delta^2)}$$

which proves the theorem.

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