

USING THE METHOD OF ORTHOGONAL
COLLOCATION FOR CERTAIN THREE-DIMENSIONAL
PROBLEMS OF STELLAR STRUCTURE?

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ABSTRACT

The method is developed for two specific problems: (i) computation of the structure of the primary component (assumed to consist of a polytropic gas) in a synchronous close binary system and (ii) search for non-axisymmetric configurations of differentially rotating polytropes. In both cases the structure equations reduce to a mildly non-linear elliptic partial differential equation in three dimensions with boundary conditions at the center, on a sphere containing the star and involving a 'free' boundary. The present method has several advantages over the 'standard' methods (namely, improvements of Chandrasekhar's perturbation analysis). The most important of these are consistency and easier application to real stars. However, the method becomes computationally inefficient when used for computing of configurations with strong angular dependence. In such cases (related) Galerkin methods offer significant advantages.

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[†]On leave from Department of Applied Mathematics, University of Cape Town.

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1. Introduction

Several successful attempts at numerical solution of stellar structure problems have been made in the past dozen years (see Strittmatter, 1969 and Papaloizou and Whelan, 1973). Perhaps the most elegant of these methods has been the one devised by Stoeckly (1965). Most of these methods suffer from serious limitations on the range of their applicability, or undue complexity and/or unproven convergence properties. Even so, none of them has ever been applied to three-dimensional problems.

In this investigation, Stoeckly's formulation of the structure problem for rotating stars is altered to include most of the other stellar structure problems (section 2). Then, his numerical method, which is in fact a special form of the method of orthogonal collocation (used extensively in theoretical chemistry; Finlayson, 1972), is improved and generalized to three dimensions (sections 3 and 4). The method is developed for polytropic models of stars and in its present form it is applicable only to polytropes with the polytropic index $n \geq 1$. However, it would be fairly straightforward to modify the method in such a way that polytropes with $n < 1$ and, also, matter more complicated than polytropes (Miketinac, 1976) could be treated. Convergence of the method is virtually assured through the work of Vainikko on perturbed Galerkin methods (1972, 1967). The method produces results (section 5) in satisfactory agreement with the known results for uniformly rotating configurations of polytropes (which is actually a two-dimensional problem). However, the method has an undesirable feature making its use on the computer much more expensive than the 'perturbed' Galerkin method. This last method is, therefore, the recommended method for computations of three-dimensional stellar structure models. It is shown in section 3 that the method of orthogonal collocation is in fact an equivalent - but computationally less efficient (shown in section 4) - form of the 'perturbed' Galerkin method. It is likely that the 'perturbed' Galerkin method as formulated in sections 3 and 4 will have applications, also, outside the stellar structure problems.

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2. Structure Equations for a Polytropic Model

It can be shown that equations describing the configuration of a self-gravitating, polytropic fluid modelling a star in equilibrium under the influence of some disturbing force reduce to

$$\nabla^2 V = -\theta^n, \quad (2.1)$$

$$\theta = V - V_c + 1 + V_D. \quad (2.2)$$

All symbols in these equations - called the structure equations - represent dimensionless quantities.

The equations are formulated in the spherical polar coordinate system (x, ϑ, φ) with the center coinciding with the center of symmetry (usually the center of mass) and with the axes oriented along the lines of symmetry of the star. In this coordinate system the Laplace operator, ∇^2 , is

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{2}{x} \frac{\partial}{\partial x} + \frac{1}{x^2} \left[\frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2}{\partial \varphi^2} \right] \quad (2.3)$$

where $\mu = \cos \vartheta$. Symmetries of the star model depend on the given function

$$V_D = V_D(\lambda, x, \mu, \varphi) \quad (2.4)$$

which is the potential describing the disturbing force. It is assumed that the function V_D vanishes at $x = 0$, and that the positive, arbitrary parameter λ - characterizing the strength of disturbance - is such that for $\lambda = 0$, $V_D \equiv 0$ (in this case the star is undisturbed and assumes a spherical shape). The function $-V(x, \mu, \varphi)$ is essentially the gravitational potential of the model and $V_c \equiv V(x=0, \mu, \varphi)$. The parameter n - called the polytropic index - is arbitrary in the range $0 \leq n \leq 5$ (it specifies the chemical composition of the model). The equation (2.2) - known as the equation of hydrostatic support - is valid only inside (defined by $\theta > 0$) the star; the star's surface is defined as that surface on which $\theta(x, \mu, \varphi) = 0$. Since θ is not known in advance, this gives the structure problem a 'free boundary' aspect. The function θ , where positive, is related to the density and is such that

$$\theta(x=0, \mu, \varphi) = 1. \quad (2.5)$$

For convenience, (2.2) will be taken to define θ throughout, but then in equation (2.1) θ can only be positive, i.e. (2.1) is replaced by

$$\nabla^2 V = -\theta_+^n, \quad (2.1')$$

where

$$\theta_+ = \begin{cases} \theta & \text{when } \theta > 0 \\ 0 & \text{when } \theta < 0 \end{cases} \quad (2.6)$$

From this it follows that $(k+2)$ -th derivative of the potential V with respect to x, μ , or φ exists at the star's surface only if $k \leq n$, otherwise $(k+2)$ -th derivative is singular there.

The structure equations must be solved under the following conditions: (i) at the center, $x = 0$, ∇V is continuous, (ii) at the surface of the star ∇V is continuous and (iii) at the outside $\nabla V \rightarrow 0$ as $x \rightarrow \infty$. A solution V_c, V, θ of the structure equations (2.1') and (2.2) and the above boundary conditions is a possible configuration of the model and is characterized by a specific value of (n, λ) . All quantities of astrophysical interest can be computed from this solution. The most interesting values of n are $n = 1.5$ (neutron stars, Cowling model) and $n = 3.0$ (supermassive stars).

Examples of astrophysical systems whose structure can be modelled with equations (2.1'), (2.2) and the associated boundary conditions include (i) (uniformly or differentially) rotating stars, (ii) magnetic stars (with or without rotation), and (iii) binary stars. In the case of uniform rotation, the function V_D is given by

$$V_D = \frac{1}{2} \Omega^2 x^2 (1 - \mu^2) \quad (2.7)$$

where the parameter λ is now the (dimensionless) angular velocity Ω . Differential (or non-uniform) rotation can be described by (Stoekly, 1965)

$$V_D = \frac{\Omega^2}{4b} [1 - e^{-2b x^2 (1 - \mu^2)}], \quad (2.8)$$

where the constant b is a parameter of non-uniformity of rotation, ranging from $b = 0$, for uniform rotation, to $b \approx 1$, which approximates spatial dependence of the centrifugal potential possibly arising during contraction from a uniformly rotating mass of initially homogeneous density. Auchmuty and Beals (1971a, 1971b) consider the question of existence and regularity of solutions for some models of rotating stars and show, using the variational method, that physically reasonable solutions exist in most cases. However, their conditions are so stringent that the (physically interesting) polytrope with $n = 3.0$ is excluded and the uniform rotation is inadmissible. Auchmuty (1974, 1975) formulates a numerical scheme based on the variational method for obtaining solutions of the structure problem for rotating stars. He shows that the method converges, but results of his computations and comparisons with other methods have yet to appear.

For the primary in a synchronous binary system there are two versions of the disturbing potential. In the so called first-order theory (Chandrasekhar, 1933)

$$V_D = \frac{mq}{4\pi X} \sum_{j=2}^4 \left(\frac{x}{X}\right)^j P_j(\sin \vartheta \cos \varphi) + \frac{m(1+q)}{4\pi X} \frac{1}{2} \left(\frac{x}{X}\right)^2 \sin^2 \vartheta, \quad (2.9)$$

where X is the separation between the centers of mass, m is the mass of the undisturbed primary (i.e. the mass when $\frac{1}{X} = 0$) and $q = \frac{\bar{m}}{m}$, \bar{m} being the mass of the secondary. In the second-order theory (Martin, 1970)

$$V_D = \frac{mq}{4\pi X} \sum_{j=2}^6 \left(\frac{x}{X}\right)^j P_j(\sin \vartheta \cos \varphi) + \frac{m(1+q)}{4\pi X} \frac{1}{2} \left(\frac{x}{X}\right)^2 \sin^2 \vartheta - \frac{3}{2} \frac{q(b+c-2a)}{4\pi X^3} \frac{x}{X} P_1(\sin \vartheta \cos \varphi), \quad (2.10)$$

where the new symbols a, c, b are the moments of inertia of the primary about the axis pointing to the secondary, the rotation axis and the third axis, respectively. All physical quantities in (2.9) and (2.10) are dimensionless, and, m and $(b+c-2a)$ in (2.10) are obtained from the first-order theory. Expressions (2.9) and (2.10) are approximate models of the real system consisting of two deformable, extended bodies interacting gravitationally while each body rotates uniformly at the same rate as it revolves; the axes of rotation and revolution being all parallel. The first order expression (2.9) includes all effects of the secondary on structure of the primary up to the order of magnitude $O(v^5)$, where $v = \frac{x_0}{X}$ with x_0 being the radius of the undistorted primary. The second-order expression (2.10) includes effects of magnitude up to $O(v^7)$. To these orders of magnitude structure of the primary is independent of the details of structure of the secondary.

Both, Chandrasekhar and Martin, combine (2.1) and (2.2) into an equation for θ , and then seek a solution in the form of a perturbation expansion

$$\theta(x, \mu, \varphi) = \theta_0(x) + \sum_{k=3}^J v^k \theta_k(x, \mu, \varphi)$$

where the cutoff J is equal to 5 in the first-order theory, 7 in the second-order theory. This expansion can not be uniformly valid throughout the primary, because $\theta_0(x) = 0$ at $x = x_0$ (the so called Emden radius) inside the primary. Martin (1970) argues, however, that the expansion with $J = 7$ is valid around $x = x_0$ if the polytropic index $n \geq \frac{8}{3}$, so that his results for the polytrope $n = 3.0$ should be quite accurate. But the same argument shows that even the $J = 5$ expansion is

valid around $x = x_0$ only if $n \geq \frac{6}{3} = 2$ leaving out the other interesting polytrope (for which $n=1.5$). This inability of the (regular) perturbation analysis (directly traceable to the non-uniformity of the above expansion) to cope with both $n = 1.5$ and $n = 3.0$ polytropes suggests that solution of the structure problem should be sought either numerically or using techniques of the singular perturbation analysis. The first alternative provides the major motivation for examining the method of orthogonal collocation in this paper. For the second alternative in the case of uniform rotation see Smith (1975, 1976).

In all cases, (2.7) to (2.10) the following operations

$$\begin{aligned}\mu &\rightarrow -\mu \\ \varphi &\rightarrow 2\pi - \varphi\end{aligned}\tag{2.11}$$

do not change the system as a consequence of symmetries built in the expressions for the disturbing potential. This means that a (numerical) solution of the structure problem need to be found only in one half of the space above the equatorial plane. A further restriction of the domain, in which a solution is sought, can be obtained by a reformulation of the outside boundary condition.

For every physically interesting solution (such solutions are assumed to exist) of the structure problem, there is a sphere of some finite radius completely containing the region where the function θ is positive. Let x_H be the greatest of these radii for a certain sequence of configurations. Then outside this sphere the potential satisfies the Laplace's equation and it is expansible in the form

$$V(x, \mu, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} V_{\ell m}^*(x) Y_{\ell m}^{(+)}(\mu, \varphi)\tag{2.12}$$

where

$$Y_{\ell m}^{(+)}(\mu, \varphi) = \sqrt{\frac{2\ell+1}{2\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^m(\mu) \cos m \varphi$$

when $m \neq 0$ and for $m = 0$

$$Y_{\ell 0}^{(+)}(\mu, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\mu)$$

and the summation in (2.12) is such that $m+\ell$ is always even. This restriction and the fact that only cosines of the angle φ appear are due to symmetries (2.11). The coefficients satisfy the following differential equation

$$\left[\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{\ell(\ell+1)}{x^2} \right] V_{\ell m}^*(x) = 0.$$

There is only one solution which remains finite as $x \rightarrow \infty$; this solution is

$$V_{\ell m}^*(x) \propto x^{-(\ell+1)}$$

and on the sphere of radius, x_H , completely outside the star it can be expressed as

$$\frac{d V_{\ell m}^*(x_H)}{dx} + \frac{\ell+1}{x_H} V_{\ell m}^*(x_H) = 0. \quad (2.13)$$

For fixed n the solution of system of equations (2.1'), (2.2), and (2.13) depends only on the parameter λ . This sequence of configurations 'starts' with $\lambda = 0$, the spherically symmetric configuration, and 'ends' with the critical configuration, when $\lambda = \lambda_c$. For $n \geq 1$ (which includes the two interesting cases) the critical configuration for uniformly rotating stars (James, 1964) and synchronous binary systems (Martin, 1970) is characterized by the fact that the sum of all forces at the equator just balances to zero, i.e.

$$\frac{\partial \theta}{\partial x}(x_e, 0, 0) = 0. \quad (2.14)$$

Locating the critical configuration is the main purpose of this investigation. It will be done by checking the left-hand side of (2.14) for every computed configuration of the sequence characterized by fixed n and adjusting the value of λ in such a way as to make it as close to zero as possible but keeping it negative.

In the case of non-uniform rotation, when the disturbing potential is given by (2.8), the sequence of axisymmetric equilibrium configurations for polytropes with $n \geq 1$ does not necessarily end with a configuration in which the effective gravity is zero at the equator (i.e. when the condition (2.14) is satisfied). Stoeckly (1965) provides some evidence that for $n = 1.5$ and b close to 1 a point of bifurcation is reached and the sequence continues with non-axisymmetric configurations. James (1964) has shown numerically that such bifurcation points exist for uniformly rotating polytropes with the polytropic index $n \leq 0.808$. The Maclaurin and Jacobi ellipsoids (which can be thought of as uniformly rotating polytropes with $n = 0$) are a famous example of bifurcation. A numerical search for bifurcation points and continuation of the sequence of solutions along the new branches could be mounted using the ideas of H. B. Keller (1976). This provides the second major motivation for looking

at the method of orthogonal collocation as a means of computing three-dimensional stellar configurations.

3. The Method of Orthogonal Collocation as a 'Perturbed' Galerkin Method

The sequence of solutions of the system of non-linear equations (2.1'), (2.2) and (2.13), characterized by fixed value of n and variable value of λ , can be generated from the spherically symmetric solution, $\lambda = 0$, (which can always be computed as an initial-value problem for a single ordinary differential equation of second order, Miketinac, 1976) by using a combination of the Newton's linearization and the trial free-boundary method (Cryer, 1976). This is possible because a 'guess' for the solution V_c, V provides, because of (2.2), also, a 'guess' for the free boundary. The procedure consists of two steps

- (i) approximating an unknown solution λ, V_c, V by $\lambda, \bar{V}_c, \bar{V}$ where \bar{V}_c, \bar{V} is a known solution with $\bar{\lambda} = \lambda - \delta\lambda$ and $\delta\lambda$ is 'small',
- (ii) improving this approximation by $\lambda, V_c + \delta V_c, V + \delta V$ where

$$\nabla^2 \delta V + \delta \theta_+^n = -\nabla^2 \bar{V} - \bar{\theta}_+^n.$$

The second step is iterated with \bar{V}_c, \bar{V} replaced by $\bar{V}_c + \delta V_c, \bar{V} + \delta V$ until certain convergences criteria are satisfied. In the second step

$$\bar{\theta} = \bar{V} + 1 - \bar{V}_c + V_D(\lambda) \quad (3.1)$$

(which defines and 'moves' the boundary), and

$$\delta \theta_+^n = n \bar{\theta}_+^{n-1} (\delta V - \delta V_c)$$

so that 'corrections' $\delta V_c, \delta V$ satisfy the linear equation

$$\nabla^2 \delta V + n \bar{\theta}_+^{n-1} (\delta V - \delta V_c) = -\nabla^2 \bar{V} - \bar{\theta}_+^n \quad (3.2)$$

and corresponding (linearized) boundary conditions on fixed boundaries.

Equation (3.2) is of the general form

$$Lu + Qu = F \quad (3.3)$$

where the solution u is a function of x and the polar angles of a point on the unit sphere,

$u = u(x, P)$, the operator L consists of two parts

$$L = L_x + \frac{1}{x^2} L_p$$

with $L_x \equiv \frac{\partial^2}{\partial x^2} + \frac{2}{x} \frac{\partial}{\partial x}$ and $Q = Q(x, P)$, $F = F(x, P)$ are known functions. Since the outside boundary conditions take the form (2.13), a numerical solution of the problem (3.3) must be sought as a truncated expansion (2.12), i.e.

$$u^N(x, P) = \sum_{k=0}^N C_k^N(x) \Phi_k(P), \quad (3.4)$$

where N is a suitable cutoff and $\Phi_k(P)$ stands (apart from a factor) for a (general) spherical harmonic $Y_{\ell m}^{(+)}(\mu, \varphi)$ with $\ell+m$ being an even number; it is assumed that

$$L_P \Phi_k(P) = \lambda_k \Phi_k(P) \quad (3.5)$$

where λ_k specializes to $-\ell(\ell+1)$ in the case of eq. (3.2). To formulate 'perturbed' Galerkin methods for obtaining coefficients $C_k^N(x)$ in (3.4), it will be assumed that the sequence of functions $\{\Phi_k(P)\}_{k=0}^N$ build an orthonormal system with the 'discrete' inner product

$$\langle \Phi_k, \Phi_s \rangle \equiv \sum_{\ell=1}^{M(N)} W_\ell \Phi_k(P_\ell) \Phi_s(P_\ell) = \delta_{ks}, \quad (3.6)$$

where $M(N)$ points P_ℓ of the unit sphere and weights W_ℓ are suitably chosen.

Existence of an inner product of the form (3.6) for a sequence of spherical harmonics, $\{Y_{\ell m}^{(+)}(\mu, \varphi), \ell+m \text{ an even number}\}_{\ell=0}^J, m=0$, can be established in the following way. It is known (see, for example, Fox and Parker, 1968) that the cosines $\{\cos m \varphi\}_{m=0}^J$ build an orthogonal system, because

$$\sum_{j=0}^J w_j \cos m \varphi_j \cos m' \varphi_j = \begin{cases} 0 & m \neq m' \\ \frac{1}{2}J & 0 < m=m' < J \\ J & m = m' = 0, J \end{cases}$$

with $\varphi_j = \frac{\pi}{J}j$ and $w_j = 1$ for $1 \leq j \leq J-1$, $w_0 = \frac{1}{2} = w_J$. For another possible choice of $(w_j, \varphi_j)_{j=0}^J$ see Dahlquist and Björck (1974). Therefore,

$$\sum_{j=0}^J w_j Y_{\ell m}^{(+)}(\mu, \varphi_j) Y_{\ell' m'}^{(+)}(\mu, \varphi_j) = \delta_{mm'} Z_m N_\ell^m N_{\ell'}^m P_\ell^m(\mu) P_{\ell'}^m(\mu),$$

where

$$Z_m = \begin{cases} \frac{1}{2}J & 0 < m < J \\ J & m = 0, J \end{cases},$$

$$N_{\ell}^m = \begin{cases} \sqrt{\frac{2\ell+1}{2\pi}} \frac{(\ell-m)!}{(\ell+m)!} & m \neq 0 \\ \sqrt{\frac{2\ell+1}{4\pi}} & m = 0 \end{cases}$$

Since the product $P_{\ell}^m(\mu) P_{\ell'}^m(\mu)$ is an even polynomial in μ of degree $\ell+\ell'$ which is even and at the most equal to $2J$, there is an exact Gaussian quadrature such that

$$\sum_{i=1}^{M_J} \bar{w}_i P_{\ell}^m(\mu_i) P_{\ell'}^m(\mu_i) = \int_{-1}^1 d\mu P_{\ell}^m(\mu) P_{\ell'}^m(\mu) = \delta_{\ell\ell'} \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$$

(M_J is the minimal number of points for which the above relationship is true). This shows that

$$\sum_{i=1}^{M_J} \sum_{j=0}^J \bar{w}_i w_j Y_{\ell m}^{(+)}(\mu_i, \varphi_j) Y_{\ell' m'}^{(+)}(\mu_i, \varphi_j) = \delta_{\ell\ell'} \delta_{mm'} / A_m,$$

where

$$A_m = \begin{cases} \frac{2\pi}{J} & m \neq J \\ \frac{\pi}{J} & m = J \end{cases}$$

indicating that the symbol $\Phi_k(P)$ actually stands for $\sqrt{A_m} Y_{\ell m}^{(+)}(\mu, \varphi)$. The number of discrete points $(\mu_i, \varphi_j)_{i=1, j=0}^{M_J, J}$ always exceeds the number of functions in the sequence $\{Y_{\ell m}^{(+)}(\mu, \varphi), m+\ell \text{ an even integer}\}_{\ell=0, m=0}^J, \ell$ for a given cutoff J ; see Table 1.

Coefficients of the 'perturbed' Galerkin approximation of the form (3.4) to the solution of (3.3)

are obtained by substituting (3.4) into (3.3) and taking 'discrete' inner products of the resulting

equation and $\Phi_k(P)$ for $k = 0, 1, \dots, N$. This gives the following system of $N+1$ equations for the

coefficients $\{C_k^N(x)\}_{k=0}^N$

$$(L_x + \frac{\lambda_k}{x^2}) C_k^N(x) + \sum_{s=0}^N Q_{ks}^N(x) C_s^N(x) = F_k^N(x), \quad (3.7)$$

where

$$F_k^N(x) = \sum_{\ell=1}^{M(N)} W_{\ell} F(x, P_{\ell}) \Phi_k(P_{\ell}) \quad (3.8)$$

and similarly,

$$Q_{ks}^N(x) = \sum_{\ell=1}^{M(N)} W_{\ell} Q(x, P_{\ell}) \Phi_s(P_{\ell}) \Phi_k(P_{\ell}). \quad (3.9)$$

For the exact (i.e. 'unperturbed') Galerkin method the matrix Q^N and the vector F^N are defined as

corresponding integrals over the sphere and may differ in value from the expressions given by (3.9)

and (3.8). The difference can be displayed by expanding the functions F and Q

J	N_J	$M(J)$
0	1	1
1	2	2
2	4	6
3	6	8
4	9	15
5	12	18
6	16	28
7	20	32
8	25	45
9	30	50
10	36	66
14	64	120
20	121	231

Table 1. N_J is the total number of coefficients in the expansion (2.12) with $\ell \leq J$. $M(J)$ is the minimal number of quadrature points in the 'discrete' inner product; $M(J) = M_J(J+1)$.

$$F(x, P) = \sum_{j=0}^{\infty} f_j(x) \Phi_j(P),$$

$$Q(x, P) = \sum_{j=0}^{\infty} q_j(x) \Phi_j(P),$$

then

$$F_k^N(x) = \sum_{j=0}^{\infty} f_j(x) \sum_{\ell=1}^{M(N)} W_{\ell} \Phi_j(P_{\ell}) \Phi_k(P_{\ell}) \quad (3.10)$$

and

$$Q_{ks}^N(x) = \sum_{j=0}^{\infty} q_j(x) \sum_{\ell=1}^{M(N)} W_{\ell} \Phi_j(P_{\ell}) \Phi_k(P_{\ell}) \Phi_s(P_{\ell}). \quad (3.11)$$

The value of vector F^N , as computed by (3.8) or (3.10), can be made arbitrarily close to its exact value by choosing the cutoff N in such a way that $f_j(x)$ for $j \geq N$ is sufficiently close to zero. However, even if for this choice of N , $q_j(x) = 0$ when $j \geq N$, the matrix Q^N as computed by (3.9) or (3.11) will contain some 'aliasing' terms.

This can be shown by assuming that the 'coordinate' functions $\Phi_k(P)$ couple in the following way

$$\Phi_j(P) \Phi_m(P) = \sum_{s=|j-m|}^{j+m} C_{jms} \Phi_s(P) \quad (3.12)$$

where

$$C_{jms} = \int dP \Phi_j(P) \Phi_m(P) \Phi_s(P).$$

Existence of such a coupling rule for the spherical harmonics $\{Y_{\ell m}^{(+)}(\mu, \varphi), \ell+m \text{ even}\}$ can be easily demonstrated (see, M. E. Rose, 1957). Then, denoting by

$$\hat{C}_{jks} = \sum_{\ell=1}^{M(N)} W_{\ell} \Phi_j(P_{\ell}) \Phi_k(P_{\ell}) \Phi_s(P_{\ell}), \quad (3.13)$$

the matrix Q^N is given by

$$Q_{ks}^N(x) = \sum_{j=0}^N q_j(x) \hat{C}_{jks}$$

where using (3.12)

$$\hat{C}_{jks} = \sum_{n=0}^{2N} C_{ksn} \left[\sum_{\ell=1}^{M(N)} W_{\ell} \Phi_j(P_{\ell}) \Phi_n(P_{\ell}) \right]$$

and

$$\hat{C}_{jks} = C_{jks} + \sum_{n=N+1}^{2N} C_{ksn} \left[\sum_{\ell=1}^{M(N)} W_{\ell} \Phi_j(P_{\ell}) \Phi_n(P_{\ell}) \right] \quad (3.14)$$

In other words, the expressions (3.13) coincide with c_{jks} when $0 \leq j \leq N$ and $0 \leq k+s \leq N$, but when $k+s > N$ there are extra terms. Practice shows (see, also, Orszag, 1974) that these aliasing terms are not a serious error. It is always possible to choose $M(N)$ so large that (3.6) will be true for $0 \leq k, s \leq 2N$ which would guarantee $\hat{C}_{jks} = c_{jks}$ for $0 \leq j, k, s \leq N$. In that case equations (3.7) would not be different from the exact Galerkin approximation to (3.3); methods of this type, known as spectral methods, have been successfully applied to numerical weather prediction (Bourke, 1972) and other hydrodynamical fluid flow problems (Orszag, 1974).

Another useful form of 'perturbed' Galerkin equations is obtained by multiplying (3.7) with $\Phi_k(P_m)$ and summing over k ; the result is

$$\sum_{k=0}^N \left(L_x + \frac{\lambda_k}{x^2} \right) C_k^N(x) \Phi_k(P_m) + Q(x, P_m) u^N(x, P_m) = F(x, P_m) \quad (3.15)$$

using the fact that

$$\sum_{j=0}^N \Phi_j(P_k) \Phi_j(P_i) = \frac{\delta_{ik}}{W_k}$$

The second term in (3.15) can be written as

$$\begin{aligned} \sum_{k=0}^N \lambda_k C_k^N(x) \Phi_k(P_m) &= \\ &= \sum_{k=0}^N \lambda_k \left[\sum_{s=1}^{M(N)} W_s \Phi_k(P_s) u^N(x, P_s) \right] \Phi_k(P_m) = \\ &= \sum_{s=1}^{M(N)} \left[\sum_{k=0}^N W_s \lambda_k \Phi_k(P_s) \Phi_k(P_m) \right] u^N(x, P_s) \end{aligned}$$

giving, finally, the following form to (3.15)

$$L_x u^N(x, P_m) + \frac{1}{x^2} \sum_{s=1}^{M(N)} B_{ms}^N u^N(x, P_s) +$$
(3.16)

$$+ Q(x, P_m) u^N(x, P_m) = F(x, P_m),$$

$$B_{ms}^N = \sum_{k=0}^N W_s \lambda_k \Phi_k(P_s) \Phi_k(P_m). \quad (3.17)$$

In this formulation the method is known as the orthogonal collocation method (see, Finlayson, 1972; Miketinac, 1976) and its main advantage over (3.7) is that the matrix B^N does not depend on x ; the disadvantage is that the number of unknowns, i.e. $\{u^N(x, P_m)\}_{m=1}^{M(N)}$, is increased to $M(N)$. Equations (3.16) can be derived in a different way starting with the standard collocation method.

The question of convergence of methods of the type (3.7) is not considered here, because there is already an extensive amount of information about it through the work of G. M. Vainikko (1972, 1967). The question is also considered by Orszag (1974).

4. Numerical Procedure

The solution of structure equations is, therefore, sought in the form

$$V^J(x, \mu, \varphi) = \sum_{\ell=0}^J \sum_{m=0}^{\ell} \substack{V_{\ell m}^J(x) Y_{\ell m}^{(+)}(\mu, \varphi) \\ (\ell+m \text{ even})}$$

and as a limit of an iterative process each step of which is a two-point boundary value problem, consisting of a system of linear ordinary differential equations. For the 'perturbed' Galerkin method

the unknowns are

$$\delta V_{00}^J(0) = \sqrt{2J} \delta V_C, \{ \delta V_{\ell m}^J(x), 0 < x \leq x_H, \ell+m \text{ even} \}_{\ell=0, m=0}^{J, \ell}; \quad (4.1)$$

the equations are for $0 < x \leq x_H$

$$\begin{aligned} & \left[\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{\ell(\ell+1)}{x^2} \right] \delta V_{\ell m}^J(x) + \\ & + \sum_{\ell'=0}^J \sum_{m'=0}^{\ell'} Q_{\ell m, \ell' m'}^J(x) \delta V_{\ell' m'}^J(x) - \\ & (\ell' + m' \text{ even}) \end{aligned} \quad (4.2)$$

$$-Q_{\ell m, 00}^J(x) \delta V_{00}^J(0) = F_{\ell m}^J(x),$$

where

$$Q_{\ell m, \ell' m'}^J(x) = n \sum_{i=1}^{M_J} \sum_{j=0}^J w_i \bar{w}_j Y_{\ell m}^{(+)}(\mu_i, \varphi_j) \bar{\theta}_+^{n-1}(x, \mu_i, \varphi_j) \cdot Y_{\ell' m'}^{(+)}(\mu_i, \varphi_j), \quad (4.3)$$

$$\begin{aligned} F_{\ell m}^J(x) = & - \left[\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{\ell(\ell+1)}{x^2} \right] \bar{V}_{\ell m}^J(x) \\ & - \sum_{i=1}^{M_J} \sum_{j=0}^J w_i \bar{w}_j \bar{\theta}_+^n(x, \mu_i, \varphi_j) Y_{\ell m}^{(+)}(\mu_i, \varphi_j). \end{aligned} \quad (4.4)$$

The boundary conditions are, in their linearized form,

$$\frac{d \delta V_{00}^J(0)}{dx} = - \frac{d \bar{V}_{00}^J(0)}{dx} \quad (4.5)$$

$$\delta V_{\ell m}^J(0) = - \bar{V}_{\ell m}^J(0), \ell \neq 0$$

and

$$\frac{d \delta V_{\ell m}^J(x_H)}{dx} + \frac{\ell+1}{x_H} \delta V_{\ell m}^J(x_H) = - \frac{d \bar{V}_{\ell m}^J(x_H)}{dx} - \frac{\ell+1}{x_H} \bar{V}_{\ell m}^J(x_H). \quad (4.6)$$

An equation determining δV_C is obtained by integrating (2.1') over the sphere and then letting $x \rightarrow 0$.

Since

$$\int_{-1}^1 d\mu \int_0^{2\pi} d\varphi \left[\frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial}{\partial \mu} + \frac{1}{1-\mu^2} \frac{\partial^2}{\partial \varphi^2} \right] V(x, \mu, \varphi) = 0$$

and

$$\int_{-1}^1 d\mu \int_0^{2\pi} d\varphi V^J(x, \mu, \varphi) = \frac{4\pi}{\sqrt{2J}} V_{00}^J(x),$$

the resulting equation is

$$3 \frac{d^2 V_{00}^J(0)}{dx^2} = - \sqrt{2J}, \quad (4.7)$$

which in the linearized form is

$$3 \frac{d^2 \delta V_{00}^J(0)}{dx^2} = -\sqrt{2J} - 3 \frac{d^2 \bar{V}_{00}^J(0)}{dx^2} . \quad (4.7')$$

For the method of orthogonal collocation the unknowns are

$$\delta V_c, \{\delta V(x, \mu_i, \varphi_j), 0 < x \leq x_H\}_{i=1, j=0}^{M_J, J} ; \quad (4.8)$$

the equations are for $0 < x \leq x_H$

$$\begin{aligned} & \left(\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} \right) \delta V(x, \mu_i, \varphi_j) + \frac{1}{x^2} \sum_{p=1}^{M_J} \sum_{q=1}^J B_{ij,pq}^J \delta V(x, \mu_p, \varphi_q) + \\ & + n[\bar{\theta}_+(x, \mu_i, \varphi_j)]^{n-1} [\delta V(x, \mu_i, \varphi_j) - \delta V_c] = R(x, \mu_i, \varphi_j), \end{aligned} \quad (4.9)$$

where

$$B_{ij,pq}^J = \sum_{\ell=0}^J \sum_{m=0}^{\ell} (-)^{\ell(\ell+1)} A_m w_p \bar{w}_q Y_{\ell m}^{(+)}(\mu_p, \varphi_q) Y_{\ell m}^{(+)}(\mu_i, \varphi_j), \quad (4.10)$$

(m+l even)

$$R(x, \mu_i, \varphi_j) \equiv -(\nabla^2 \bar{V}^J + \bar{\theta}_+^n)_{\mu=\mu_i, \varphi=\varphi_j} \quad (4.11)$$

The boundary conditions are, in their unlinearized form,

$$\sum_{i=1}^{M_J} \sum_{j=0}^J w_i \bar{w}_j \frac{\partial V^J(0, \mu_i, \varphi_j)}{\partial x} = 0, \quad (4.12)$$

$$\frac{dV^J(x_H, \mu_i, \varphi_j)}{dx} + \frac{1}{x_H} \sum_{p=1}^{M_J} \sum_{q=0}^J C_{ij,pq}^J V^J(x_H, \mu_p, \varphi_q) = 0, \quad (4.13)$$

where

$$C_{ij,pq}^J = \sum_{\ell=0}^J \sum_{m=0}^{\ell} A_m^{(\ell+1)} w_p \bar{w}_q Y_{\ell m}^{(+)}(\mu_i, \varphi_j) Y_{\ell m}^{(+)}(\mu_p, \varphi_q). \quad (4.14)$$

To determine V_c one more equation must be supplied; it is given as another form of equation (4.7)

$$3 \sum_{i=1}^{M_J} \sum_{j=0}^J w_i \bar{w}_j \frac{d^2 V^J(0, \mu_i, \varphi_j)}{dx^2} = -2J. \quad (4.15)$$

These two boundary value problems are singular, but there is a variety of methods for obtaining their numerical solution (see, de Hoog and Weiss, 1975; also, Parter, 1965). Perhaps, the simplest method is to use central finite differences on an equidistant grid $x_h = h \Delta x$, $h = 0, 1, 2, \dots, H$; this approach is adopted in this investigation. To improve the accuracy differential operators involving the corrections (4.1) on the left-hand side of equations (4.2), (4.7'), (4.5) and (4.6) are written out

using the simplest central finite difference formulas, while the same differential operators involving the previous iterate

$$\bar{V}_c^J, \{\bar{V}_{\ell m}^J(x_h), h = 1, 2, \dots, H, \ell + m \text{ even}\}_{\ell=0, m=0}^J, \ell$$

on the right-hand side of these equations are represented by more accurate approximations in terms of central differences. This amounts to a fusion of Newton corrections (4.1) and deferred corrections into a single iterative procedure. In this way one obtains the following system of simultaneous linear algebraic equations

$$\delta V_{00}^J(0) - \delta V_{00}^J(1) = (\Delta x)^2 F_{00}^J(0), \quad (4.16)$$

and for $h = 1, 2, \dots, H-1$

$$\begin{aligned} & -(\Delta x)^2 Q_{\ell m, 00}^J(x_h) \delta V_{00}^J(0) + (1 - \frac{1}{h}) \delta V_{\ell m}^J(x_{h-1}) - [2 + \frac{\ell(\ell+1)}{h^2}] \delta V_{\ell m}^J(x_h) + \\ & + (1 + \frac{1}{h}) \delta V_{\ell m}^J(x_{h+1}) + (\Delta x)^2 \sum_{\ell'=0}^J \sum_{\substack{m'=0 \\ (\ell'+m' \text{ even})}}^{\ell'} Q_{\ell m, \ell' m'}^J(x_h) \delta V_{\ell' m'}^J(x_h) = (\Delta x)^2 F_{\ell m}^J(x_h); \end{aligned} \quad (4.17)$$

the final equation is

$$\begin{aligned} & 2\delta V_{\ell m}^J(x_{H-1}) - [2 + \frac{\ell(\ell+1)}{H^2} + 2 \frac{\ell+1}{H^2} (H+1)] \delta V_{\ell m}^J(x_H) = \\ & = (\Delta x)^2 F_{\ell m}^J(x_H) + 2(1 + \frac{1}{H}) \Delta x G_{\ell m}, \end{aligned} \quad (4.18)$$

where

$$G_{\ell m} = \frac{d \bar{V}_{\ell m}^J(x_H)}{dx} + \frac{\ell+1}{x_H} \bar{V}_{\ell m}^J(x_H). \quad (4.19)$$

The right-hand sides are derived in the appendix. A similar process can be used to obtain a numerical solution of equations (4.9), (4.12), (4.13) and (4.15).

The coefficient matrix for the algebraic system is block-tridiagonal except for the full first column when natural ordering of unknowns is used (see Figure 1). Such systems can be inverted directly by either treating the coefficient matrix as a band matrix or by exploiting the block-tridiagonal structure. However, the standard methods such as those described by Martin and Wilkinson (1967) or Isaacson and Keller (1966) must be modified to take into account the full first column. These modifications are particularly simple, if the ordering of unknowns is reversed. Both algorithms have

using the simplest central finite difference formulas, while the same differential operators involving the previous iterate

$$\bar{V}_C^J, \{\bar{V}_{\ell m}^J(x_h), h = 1, 2, \dots, H, \ell + m \text{ even}\}_{\ell=0, m=0}^J, \ell$$

on the right-hand side of these equations are represented by more accurate approximations in terms of central differences. This amounts to a fusion of Newton corrections (4.1) and deferred corrections into a single iterative procedure. In this way one obtains the following system of simultaneous linear algebraic equations

$$\delta V_{00}^J(0) - \delta V_{00}^J(1) = (\Delta x)^2 F_{00}^J(0), \quad (4.16)$$

and for $h = 1, 2, \dots, H-1$

$$\begin{aligned} & -(\Delta x)^2 Q_{\ell m, 00}^J(x_h) \delta V_{00}^J(0) + (1 - \frac{1}{h}) \delta V_{\ell m}^J(x_{h-1}) - [2 + \frac{\ell(\ell+1)}{h^2}] \delta V_{\ell m}^J(x_h) + \\ & + (1 + \frac{1}{h}) \delta V_{\ell m}^J(x_{h+1}) + (\Delta x)^2 \sum_{\ell'=0}^J \sum_{m'=0}^{\ell'} Q_{\ell m, \ell' m'}^J(x_h) \delta V_{\ell' m'}^J(x_h) = (\Delta x)^2 F_{\ell m}^J(x_h); \\ & (\ell' + m' \text{ even}) \end{aligned} \quad (4.17)$$

the final equation is

$$\begin{aligned} & 2\delta V_{\ell m}^J(x_{H-1}) - [2 + \frac{\ell(\ell+1)}{H^2} + 2 \frac{\ell+1}{H^2} (H+1)] \delta V_{\ell m}^J(x_H) = \\ & = (\Delta x)^2 F_{\ell m}^J(x_H) + 2(1 + \frac{1}{H}) \Delta x G_{\ell m}, \end{aligned} \quad (4.18)$$

where

$$G_{\ell m} = \frac{d \bar{V}_{\ell m}^J(x_H)}{dx} + \frac{\ell+1}{x_H} \bar{V}_{\ell m}^J(x_H). \quad (4.19)$$

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been used in actual computations on the computer taking about the same amount of computing time to invert a given coefficient matrix; the results have been practically identical with the Isaacson-Keller algorithm being more efficient because the amount of data necessary to transfer to and from the massive storage is halved. In spite of this the Martin-Wilkinson algorithm is to be preferred because it has not been possible to show that the condition guaranteeing numerical stability of the Isaacson-Keller algorithm (see, Varah, 1972) is satisfied; in fact, a simple application of the Gerschgorin theorem shows that the condition is not satisfied, but it is very close to it.

$$\begin{bmatrix} A_1 & C_1 & & & & \\ B_2 & A_2 & C_2 & & & \\ c_3 & B_3 & A_3 & C_3 & & \\ c_4 & & B_4 & A_4 & C_4 & \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & B_H & A_H \end{bmatrix}$$

Figure 1. The coefficient matrix for equations (4.16), (4.17)

and (4.18). All blocks are square matrices of order

N_J (A_1 has one additional row and column) except

B_2 and C_1 . The order of the coefficient matrix is

$H \cdot N_J + 1$; c_3, c_4, \dots, c_{H-1} form the column vector in

(4.17). Blocks B_i ($i > 2$) and C_i ($i > 1$) are diagonal matrices.

When considering the computer implementation of the two methods described above, it is necessary to take into account the cost of the following major operations; (i) setting up the coefficient matrix, (ii) solving the linear system, and (iii) storing and retrieving data from the massive storage. There are $H \cdot 2(N_J)^2 M_J(J+1)$ multiplications involved in setting up the coefficient matrix for the 'perturbed' Galerkin method and $H(N_J)^3$ multiplications needed to solve the corresponding linear system using the Martin-Wilkinson algorithm; the total is

$$H(N_J)^2 [2M_J(J+1) + N_J] . \quad (4.20)$$

For the method of orthogonal collocation the number of multiplications is of the order of magnitude

$$H[M_J(J+1)]^3 , \quad (4.21)$$

since setting up the coefficient matrix is not a major contribution in this case. Although the number (4.21) is greater than (4.20) for $J = 7$ and $J = 10$, which were actually used on the computer, the method of orthogonal collocation should not be used for stellar structure computations in three dimensions, because the cost of the third major operation mentioned above becomes prohibitively expensive very rapidly with increasing J . Also, as J increases it becomes necessary to use double precision when solving the linear system much sooner for the method of orthogonal collocation bringing a further setback to the method.

Even for two-dimensional calculations (Stoeckly, 1965), where the number of coefficients equals the number of quadrature points, the 'perturbed' Galerkin method is an almost equally efficient alternative to the method of orthogonal collocation. This is because the cost of the major operations (i) and (ii) is about equal (the factor is about 1.5 if symmetry of the matrix Q^N is taken into account), while the cost of the third major operation does not have to be considered (even though it is the same).

5. Results of Numerical Experiments

A computer implementation of the method of orthogonal collocation and the 'perturbed' Galerkin method has been developed for the problem of uniform rotation with the disturbing function given by (2.7). This is very convenient for two reasons. The rotating configurations should be axially symmetric, because of the form of V_D (at least for small Ω , see James, 1964), so that the problem is actually two-dimensional and as such its numerical solution is well-known. So the present methods will be thoroughly tested in providing a close agreement with the known solution. And, finally, the treatment of the binary problem using the same program is equivalent to a simple replacement of the formula (2.7) by (2.9) or (2.10) in addition to some other (minor) complications. However, for reasons already mentioned only the 'perturbed' Galerkin method has been used for actual computations in the

binary problem. In fact, the method of orthogonal collocation has been programmed for, mostly, sentimental reasons generated by its superb performance in two dimensions (Miketinac, 1976), due in part to the fact that there the number of coefficients in the expansion (3.4) equals the number of quadrature (or collocation) points in (3.6). For both methods the basic computations can be organized as in the flow chart in Figure 2. The possibility of using Δx as the parameter p in the flow chart can be exploited in two ways: (i) the mesh could be refined (with all other parameters kept fixed) giving an idea about the accuracy of an obtained solution, and (ii) the mesh could be coarsened to ensure that the maximal radius of a particular solution is smaller than $H \Delta x$.

All numerical experiments reported here have been performed for the polytropic index $n = 3.0$, with H equal to 60, and J has been chosen to be equal to 7 except for a few experiments when J has been equal to 10. Table 2 contains some pertinent information about the storage requirements and computing times involved in an iteration cycle for the 'perturbed' Galerkin method. In the case of the method of orthogonal collocation it has been necessary to use double precision even for $J=7$; the computing time per iteration cycle has been about 1 minute. From the Table 2 it is clear that frequent use of the massive storage must be made. This is achieved by modifying the Martin-Wilkinson algorithm (for which the coefficient matrix is a band matrix of band width $2N_J+1$ and having $60 \cdot N_J+1$ rows) in such a way that: (i) the coefficient matrix is set up in consecutive blocks containing N_J rows each (except the first block which contains N_J+1 rows), (ii) the fast core contains only two such blocks at any one time and (iii) results of the elimination are transferred to and from the massive storage in blocks, too. For all values of the parameter p in Figure 2 that have been tried, the number of iterations necessary for convergence (convergence criteria are that $\max_{\ell, m, h} |R_{\ell m}(x_h)| \leq 0.000001$ and $\max_{\ell, m, h} |\delta V_{\ell M}(x_h)| \leq 0.0005$), seldom exceeded three.

Computation of uniformly rotating configurations has served (as indicated at the beginning) as a test problem in two ways. The accuracy of a method can be judged by (i) looking for dependence of the results on the angle φ , and (ii) agreement with the corresponding two-dimensional results. For the 'perturbed' Galerkin method no dependence on φ is found to seven decimal digits and the results agreed with the two-dimensional results (obtained by the method of orthogonal collocation

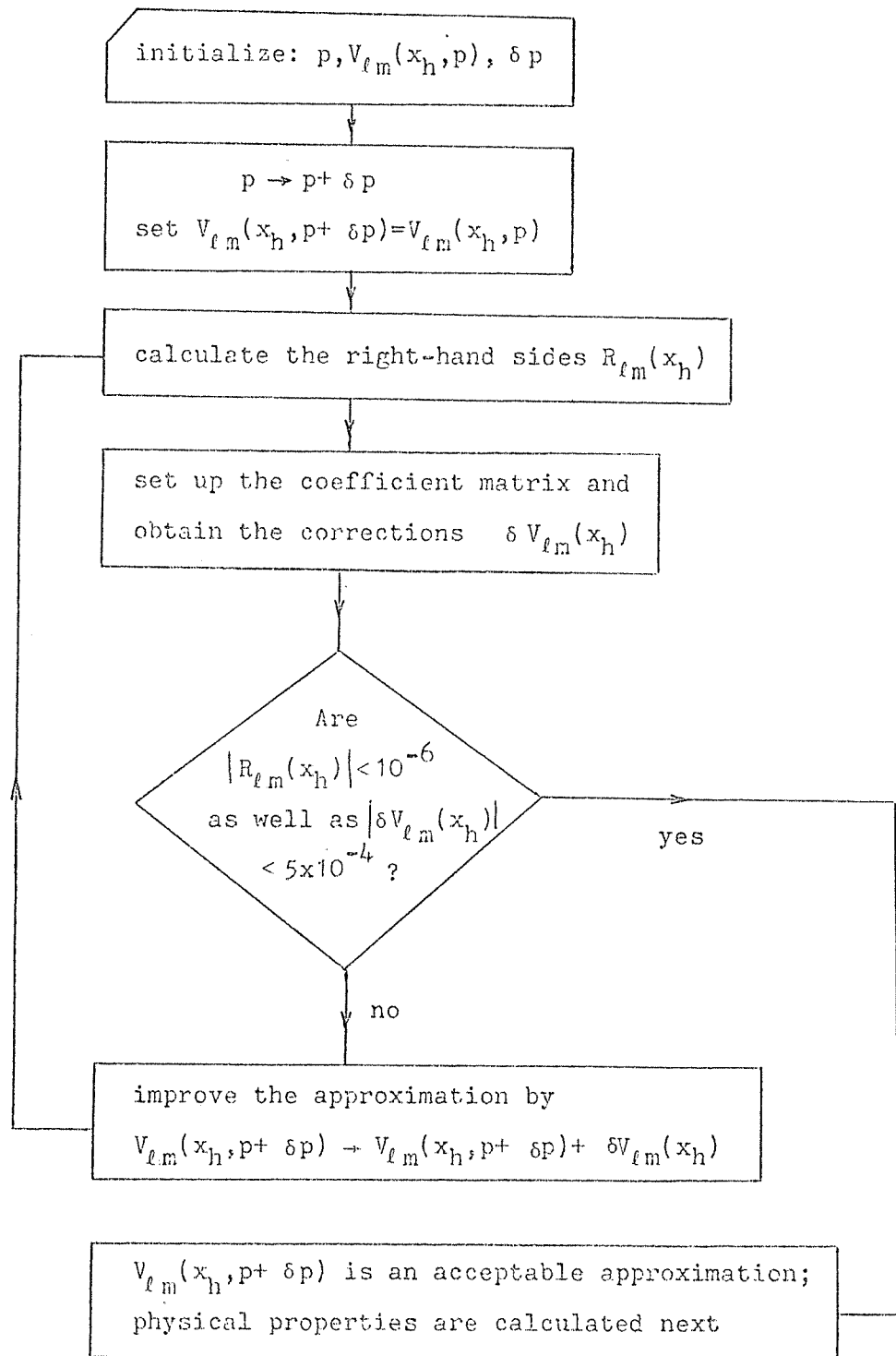


Figure 2: The flow chart contains only the basic portions of the iteration procedure. The symbol p could be the parameter λ , but it could, also, be Δx or n changed by a 'small' amount δp .

J	N_J	N_D	T_S	T_R	T
7	20	50 360	32	8	40
10	36	159 768	158	29	187

Table 2. The new symbols N_D, T_S, T_R , and T have the following meanings: N_D is the number of non-zero elements for the Martin-Wilkinson algorithm, T_S is the time taken for setting up the coefficient matrix and solving for the corrections, T_R is the time taken for calculating the right-hand sides, and T is the total time per iteration cycle; all times are in seconds (the machine is UNIVAC 1110).

formulated for two-dimensional computations, Miketinac and Barton, 1972) to four decimal digits. These results have been slightly bettered by the method of orthogonal collocation but with the Gauss elimination done in double precision. Both computations have been performed for $J = 7$ made possible by the fact that the magnitude of the coefficients $V_{\ell m}(x_h)$ becomes very small for $\ell \geq 5$. Martin (1970), also, tested his perturbation expansion on the example of uniformly rotating polytropes. His results for $n = 3.0$ agreed very well with the results obtained by Miketinac and Barton (1972) using the method of orthogonal collocation; the results are shown in Table 3.

Ω_c	4.070×10^{-3}	4.076×10^{-3}
x_e	10.05	10.065

Table 3. Equatorial radius, X_e , for the polytrope of index $n = 3.0$ rotating uniformly with the critical angular velocity, Ω_c . The second column contains results obtained by Miketinac and Barton, while the third column contains Martin's results.

Computation of the structure of the primary in a close binary system has been performed, so far, for only one value of the ratio of masses, $q = 0.5$ (with $n = 3.0$). The first-order theory has been used, J being equal to 7 and the critical separation X_c is found by testing (2.14). The results are collected in the second column of Table 4, which contains (in the third column) the corresponding results obtained by P. G. Martin (1970) using the second-order theory and the perturbation method. The fact that results do not agree much more closely can probably be accounted for by the difference between the first-order theory and the second-order theory. For the critical configuration the absolute magnitude of coefficients $V_{lm}(x_h)$ becomes smaller than 0.000 002 for $l \geq 6$ showing that the cutoff $J = 7$ is reasonable. An intermediate configuration, belonging to $X^{-1} = 0.03$, has been computed, also, with $J = 10$ (and still using the first-order theory), the results agreed to five decimal digits. Another intermediate configuration, belonging to $X^{-1} = 0.06$, has been obtained in two different ways: (i) as an end of a sequence of configurations with X^{-1} making 'small jumps' $0.00 \rightarrow 0.03 \rightarrow 0.04 \rightarrow 0.05 \rightarrow 0.06$, and (ii) directly from $X^{-1} = 0.00$ to $X^{-1} = 0.06$; the results agreed again to five decimal digits.

The immediate future work on the binary problem should have the following two objectives. First, critical configurations for a range of values of the mass ratio, q (including $q = 0.5$) should be obtained using both the first-order and the second-order theory with $J = 10$ and the results should then be compared with those obtained by Martin (1970); a close agreement of the results is expected. In these computations the polytropic index would have to be equal to $n = 3.0$, since Martin's method can not be extended to the other interesting polytrope for which $n = 1.5$. The second objective should be computation of critical configurations for this other polytrope using again both the first-order and the second-order theory with $J = 10$. The results could then be compared with the first-order results of Naylor and Anand (1970), but large discrepancy of the results would not be surprising on account of remarks made in Section 2. It is expected that $H = 60$ will have to be replaced by, probably, $H = 100$ in order to maintain $\Delta x \leq 0.2$ and yet accommodate the configurations for the $n = 1.5$ polytrope, which is known to be less centrally condensed than the $n = 3.0$ polytrope. This change in H will make the computations more expensive and an increased efficiency will have to be achieved. For example, the cost of computing critical configurations using the second-order theory

X_c^{-1}	0.064	0.062
x_e	0.583	0.571
x_t	0.459	0.439
x_n	0.497	0.468
x_p	0.428	0.414
g_p	1.092	1.094
g_t	0.756	0.780
g_n	0.533	0.601
V_s	2.291	2.961

Table 4. A comparison of results for the primary of a close binary system. The symbols x_e, x_t, x_n , and x_p are the radii in the following (μ, φ) -directions: $(0, 0)$, $(0, \pi/2)$, $(0, \pi)$, and $(1, \varphi)$. The gradient of the (total) potential (i. e. $V+V_D$) at the surface is denoted by g ; the quantities g_t and g_n are expressed in the units of g_p and g_p is itself normalized by the gravity, g_0 , on the undistorted Emden sphere. The (total) potential V_s at the surface is the potential normalized by Gm/R .

with $J = 10$ can be lowered considerably by first locating the critical configuration in the first-order theory with $J = 7$ and then refining the results using the second-order theory and $J = 10$.

The future work on the 'perturbed' Galerkin method should include the proof of convergence. Cryer (1976) reports that no proof of convergence of a trial free-boundary method has been given. From the applications point of view the most important future problems would be the bifurcation problem for the disturbing potential (2.8) and the possibility of computing the critical configurations of more realistic models of stars.

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Appendix

From equations (4.5) and (4.7') it follows that the right-hand side in (4.16) is given by

$$F_{00}^J(0) = \frac{1}{6} \sqrt{2J} + \frac{1}{2} \frac{d^2 \bar{V}_{00}^J(0)}{dx^2} + \frac{1}{\Delta x} \frac{d \bar{V}_{00}^J(0)}{dx}.$$

The right-hand side for (4.17) is defined in (4.4), and the final right-hand side in (4.18) is a combination of (4.4) and (4.19).

Differential operators in these right-hand sides are approximated using the following formulas

$$\left. \frac{dy}{dx} \right|_{x=x_i} = \frac{1}{12\Delta x} (-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2})$$

and

$$\left. \frac{d^2 y}{dx^2} \right|_{x=x_i} = \frac{1}{12(\Delta x)^2} (-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}).$$

Then one can show that the right-hand sides are given by

$$(\Delta x)^2 F_{00}^J(0) = \frac{1}{6} (\Delta x)^2 \sqrt{2J} + \frac{1}{24} [-3 \bar{V}_{00}^J(x_2) + 32 \bar{V}_{00}^J(x_1) - 30 \bar{V}_{00}^J(0) + \bar{V}_{00}^J(x_{-2})],$$

$$\begin{aligned} (\Delta x)^2 F_{\ell m}^J(x_h) = & \frac{1}{12} [(1 + \frac{2}{h}) \bar{V}_{\ell m}^J(x_{h+2}) - 16(1 + \frac{1}{h}) \bar{V}_{\ell m}^J(x_{h+1}) + \\ & + 30 \bar{V}_{\ell m}^J(x_h) - 16(1 - \frac{1}{h}) \bar{V}_{\ell m}^J(x_{h-1}) + (1 - \frac{2}{h}) \bar{V}_{\ell m}^J(x_{h-2})] + \\ & + \frac{\ell(\ell+1)}{h^2} \bar{V}_{\ell m}^J(x_h) - (\Delta x)^2 \sum_{i=1}^M \sum_{j=0}^J w_i \bar{w}_j \bar{\theta}_+^n(x_h, \mu_i, \varphi_j) Y_{\ell m}^{(+)}(\mu_i, \varphi_j) \end{aligned}$$

for $h = 1, 2, \dots, H-1$, and

$$\begin{aligned} (\Delta x)^2 F_{\ell m}^J(x_H) + 2(1 + \frac{1}{H}) \Delta x G_{\ell m} = & \frac{1}{12} [-\bar{V}_{\ell m}^J(x_{H+2}) + 30 \bar{V}_{\ell m}^J(x_H) - 32 \bar{V}_{\ell m}^J(x_{H-1}) + \\ & + 3 \bar{V}_{\ell m}^J(x_{H-2})] + \frac{\ell+1}{H^2} [\ell + 2(H+1)] \bar{V}_{\ell m}^J(x_H). \end{aligned}$$

These expressions can be used only if a prescription is given for computing $\bar{V}_{00}^J(x_{-2})$, $\bar{V}_{\ell m}^J(x_{-1})$ and $\bar{V}_{\ell m}^J(x_{H+1})$, $\bar{V}_{\ell m}^J(x_{H+2})$. The last two quantities are obtained from the outside boundary condition by setting in

$$\frac{\bar{V}_{\ell m}^J(x_{M+1}) - \bar{V}_{\ell m}^J(x_{M-1})}{2\Delta x} + \frac{\ell+1}{M\Delta x} \bar{V}_{\ell m}^J(x_M) = 0$$

$M = H$ and $M = H+1$. An expression for $\bar{V}_{\ell m}^J(x_{-2})$ is obtained by putting $h = -1$ in the equation

$$-(1 + \frac{1}{h})\bar{V}_{00}^J(x_{h+1}) + 2\bar{V}_{00}^J(x_h) - (1 - \frac{1}{h})\bar{V}_{00}^J(x_{h-1}) =$$

$$= (\Delta x)^2 \sum_{i=1}^{M_J} \sum_{j=0}^J w_i \bar{w}_j Y_{00}^{(+)}(\mu_i, \varphi_j) \bar{\theta}_+^n(x_{-1}, \mu_i, \varphi_j),$$

where

$$\bar{\theta}(x_{-1}, \mu, \varphi) = \bar{V}(x_{-1}, \mu, \varphi) - \bar{V}_C + 1 + V_D(\lambda, x_{-1}, \mu, \varphi),$$

giving

$$\bar{V}_{00}^J(x_{-2}) = \bar{V}_{00}^J(x_{-1}) - \frac{1}{2} \frac{(\Delta x)^2}{\sqrt{2J}} \sum_{i=1}^{M_J} \sum_{j=0}^J w_i \bar{w}_j \bar{\theta}_+^n(x_{-1}, \mu_i, \varphi_j).$$

It follows from (4.1) that when $\ell = 0$ one can set

$$\bar{V}_{\ell m}^J(x_{-1}) = (-)^{\ell} \bar{V}_{\ell m}^J(x_1).$$

The validity of this formula for $\ell \neq 0$ follows from the known solution (Weinberger, 1965) of

$$-\frac{d^2 \bar{V}_{\ell m}}{dx^2} + \frac{2}{x} \frac{d \bar{V}_{\ell m}}{dx} - \frac{\ell(\ell+1)}{x^2} \bar{V}_{\ell m} = -1$$

near $x = 0$.