PROPERTIES OF CONFLICT FREE AND PERSISTENT
PETRI NETS

by
L. H. Landweber
and
E. L. Robertson

Computer Sciences Technical Report #264
December 1975
PROPERTIES OF CONFLICT-FREE AND PERSISTENT PETRI NETS

by

L. H. Landweber
Computer Sciences Department
University of Wisconsin--Madison

and

E. L. Robertson
Computer Science Department
Pennsylvania State University

*This research was partially supported by NSF Contract #GJ33087.
Abstract

Petri nets have been extensively studied because of their suitability as models for asynchronous computing. Despite this effort, the mathematical properties of Petri nets are not very well understood. In this paper, we investigate two important special types of Petri nets, the conflict free and the persistent nets, the former being a proper subset of the latter. Our results completely characterize the sets of reachable markings attainable by such nets. Reachability sets of persistent nets are shown to be semi-linear. A stronger result is obtained for conflict free nets which results in an exponential time algorithm for deciding boundedness of such nets. The best known upper bound for deciding boundedness of arbitrary nets is Ackermann's function. We conclude with a proof that all reachability sets of Petri nets may be realized with a minimal amount of non-persistence.
1. Introduction

Petri nets and vector addition systems have been extensively studied because of their suitability as models of asynchronous computing. Despite this effort, the mathematical properties of these equivalent models are not very well understood. For example, it is not known whether the reachability and liveness problems are decidable although the recursive equivalence of these problems has been demonstrated [2]. It is known [5] that if reachability is decidable, then the time required by the decision procedure must be at least exponential in the size of the inputs (i.e., the size of a representation of the Petri net plus the marking to be reached).

In this paper, we investigate two important special types of Petri nets, the conflict free and the persistent nets, the former being a proper subset of the latter. Our results completely characterize the sets of reachable markings attainable by such nets. Persistence for Petri nets is very similar to the persistence property used by Lipton, Miller and Snyder [6] in their study of linear asynchronous structures. It is also similar to a notion used by Muller to study various types of switching circuits.

Liveness is known to be decidable for persistent nets [4] while both liveness and reachability are decidable for conflict free nets [1]. Our first main result (Section 4) shows that persistent nets have semi-linear sets of reachable markings. Since conflict free nets are persistent, this answers a question left open in [1]. Our proof does not yield a decision procedure for reachability in persistent nets because the construction is not effective. However, it does indicate a strong probability that such a decision procedure does exist. Semi-linearity has played an important role in the study of Petri nets.

For example, Van Leeuwen's [9] decision procedure for the reachability problem for 3 coordinate vector addition systems involves the construction of semi-linear representations for sets of reachable points. Rabin's proof that containment of sets of reachable points for arbitrary Petri nets is undecidable involves building nets which have non semi-linear reachability sets. We believe that the relationship we exhibit between persistence and semi-linearity, plus the fact that only a "small" amount of non-persistence is necessary to achieve any (non-semi-linear) reachable set of markings (Section 6), clearly indicates why it has been so difficult to obtain significant results for arbitrary Petri nets.

We next (Section 5) investigate properties of conflict free nets. In particular, we show that for any conflict free net, there is a constant $c$ such that for an arbitrary initial marking with $x$ tokens, any place in the net receives either at most $cx$ tokens or an unbounded number of tokens. A corollary to this result is an exponential time algorithm for deciding boundedness for conflict free nets. This may be contrasted with Karp and Miller's [3] algorithm for deciding boundedness of arbitrary nets which has Ackermann's function as a time complexity bound.

In Section 2 we give some basic definitions and notation. Section 3 contains some combinatorial lemmas for various classes of Petri nets. Sections 4 and 5 deal with persistent and conflict free nets respectively. In the last section, we discuss some open problems and give examples which illustrate the central role that persistence and semi-linearity play in the study of Petri nets.
2. Definitions and Notation

A Petri net is a quadruple.
\[ P = (P, T, A, M_0) \]

where \( P \) is a finite set of places; \( T \) is a finite set of transitions
or firing bars; \( A \) is a finite set of arcs, \( A \subseteq (P \times T) \cup (T \times P) \); and
\( M_0 : P \rightarrow N \), \( N \) the set of natural numbers, is the initial marking.

Initially each place \( p \) of the Petri net contains \( M_0(p) \) tokens.

Let \( t \) be a transition. Then \( (p, (p, t) \epsilon A) \) and \( (p, (t, p) \epsilon A) \) are
called the input places (inputs) and output places (outputs) respectively
of \( t \).

Transition \( t \) is enabled or fireable when each input place of \( t \)
contains at least one token. If \( t \) is enabled, then it may be fired
which results in the removal of one token from each input place of
\( t \) and the addition of one token to each output place of \( t \). If \( t \)
is not enabled, then it is disabled. Write \( M_1, t \rightarrow M_2 (M_1 \epsilon T^*) \)
to indicate that \( t \) is enabled by the marking \( M_1 \) and that the firing of
\( t \) yields marking \( M_2 (t \) is enabled by the marking \( M_1 \)). Extend the
notation and definitions to sequences of transitions, \( \sigma \in T^* \), called
firing sequences.

The set of reachable markings or the reachability set \( R_p \) of the
Petri net \( P = (P, T, A, M_0) \) is \( \{ M \mid M_0 \epsilon M, \text{ for some } \sigma \epsilon T^* \} \). If
\( M \epsilon R_p \) we say that \( M \) is reachable in \( P \). The reachability problem
for a class \( C \) of Petri nets is the problem of deciding, given an
arbitrary \( P \epsilon C \) and marking \( M \), whether \( M \epsilon R_p \).

A place in a Petri net is bounded if there is a \( c \epsilon N \) such that
for all reachable markings \( M \), \( M(p) \leq c \). A Petri net is bounded if
each place in the net is bounded. A Petri net (place in a Petri net)
is unbounded if the net (the place) is not bounded. A subset \( P_1 \) of
the set of places is simultaneously unbounded if for each \( n \epsilon N \) there
is a reachable marking \( M \) such that \( M(p) \geq n \) for each \( p \epsilon P_1 \).

A Petri net is persistent if for all \( t_1, t_2 \epsilon T \), \( t_1 \neq t_2 \) and
any reachable marking \( M, M_{t_1} \) and \( M_{t_2} \) imply \( M_{t_1 \cdot t_2} \). I.e., if
\( t_1 \) and \( t_2 \) are enabled at a reachable marking, then the firing of
one cannot disable the other.

A place \( p \) and a transition \( t \) are or a self loop if \( p \) is both
an input place and an output place of \( t \).

A Petri net is conflict free if every place which is an input of
more than one transition is on a self loop with each such transition.
Conflict free nets are persistent though the converse need not be
true.

A set \( M \) of markings is linear if there is a finite set of functions
\( \{ f_i \mid f_i : P \rightarrow N, (0 \leq i \leq n) \} \) such that
\[ M = \{ f_0 + \sum_{i=1}^{n} c_i f_i \mid c_i \geq 0 \} \]

\( M \) is semi-linear if it is a union of a finite number of linear sets.
3. Combinatorial Properties

In this section we obtain some combinatorial properties of persistent and arbitrary Petri nets. In the following, assume that \( P = (P, \cdot, A_0, A_0^0) \) is a fixed but arbitrary Petri net with transitions \( T = \{ t_1, \ldots, t_k \} \). Define the Parikh map (see [7], for the first use of this important idea) \( \text{PK}: T^* \rightarrow \mathbb{N}^k \) so that \( \text{PK}(\sigma)_i \) is the number of occurrences of \( t_i \) in \( \sigma \). The corresponding Parikh space \( T \) of \( P \) is

\[ \{ \text{PK}(\sigma) | \sigma \in T^*, \sigma \text{ is fireable} \}. \]

Observe that we have discarded certain information with this map, namely the sequence in which the transitions fire. We first show that the Parikh space of a persistent net is a lattice with respect to the partial order defined by: \( x \leq y \) (\( x, y \in \mathbb{N}^k \)) if all coordinates of \( x \) are less than or equal to the corresponding coordinates of \( y \). Also \( x < y \) if \( x \leq y \) and \( x \neq y \). In the following all arithmetic operations on vectors are to be interpreted as being performed independently on all corresponding coordinates of the vectors used.

For \( \sigma, \tau \in T^* \), \( \sigma = a_1 \ldots a_n \) define \( (\sigma + \tau) \) as follows:

Let \( \tau_0 \) be \( \tau \). Obtain \( \tau_{i+1} \) by deleting the leftmost occurrence of \( a_i \) from \( \tau_i \), if \( a_i \) occurs in \( \tau_i \). If not, then \( \tau_{i+1} = \tau_i \). Define \( (\sigma + \tau) \) to be \( \tau_n \).

Following an argument similar to that used by Keller [4] we have:

Lemma 3.1. Let \( \sigma \) and \( \tau \) be fireable sequences in a persistent net. Then there is a fireable sequence \( \beta \) such that

\[ \text{PK}(\beta) = \max \{ \text{PK}(\sigma), \text{PK}(\tau) \}. \]

Moreover, \( \beta \) may be constructed so that \( \beta = \sigma \cdot (\sigma + \tau) \).

The following trivial example (Figure 1) shows that the same result does not hold for \( \text{min} \).

Clearly both \( t_1t_2 \) and \( t_3t_2 \) are fireable sequences with corresponding Parikh space points \( (1,1,0) \) and \( (0,1,1) \). But the transition \( t_2 \) corresponding to \( (0,1,0) \) is not itself fireable.

Theorem 3.2. For a persistent Petri net, the corresponding Parikh space is a lattice under the natural ordering of vectors (\( x \leq y \) iff \( x_i \leq y_i \) for all \( i \), \( 1 \leq i \leq k \)).

Proof. Join in the lattice is simply \( \text{max} \) by Lemma 3.1. Meet is not necessarily \( \text{min} \), by the above remark, but can be shown unique. Consider two points \( x, y \) of the Parikh space of the net. If two points \( u, v \) of the Parikh space satisfy \( u \leq x, u \leq y, v \leq x, v \leq y \), then \( w = \text{max}(u, v) \), satisfies \( w \leq x, w \leq y \) so \( w \) is also a lower bound of \( x, y \). But by Lemma 3.1, \( w \) is in the Parikh space of the net.

Figure 2 shows an example of a non-persistent net whose Parikh space is \( \{(0,0),(1,0),(0,1)\} \). Since \( (1,0) \) and \( (0,1) \) do not have a join, the space is not a lattice.

Figure 2
The following lemmas will be useful in Sections 4 and 5. A sequence \( \alpha = t_1 t_2 \ldots \in \omega \) is an \( \omega \)-firing sequence if every finite prefix of \( \alpha \) is a fireable firing sequence. If there exists an \( \omega \)-firing sequence \( \alpha \) such that arbitrarily large markings of \( p \) (of every member of \( P_1 \subseteq P \)) may be obtained by firing finite prefixes of \( \alpha \), then we say that \( p \) \((P_1)\) is unbounded (simultaneously unbounded) on \( \alpha \). The reader should compare this with the definitions of "unbounded place" and "simultaneously unbounded set of places" given in Section 2.

**Notation:** \( a_i \) is the \( i \)-th element of \( \alpha = a_1 a_2 \ldots \).

\( a[i = a_1 \ldots a_i \) and \( a[ij = a_1 \ldots a_j \).

It is possible to give a weaker definition of simultaneously unbounded on \( \alpha \), corresponding to the case where finite prefixes of an \( \omega \)-firing sequence \( \alpha \) mark places \( p_1 \) and \( p_2 \) with arbitrarily large values, but whenever the marking of \( p_1 \) is large, then the marking of \( p_2 \) is small and vice versa. The following lemma shows that the two notions coincide for arbitrary nets.

**Lemma 3.3.** Let \( A \) and \( B \) be sets of places of an arbitrary Petri net. Let \( A \) and \( B \) be simultaneously unbounded on some \( \omega \)-firing sequence \( \alpha \). Then there is an \( \omega \)-firing sequence \( \bar{\alpha} \) such that \( A \cup B \) is simultaneously unbounded on \( \bar{\alpha} \).

**Proof.** The idea is to find repeatable finite subsequences in \( \alpha \) which increase A's markings without decreasing B's marking and vice versa. Let \( M_i \) be the marking at time \( i \), i.e., after \( \alpha[i \) has fired. Since \( A \) and \( B \) are each simultaneously unbounded on \( \alpha \), there must be times \( a_1 < b_1 < a_2 < b_2 < \ldots \) such that

a) \( M_{a_1+1}(x) > M_{a_1}(x) \) for each \( x \in A \)

b) \( M_{b_1+1}(x) > M_{b_1}(x) \) for each \( x \in B \).

Let \( p_1, \ldots, p_r \) be an enumeration of \( \mathcal{P} - A \) and \( q_1, \ldots, q_s \) an enumeration of \( \mathcal{P} - B \). Choose an infinite subsequence \( \{a_i\} \) of the \( a_i \)'s such that \( M_{a_{i+1}'}(p_i) = M_{a_i}(p_i) \) for \( i \geq 1 \). Similarly choose an infinite subsequence \( \{b_i\} \) of the \( b_i \)'s such that \( M_{b_{i+1}'}(q_i) = M_{b_i}(q_i) \) for \( i \geq 1 \). Iterate this procedure for \( p_1, \ldots, p_r, q_1, \ldots, q_s \). This yields infinite subsequences \( \{\bar{a}_i\} \) and \( \{\bar{b}_i\} \) of \( \{a_i\} \) and \( \{b_i\} \) respectively such that \( M_{\bar{a}_{i+1}'} \geq M_{\bar{a}_i} \)

and \( M_{\bar{b}_{i+1}'} \geq M_{\bar{b}_i} \) for \( i \geq 1 \). From \( \{\bar{a}_i\}, \{\bar{b}_i\} \) choose times \( s_1 < t_1 < u_1 < v_1 < s_2 < v_1 < s_3 < v_1 < s_4 < v_1 < s_5 \) for \( i \geq 1 \) which satisfy:

a) \( M_{t_i'}(x) > M_{s_i'}(x) \) for each \( x \in A' \)

b) \( M_{u_i'}(x) > M_{v_i'}(x) \) for each \( x \in B' \).

Let \( B_i = (a|s_i u_i) \)

\( Y_i = (a|t_i v_i) \)

\( \Delta_i = (a|u_i v_i) \)

\( \epsilon_i = \alpha|v_i s_{i+1} \).

Note that \( \alpha = (a|s_1) \beta_1 Y_1 \delta_1 \epsilon_1 \beta_2 Y_2 \delta_2 \epsilon_2 \ldots \). Then the required \( \bar{\alpha} \) is

\( \bar{\alpha} = (a|s_1) \beta_1 Y_1 \delta_1 \epsilon_1 \beta_2 Y_2 \delta_2 \epsilon_2 \ldots \). Each new \( B_i(\delta_i) \)

increases the marking of each place in \( A \) (each place in \( B \)) without decreasing the markings of places in \( B \) (places in \( A \)). At all times after \( \epsilon_i \) has been fired, all places in \( A \cup B \) will have at least \( i \) tokens. Hence \( A \cup B \) is simultaneously unbounded on \( \bar{\alpha} \). \( \square \)

**Corollary 3.4.** Let \( A \) be a set of places which is simultaneously unbounded on some \( \omega \)-firing sequence \( \alpha \). Then there is an \( \omega \)-firing
sequence $\alpha$ such that for all $n \in \mathbb{N}$, there is an $i_n$, such that for $i > i_n$, $\alpha_i$ marks each member of $A$ with at least $n$ tokens.

Proof. This follows immediately from the proof of Lemma 3.3. $\square$

The next two results are true for persistent nets.

Theorem 3.5. Let $P$ be a persistent net with $A \subseteq P$, $B \subseteq P$. Then the following are equivalent.

1. $A$ and $B$ are simultaneously unbounded on some $\omega$-firing sequence.
2. $A$ and $B$ are simultaneously unbounded.
3. $A \cup B$ is simultaneously unbounded.
4. $A \cup B$ is simultaneously unbounded on some $\omega$-firing sequence.

Proof. $2 \Rightarrow 1$: Assume $A$ and $B$ are simultaneously unbounded. We construct an $\omega$-firing sequence on which both $A$ and $B$ are simultaneously unbounded. Let $\alpha_1$ be a finite fireable firing sequence which marks each place in $A$ with $\geq 1$ token. Let $\beta_1$ be a finite fireable firing sequence which marks each $B$ member with $\geq 1 + |\alpha_1|$ tokens where $|\alpha_1|$ is the length of $\alpha_1$. By Lemma 3.1, because $P$ is persistent, $\alpha_1 \cdot (\beta_1 \cdot \alpha_1)$ is fireable. But $\alpha_1$ marks each member of $B$ with $\geq 1$ token. Assume $\alpha_i$, $\beta_i$, $\sigma_i$ have been defined for $1 \leq i < n$ and satisfy:

a) $\alpha_i$ is fireable and marks each $A$ member with $\geq 1$ tokens.

b) $\beta_i$ is fireable and marks each $B$ member with $\geq 1 + |\alpha_i|$ tokens.

c) $\sigma_i = \alpha_i \cdot (\beta_i \cdot \alpha_i)$ is fireable and marks each $B$ member with $\geq 1$ tokens.

d) $\sigma_i$ is an initial segment of $\sigma_{i+1}$ for $1 \leq i < n - 1$.

Let $\gamma$ be a fireable sequence which marks each $A$ member with $\geq n + |\sigma_{n-1}|$ tokens. By Lemma 3.1, $\sigma_n = \sigma_{n-1} \cdot (\gamma \cdot \sigma_{n-1})$ is fireable and marks each member of $A$ with $\geq n$ tokens so $\sigma_n$ satisfies a). Now let $\beta_n$ be a fireable sequence which marks each $B$ member with $\geq n + |\sigma_{n-1}|$ tokens (satisfying b)). By Lemma 3.1, $\sigma_n = \sigma_n \cdot (\beta_n \cdot \sigma_n)$ is fireable and marks each $B$ member with $\geq n$ tokens (satisfying c)). Moreover $\sigma_n$ and $\sigma_{n-1}$ satisfy d). In the limit, we obtain an $\omega$-firing sequence on which both $A$ and $B$ are simultaneously unbounded.

$3 \Rightarrow 2$: obvious from definitions.

$4 \Rightarrow 3$: obvious from definitions.

$1 \Rightarrow 4$: Lemma 3.3 $\square$

Corollary 3.6. Let $P$ be a persistent Petri net with $A \subseteq P$. Then $A$ is simultaneously unbounded if and only if $A$ is simultaneously unbounded on some $\omega$-firing sequence.

Lemma 3.7. Let $P$ be an arbitrary Petri net. A transition $t$ occurs $\omega$ times in some $\omega$-firing sequence $\alpha$ if and only if either:

a) $t$ is on a loop all of whose transitions occur $\omega$ times in $\alpha$ (t is on an $\alpha$-$\omega$-loop); or

b) $t$ is on a path from a loop such that all transitions on the loop and all transitions on the path up to and including $t$ occur $\omega$ times in $\alpha$ (t depends on an $\alpha$-$\omega$-loop); or

c) $t$ is on a path all of whose transitions, including $t$, occur $\omega$ times in $\alpha$ and the first transition on the path has no input places (t is on an $\alpha$-$\omega$-path).

Proof. $\Leftarrow$ trivial

$\Rightarrow$ Assume $t$ occurs $\omega$ times in $\alpha$. Each input place to $t$ must be an output place for some transition which occurs $\omega$ times in $\alpha$. Iterate the backward search. Eventually, either a transition repeats (cases a or b) or a transition is reached which has no input places (case c). $\square$

Lemma 3.8. Let $P$ be a persistent net with transitions $t_1, t_2 \in T$. The following are equivalent:
1. For each \( k \geq 1 \), there are finite fireable sequences \( A^k_1 \) and \( A^k_2 \) having at least \( k \) occurrences of \( t_1 \) and \( t_2 \) respectively.

2. For each \( k \geq 1 \) there is a finite fireable sequence \( \beta \) in which \( t_1 \) and \( t_2 \) each occur at least \( k \) times.

3. \( t_1 \) and \( t_2 \) occur infinitely often on some \( \omega \)-firing sequence \( \alpha \).

4. There are \( \omega \)-firing sequences \( \alpha_1 \) and \( \alpha_2 \) such that \( t_1 \) and \( t_2 \) occur infinitely often on \( \alpha_1 \) and \( \alpha_2 \) respectively.

Proof. The proof is similar to proofs given in [4]. \( \square \)

Lemma 3.9. Let \( \alpha t \alpha' \) be a fireable sequence of a conflict free Petri net \( P \) where \( t, t' \in T \) and \( \delta \in T^* \). Then there is a fireable sequence \( \alpha t \delta t' \) such that neither \( t \) nor \( t' \) occurs in \( \beta \).

Proof. The proof is by induction on \( L(t, \delta, t') \), the maximum length of directed paths in \( P \) from \( t \) to \( t' \) which are subpaths of \( \delta \). Let \( L(t, \delta, t') = 0 \) if no directed path from \( t \) to \( t' \) is a subpath of \( \delta \).

If \( L(t, \delta, t') = 0 \), then no path from \( t \) to \( t' \) is a subpath of \( \delta \). First delete all occurrences of \( t \) in \( \delta \). Then iterate the process of deleting occurrences of other transitions in \( \delta t \) which have become disabled. Call the portion of \( \delta \) that remains \( \beta \). The sequence \( \alpha t \delta t' \) is fireable because, for every deleted transition \( \epsilon \), the firing of \( \epsilon \) could not have been necessary to the firing of \( t' \). (Else some subpath of \( \delta \) would connect \( t \) to \( t' \).)

Now let \( \delta \) be the largest initial segment of \( \beta \) not containing \( t' \).

Assume the lemma is true for all transitions \( \epsilon \) and all sequences \( \gamma \) such that \( L(t, \gamma, \epsilon) \leq n \) and let \( L(t, \delta, t') = n+1 \). Assume that \( t' \) does not occur in \( \delta \) (if it does, then let \( \delta \) be the longest initial segment of \( \delta \) not containing \( t' \)). Let \( t_1, \ldots, t_n \) be those transitions which occur in \( \delta \) and which have output places that are also input places of \( t' \). If there are no such transitions, then \( \alpha t \delta t' \) is fireable. Let \( \delta_i \) be the initial segment of \( \delta \) which precedes the last occurrence of \( t_i \) in \( \delta \). Then, for each \( i \), \( \alpha \delta_i t_i \) is fireable and \( L(t, \delta_i, t_i) = n \) so by the induction hypothesis, there is a \( \beta_i \) not containing \( t \) such that \( \alpha t \delta_i t_i \) is fireable. But then by Lemma 3.1, there is a fireable sequence \( \sigma \) which begins with \( \omega t \) and which satisfies

\[
PK(\sigma) = \max \{ PK(\alpha t \delta_i t_i) \}.
\]

In particular, \( \sigma \) is of the form \( \alpha t \beta \) where \( \beta \) contains no occurrences of \( t \) or \( t' \) and at least one occurrence of each \( t_i \). But the net is conflict-free so \( \alpha t \beta \) enables \( t' \) and hence \( \alpha t \delta t' \) is fireable. The Lemma is therefore true for \( L(t, \delta, t') = n+1 \) and so by induction it is true for all \( n \). \( \square \)

Lemma 3.10. Let \( P \) be a conflict-free Petri net. Let \( \alpha t t' \) be transitions which occur infinitely often in the \( \omega \)-firing sequence \( \alpha \). Then there is an \( \omega \)-firing sequence \( \alpha' \) in which \( t \) and \( t' \) occur infinitely often where

\[
\alpha' = \gamma t \beta_1 t_1 \gamma_1 t \beta_2 t_2 \gamma_2 t_3 \ldots
\]

and \( t \) does not occur in \( \gamma \), \( \beta_i \) and \( \gamma_i \) (i\( \in \mathbb{N} \)).

Proof. The sequence can be written in the form

\[
\alpha = \gamma t \delta t' \tilde{\alpha}
\]

where \( \gamma \) does not contain \( t \) and \( \delta \) does not contain \( t' \). The sequence \( \alpha' \) is constructed by successive applications of Lemma 3.1 and 3.9.

Begin by applying Lemma 3.9 to obtain a fireable sequence \( \gamma t \delta t' \beta \) where \( \beta \) does not contain \( t \). By Lemma 3.1, the \( \omega \)-sequence \( \sigma \) given by

\[
\sigma = \gamma t \delta t' \cdot (\gamma t \delta t' \gamma t \delta t') \tilde{\alpha}
\]

is fireable where \( \tau \) does not contain \( t \) and \( \xi \) does not contain \( t' \). Now repeat the above procedure using \( t \xi t' \) in \( \sigma \). The iteration of this procedure yields the required sequence \( \alpha' \). \( \square \)
4. Persistent Nets

In this section we show that the set of reachable markings of a persistent Petri net is semi-linear. In the following, let \( P = \langle P, T, A, M_0 \rangle \) be a fixed but arbitrary persistent Petri net with \( k \) transitions and \( \lambda \) places. For convenience, we introduce an extended Parikh map, \( \text{EPK}: T^+ \rightarrow N^{k+2} \). Let \( \Sigma: T^+ \rightarrow N^{\lambda} \) be defined by \( \Sigma(\theta) = (m_1, \ldots, m_\lambda) \), if \( \theta \) is a fireable firing sequence and \( m_i \) is the change in the number of tokens in the place \( p_i \) as a result of firing \( \theta \). Then, for a fireable sequence \( \theta \),

\[
\text{EPK}(\theta) = \text{PK}(\theta) \times \Sigma(\theta).
\]

\( \text{EPK}(\theta) \) is a \( k + \lambda \)-tuple whose coordinates give the number of occurrences of each transition in \( \theta \) and the change in the marking of each place which results from firing \( \theta \). Recall that for \( b, c \in N^{k+2} \), \( b \preceq c \) if every coordinate of \( b \) is \( \leq \) the corresponding coordinate of \( c \).

**Definition.** For the marking \( v \in N^\lambda \), let \( S_v \) be the set of points \( b \in N^{k+2} \) for which there is a fireable \( \theta \in T^+ \) satisfying \( \text{EPK}(\theta) = b \) and \( \Sigma(\theta) \preceq 0 \).

**Definition.** For each marking \( v \in N^\lambda \), let \( F_v \) be the set of minimal points in \( S_v \). I.e.,

1. \( F_v \subseteq S_v \)
2. If \( b_1, b_2 \in F_v \), then \( b_1 \) and \( b_2 \) are incomparable (\( b_1 \not\preceq b_2, b_2 \not\preceq b_1 \))
3. For each \( c \in S_v \), there is a \( b \in F_v, b \preceq c \).

**Lemma 4.1.** Each \( F_v \) is finite.

**Proof.** By Koenig's infinity Lemma, every set of pairwise incomparable vectors on \( N^{k+2} \) is finite. Hence each \( F_v \) is finite. \( \Box \)

**Lemma 4.2.** For \( u, v \in N^\lambda \), if \( u \preceq v \), then \( (\text{Vax} F_u)(\text{Vax} F_v)[\text{Vax} b]a \).

**Proof.** Let \( a \in F_v \). Then \( \text{EPK}(\theta) = a \) for some \( \theta \) fireable at \( v \), but \( \theta \) is also fireable at \( u \) so either \( a \in F_u \) or some \( b < a \) is in \( F_u \). \( \Box \)

Observe that \( F_u \) can only contain points that are smaller than or incomparable with the points in \( F_v \). This is true because when the marking is increased from \( v \) to \( u \), all previously enabled sequences remain enabled while some additional sequences may become enabled.

**Lemma 4.3.** \( F = \cup_{v \in S} F_v \) is finite.

**Proof.** Let \( MF \) be the set of minimal points of \( F \). Koenig's Infinity Lemma implies that \( MF \) is finite. Hence there is a finite \( S \subseteq N^\lambda \) such that \( MF \subseteq \cup_{v \in S} F_v \).

Let \( w = \max S \) (componentwise maximum). We first show that \( MF = F_w \).

1. \( MF \subseteq F_w \): Let \( a \in MF \). Then \( a \in F_v \) for some \( v \in S \). Since \( w \preceq v \), by Lemma 4.2, there is a \( b \in F_w, b \preceq a \). But \( b < a \) is impossible because \( a \in MF \). Therefore \( a = b \) so \( a \in F_w \).

2. \( F_w \subseteq MF \): Let \( a \in F_w \). Since \( MF \) is the minimal set for \( \cup_{v \in S} F_v \), there is a \( b \in MF \) such that \( b \preceq a \). But then \( b \in F_v \) for some \( v \in S \). Since \( w \preceq v \), there is a \( c \in F_w, c \preceq b \).

Combining the above we get \( c \preceq b \preceq a \) where \( a, c \in F_w, b \in MF \). But \( F_w \) is a set of minimal points so \( a \preceq b \preceq c \) and \( a \in MF \).

Because \( F_w = MF \), \( u \preceq w \) implies that \( F_u = MF \). Hence \( F = \cup_{v \in S} F_v = \cup_{v \in S} F_v \)

so \( F \) is finite by Lemma 4.1. \( \Box \)

The next Lemma shows that the effect of any firing sequence can be achieved by concatenating canonical sequences whose images under EPK are in the finite set \( F \).
Lemma 4.4. Let \( \sigma \in T^* \) be fireable at some marking \( v \in \mathbb{N}^S \) and assume \( \Sigma(\sigma) > 0 \). Then there is a sequence \( \xi_1 \ldots \xi_r, r \geq 1 \), \( \xi_i \in T^* \) which satisfies

1. \( \Sigma(\xi_i) \in F \quad 1 \leq i \leq r \)
2. \( \Sigma(\xi_1 \ldots \xi_r) = \Sigma(\sigma) \)
3. \( \xi_1 \ldots \xi_r \) is fireable at \( v \).

Call \( \xi_1 \ldots \xi_r \) a decomposition of \( \sigma \).

Proof. The proof is by induction on the length of \( \sigma \).

If \( |\sigma| = 1 : EPK(\sigma) \) is minimal so \( \sigma = \xi_1 \).

Assume the result holds for sequences of length \( < n \) and let \( |\sigma| = n \). If \( EPK(\sigma) \in F \), then \( EPK(\sigma) \in F \) and we are done. If \( EPK(\sigma) \notin F \), then there is some \( \eta \) fireable at \( v \) such that \( EPK(\eta) \in F \) and \( EPK(\eta) < EPK(\sigma) \) (\( F \) is the set of minimal points in \( \mathbb{N}^{k+2} \) corresponding to sequences that are fireable at \( v \) and which do not decrease the marking.). Because the net is persistent, Lemma 3.1 implies that there is a \( \delta \in T^* \) such that \( \eta \delta \) is fireable at \( v \) and \( EPK(\delta) = EPK(\eta \delta) \). Since \( \eta \delta \) is fireable at \( v \), \( \delta \) is fireable at \( v \). Also \( EPK(\eta) < EPK(\sigma) = EPK(\eta \delta) \) and \( \Sigma(\sigma) > 0 \) implies \( \Sigma(\delta) > 0 \). Since \( |\delta| < n \), the induction hypothesis gives a decomposition \( \xi_1 \ldots \xi_r \) for \( \delta \). But then \( \eta \xi_1 \ldots \xi_r \) is the required decomposition of \( \sigma \).

Theorem 4.5. The set of reachable markings of a persistent Petri net is semi-linear.

Proof. Get \( G = \{\xi_1, \ldots, \xi_n\} \) be the set of all firing sequences \( \sigma \) which satisfy \( EPK(\sigma) \in F \). By Lemma 4.3, \( G \) is finite. For each \( S = \{\xi^S_1, \ldots, \xi^S_r\} \subseteq G \), let \( A_S \) be the set of points \( b \in \mathbb{N}^{k+2} \) such that \( 1) b = (PK(\sigma), M) \) for some fireable firing sequence \( \sigma \) where \( M_0 \subseteq M \) and \( 2) \) every sequence in \( S \) is fireable at \( M \).

Let \( B_S \) be the finite set of minimal points of \( A_S \). Let \( M_S \) be the finite set of markings which are associated with the points of \( B_S \). We claim that the set of reachable markings is semi-linear and may be defined by

\[
R_S = \bigcup_{S \subseteq G} \bigcup_{M \in M_S} \{ M + \Sigma c_i \Sigma(\xi^S_i) | c_i \geq 0, 1 \leq i \leq r \}.
\]

First show that every marking in \( R \) is reachable. Fix \( S \subseteq G, M \in M_S \) and \( R = M + \Sigma c_i \Sigma(\xi_i^S) \) for \( i \in [1, r] \).

By definition, \( M \) is a reachable marking. Moreover, every member of \( S \) is fireable at \( M \).

Let \( \xi_i \in S \) be an arbitrary number of times from \( M \). Thus \( R \) is a reachable marking.

Let \( R \) be an arbitrary reachable marking and show that \( \sigma \in R \).

Let \( S \) be the set of sequences in \( G \) that are fireable at \( \sigma \).

Then \( \sigma \) is part of a point \( a = (PK(\gamma), R) \) in \( A_S \). If \( R \in B_S \), then we are done. If not, then there is an \( M' \subseteq M \) such that \( M' \subseteq M \) is part of a minimal \( b = (PK(\sigma), M') \), \( b < a \), where \( M_0 \subseteq M \) and \( M_0 \subseteq M' \) with \( PK(\sigma) \times M' \subseteq B \). Then \( PK(\sigma) \subseteq PK(\gamma) \) and because the net is persistent, Lemma 3.1 implies that there is a fireable \( \tau \) such that \( PK(\tau) = \max(PK(\sigma), PK(\gamma)) \). Hence \( M_0 \subseteq M' \subseteq M \) a \( \tau \). But \( M' \subseteq M \) implies that \( \Sigma(\tau) > 0 \) so by Lemma 4.4, there is a sequence \( \delta \in S^* \) such that \( M' \subseteq \delta \subseteq M' \).

This implies that

\[
R_S = \bigcup_{S \subseteq G} \bigcup_{M \in M_S} \{ M + \Sigma c_i \Sigma(\xi^S_i) \},
\]

for some \( c_i \geq 0 \) where \( S = \{\xi_i^1, \ldots, \xi_i^r\} \) so \( \sigma \in R \).
Theorem 4.5 shows that the set of reachable markings of a persistent net is semi-linear. We believe that this indicates why persistent nets have been more tractable than arbitrary Petri nets. In Section 6 we give examples which indicate that only a "small amount" of non-persistence allows us to simulate arbitrary Petri nets.

5. Conflict Free Nets

Recall that a Petri net is conflict free if each place $p$ of the net either is an input for at most one transition or all such transitions are on self loops with $p$. In this section, we show that for conflict-free nets the maximum number of tokens that any place can receive is either unbounded or is linearly bounded by the size of the initial marking. I.e., there is a constant $c$ such that for an arbitrary initial marking with $x$ tokens, the maximum marking of each place is either unbounded or is bounded by $cx$. The proof of this theorem yields an exponential time algorithm for deciding whether a conflict free net is bounded. The best known algorithm for deciding boundedness of arbitrary nets [3] requires a computing time which is bounded above by Ackermann's function.

Our first result relates persistent and conflict free nets.

Theorem 5.1. A Petri net is conflict free if and only if it is persistent for all initial markings.

Proof. $\Rightarrow$ obvious

$\Leftarrow$ Assume that a Petri net is not conflict free.

Pick any two transitions $t_1$ and $t_2$ having an input place $p$ in common where $t_1$ is not on a self loop with $p$. Put 1 token on each input place of $t_1$ and 1 token on each input place of $t_2$ which is not an input place of $t_1$. No other places are given tokens. Then both $t_1$ and $t_2$ are fireable but the firing of $t_1$ disables $t_2$ so the net is not persistent. 

Definition. A transition $t$ of a Petri net is $\omega$-linearly bounded if there is a constant $c$ such that for any initial marking having a total of $x$ tokens, either: 1) $t$ occurs an unbounded number of times in fireable firing sequences or 2) $t$ occurs at most $cx$ times in fireable firing sequences. A place $p$ of a Petri net is $\omega$-linearly bounded if there is a constant $c$ such that for any initial marking having a total of $x$ tokens, either $p$ is unbounded or $M(p) \leq cx$ for any reachable marking $M$. 


Lemma 5.2: Every transition of a conflict free Petri net is \(\omega\)-linearly bounded.

Proof. The proof is by induction on the number of arcs from places to transitions. Let \(P\) be a conflict free Petri net. If \(P\) has no arcs from places to transitions, then the result is obvious, with \(c = 1\) since each transition can fire an unbounded number of times.

Assume the theorem holds for all conflict free nets with \(n\) arcs as above and let \(P\) be a conflict free net with \(n + 1\) arcs from places to transitions. Let \(t\) be an arbitrary transition of \(P\). If \(t\) has no input places, the result is immediate. Assume that \(p\) is an input place of \(t\) in \(P\). By the induction hypothesis, the theorem is true for the conflict free net \(P'\) obtained from \(P\) by deleting the arc from \(p\) to \(t\). Let \(c\) be a constant which \(\omega\)-linearly bounds the firings of the transitions of \(P'\) and let the initial marking of \(P\) have \(x\) tokens. We must show that the transition \(t\) of \(P\) is \(\omega\)-linearly bounded. Because of the induction hypothesis, there are two cases:

Case 1. If \(t\) could fire at most \(c_1\) times in \(P'\), then \(t\) can fire at most \(c_1\) times in \(P\).

Case 2. Assume \(t\) could fire an unbounded number of times in \(P'\). If each of the \(c_1\) transitions having \(p\) as an output place could fire at most \(c_1\) times in \(P'\), then the same is true of these transitions in \(P\). Then \(p\) receives at most \((c_1 + 1)x\) tokens in \(P\) so \(t\) can fire at most \((c_1 + 1)x\) times in \(P\). Assume that some transition \(t'\) having \(p\) as an output place could fire an unbounded number of times in \(P'\). By Lemma 5.3, \(t\) and \(t'\) occur infinitely often in some \(\omega\)-firing sequence \(\alpha\) of \(P'\). By Lemma 5.10, \(\alpha\) can be chosen so that

\[\alpha = \delta t_1 \gamma_1 t_2 \delta_2 t_2 \gamma_3 \ldots\]

where \(t\) does not occur in \(\delta, \gamma_1\) and \(\delta_1 \{i \geq 1\}\). Let \(\alpha(k)\) be the initial segment of \(\alpha\) up to and including the \(k\)-th occurrence of \(t\). There are two subcases:

Subcase a: If \(t\) cannot fire in \(P\), then we are done.

Subcase b: Let \(\beta\) be the shortest fireable sequence (possibly empty) of \(P\) which enables \(t\). Since \(\beta\) is fireable in \(P\), it is also fireable in \(P'\). But then by Lemma 5.1, for any \(k\), \(\sigma_k = \beta \cdot (\alpha(k) \cdot \beta)\) is fireable in \(P'\) and contains \(k\) occurrences of \(t\). The \(i\)-th occurrence of \(t\) in \(\sigma_k\) is preceded by \(\beta\) plus at least \(i - 1\) occurrences of \(t\) so before the \(i\)-th firing of \(t\) in \(\sigma_k\) (in \(P'\)), (in \(P'\)), \(p\) contains at least \(i\) tokens. Thus implies that \(\sigma_k\) is fireable in \(P\) because \(P\) is obtained from \(P'\) by adding an arc from \(p\) to \(t\). Hence in this case, \(t\) is unbounded in \(P\).

Definition. A loop in a Petri net is a non-trivial path, all of whose arrows point in the same direction, which begins and ends with the same place. A loop is unbounded if at least one place on the loop is unbounded. A transition is unbounded if it occurs an unbounded number of times on fireable firing sequences. A loop is immortal if every transition on the loop is unbounded.

Lemma 5.3. Consider a persistent net and a place \(p\) such that \(p\) is only on unbounded, immortal loops. Further assume that \(p\) is on such a loop. Then \(p\) is unbounded.

Proof. Assume that \(p\) is bounded. By Theorem 4.5, for each \(i \geq 1\), there is a finite firing sequence which marks each unbounded place of the net with at least \(i\) tokens. Let \(k_i\) be the maximum \(p\) marking for such sequences. Then let \(k\) be the maximum value which occurs infinitely often among \(k_i\) and let \(\alpha(\delta_1, (i \geq 1,i \leq j < \ldots)\) be a finite firing sequence which marks each unbounded place with \(i\) tokens and \(p\) with \(k\) tokens. Choose a subset \(\{\beta_1, \beta_2, \ldots\}\) of \(\alpha(\delta_1, (i \geq 1,i \leq j < \ldots)\) such that \(M(i+1) > M_i\) for \(i \geq 1\), where \(M_i\) is the marking after \(\beta_i\) has fired. Since \(p\) is on an immortal loop, by Lemma 5.3, there is an extension of \(\beta_1\) which fires a transition \(t'\) which has \(p\) as an output place. Because \(M(i+1) > M_i\) for \(i \geq 1\), the same extension can be appended to each \(\beta_i\). But the definitic of \(k\) requires that (after \(\beta_i\) has fired) some \(t\), having \(p\) as an input place must be fired before \(t'\) is fired. This means that \(p\)
is on a loop none of whose places, except \( p \), contains a token after 
\( \beta_1 \) has fired. (If some place, other than \( p \), of every loop that contains 
\( p \) has a token, then \( t' \) need not be fired to enable \( t' \).) Because \( p \) is only on unbounded loops, this contradicts the definition of \( \beta_1 \).
Therefore, \( p \) is unbounded. \( \Box \)

**Theorem 5.4.** Each place of a conflict-free Petri net is \( \omega \)-linearly bounded.

**Proof.** Let \( p \) be an arbitrary place of the net. Let \( c \) be the 
condant given by Lemma 5.2 which \( \omega \)-linearly bounds the transitions 
of \( P \). Assume that the initial marking of \( P \) has \( x \) tokens. There 
are four cases:

1. If all transitions into \( p \) are bounded, then the maximum marking 
on \( p \) is \( (r+1)c x \) where there are \( r \) arrows into \( p \).
2. If at least one transition into \( p \) is unbounded and is not on 
an immortal loop with \( p \), then \( p \) is unbounded.
3. If all unbounded transitions into \( p \) are only on immortal, 
unbounded loops with \( p \) and there is at least one such loop, 
then \( p \) is unbounded by Lemma 5.3.
4. Assume that some unbounded transition into \( p \) is on a bounded, 
immortal loop with \( p \). Then \( p \) is bounded by the definition of 
boundedness for loops. The only way tokens can be added to the 
places of a loop in a conflict-free net is by the firing of a 
transition not on the loop one of whose output places is on the 
loop. Moreover, the number of tokens on such a loop can never 
decrease. But then, because the loop is bounded, every transition 
into the loop may fire at most \( cx \) times so the maximum marking 
\( p \) can receive is \((r+1)x \) where \( r \) is the number of such transitions. \( \Box \)

A careful analysis of the proofs of Lemmas 5.2 and 5.3 and
and Theorem 5.4 yields the following:

**Theorem 5.5.** There is an exponential time (in the size of the net) 
algorithm for deciding whether an arbitrary conflict-free net is bounded.

Theorem 5.5 provides a substantial improvement over Karp and 
Miller's [3] algorithm for arbitrary nets which has Ackermann's function 
as an upper bound.
6. Examples and Conclusion

In this section we give some examples which illustrate the importance of persistence to the study of Petri nets. We also pose some open problems for future study.

Van Leeuwen's [9] reachability proof for 3-coordinate vector addition systems involves showing that such systems have semi-linear sets of reachable points. Since his methods do not generalize, the question arises as to how many coordinates (or places in a Petri net) are needed to generate non-semi-linear sets. Figure 3 gives a Petri net with 3 places whose reachability set is not semi-linear.

![Figure 3](image)

Any reachable marking $M$ of the net of Figure 3 satisfies:

1. $M(a) + M(b) = 1$
2. $M(c) + M(d) =$ number of firings of 2 and 4
3. $M(e) =$ total number of transitions fired

We claim that if $M(c) + M(d) = n$, then the maximum value $M(e)$ can have is $\sum_{i=1}^{n+1} i - 1 = (n+1)(n+2)/2 - 1$. This is achieved as follows:

Hence the set of reachable markings is not semi-linear. Notice that the net is conflict free (persistent) except for places $a$ and $b$. Our next example will show that no additional non-persistence is needed to achieve any set of reachable markings.

The net of Figure 3 translates to a vector addition system with seven coordinates. The two additional coordinates are needed because vector addition systems do not have self loops. An interesting problem, given Van Leeuwen's results, is to determine how many places (coordinates) are needed to generate non-semi-linear sets.

The next three constructions illustrate the central role that persistence plays in Petri nets. We first show that, in a very restricted sense, reachability is reducible to persistence. Together with the results of [10], this proves that persistence (in the restricted sense) is equivalent to reachability. It is important to note that this does not solve the difficult open problem regarding the equivalence of persistence and reachability. The paper concludes with a construction that shows that only a minimal amount of non-persistence is required to achieve any reachable set of markings.

Two transitions $t_1$ and $t_2$ are non-persistent if for some reachable marking $M$, $M \xrightarrow{t_1} M \xrightarrow{t_2}$ but not $M \xrightarrow{t_1t_2}$.

Let $P$ be an arbitrary Petri net. Let $p$ be an arbitrary place of $P$. We show that if the non-persistence of two arbitrary transitions in a Petri net is decidable, then we can decide whether $P$ has

---

A:Fire 2
Fire 3 until $c$ has no tokens
Fire 4
Fire 1 until $d$ has no tokens
Goto A

The statement of this result was communicated to us by M. Hack. The construction given is ours.
a reachable marking $M$ such that $M(p) = 0$. To accomplish this, modify $P$ as follows (Figure 4)

1. For each $t_i \in T$, add a new place $p_i$. Add $p_i$ as an input place and as an output place of $t_i$. Initially $p_i$ contains one token.

2. Add new transitions $t, \tilde{p}, \tilde{t}$ and new places $\tilde{p}, \tilde{t}$. Let $\{p_i \mid t_i \in T\}$ be the input set of $t$, and let $\{\tilde{p}, \tilde{t}, p\}$ be the output set of $t$. Let $\tilde{t}(t)$ have input places $\tilde{p}$ and $p$ ($p$ and $\tilde{p}$).

Then $\tilde{t}$ and $\tilde{p}$ are non-persistent iff $M(p) = 0$ for some reachable marking $M$.

Let $P$ be a Petri net which has $r$ non-persistent transition where $2^{m-1} < r \leq 2^m, r > 6$. Figure 5 shows how the number of non-persistent transitions can be reduced to $2m$. By iterating this procedure, a new net is obtained with at most six non-persistent transitions ($m$ non-persistent places).

The net $P'$ obtained as in Figure 5 has additional places not occurring in $P$. However, if a marking $M$ is reachable in $P'$, then the restriction of $M$ to the places of $P$ is reachable in $P$. Conversely, if $M$ is reachable in $P$, then some extension of $M$ is reachable in $P'$. Similar results are true of fireable firing sequences.

Consider Figure 5 where $v_0, \ldots, v_2^m$ are the non-persistent transitions of the original net. Add transitions $t_0^0, t_1^0, t_2^0, \ldots, t_m^0, t_0^1, t_1^1, t_2^1, \ldots, t_m^1$ and places $p_0^0, p_1^0, p_2^0, \ldots, p_m^0, p_0^1, p_1^1, p_2^1, \ldots, p_m^1$ as shown in Figure 4 with each of $a_1^0, \ldots, a_m^0$ initially having one token. Now one of $t_i^0$ or $t_i^1$ (for $i = 1, \ldots, m$) can fire, placing a token in $t_i^0$ or $t_i^1$ (for $i = 1, \ldots, m$) respectively. Assume each $v_i^j$ is indexed by an $m$ place binary number.
The firing of the combination of the $t_i$'s whose superscripts correspond to this binary number will permit $v_j$ to fire (when all inputs to $v_j$ in the original net have tokens). After $v_j$ fires, tokens are again placed in $a_1, \ldots, a_m$. Note that at most one of the $v_i$'s can be enabled at any time and that that proper choice of $t_i$'s will permit the enabling of any particular $v_j$. Hence the only non-persistent transitions in the modified net are the $t_i$'s which have been added.
Figure 6 illustrates how the six non-persistent transitions can be further reduced to two. The transitions $t_0, t_1, \ldots, t_3$ are the non-persistent transitions which result from iterating the process illustrated in Figure 5. The places $a_1, a_2$, and $a_3$ are as introduced in the last step of this process. One of $t_0^0$ or $t_1^1$ must fire before $t_0^1$ or $t_1^2$ can fire. Similarly $t_0^1$ or $t_1^2$ must fire before $t_0^2$ or $t_1^3$ can fire. Since only one of $d_0$ or $d_1$ can contain a token, only one of $t_i^0$ or $t_i^1$ ($1 \leq i \leq 3$) can be enabled at any particular time. Therefore the $t_i$'s can no longer lead to non-persistence. Only $u_0$ and $u_1$ are not persistent.

The results of this section show that all reachable sets (modulo a projection) can be realized by nets with only two non-persistent transitions. This is particularly interesting in light of the results of Section 4 that persistent nets only generate semi-linear sets. We believe that a thorough understanding of persistence will be necessary if the reachability problem is to be solved. Some interesting (and difficult) open problems are:

1. Is reachability decidable for persistent nets?
2. Is there an algorithm which decides whether an arbitrary net is persistent?
3. Is reachability decidable for arbitrary nets?
4. It has been shown that persistence is reducible to reachability [10]. Is the converse true?
5. Can Lipton's exponential time lower bound for the reachability problem be improved?
6. Can the Karp-Miller algorithm for deciding boundedness be improved?

Acknowledgments: We gratefully acknowledge several stimulating conversations with P. S. Thiagarajan and in particular his suggestion that semi-modular lattices might be important in studying persistent Petri nets.

References

5. Lipton, R. J. The reachability problem and the boundedness problem for Petri nets are exponential-space hard, Conf. on Petri Nets & Related Methods, MIT, July, 1975.