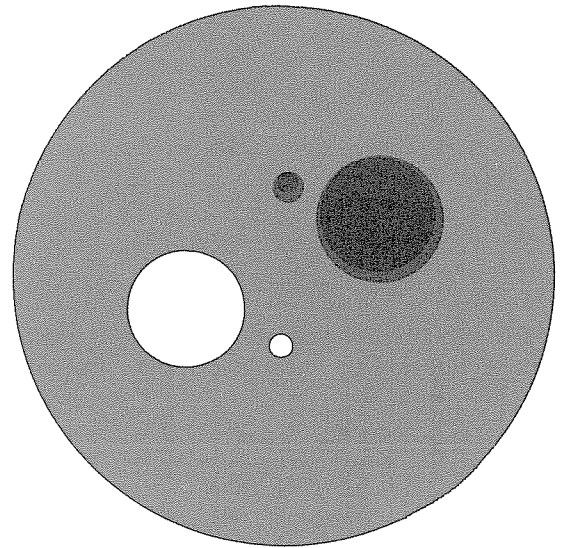


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SOLUTION OF LINEAR COMPLEMENTARITY PROBLEMS BY LINEAR PROGRAMMING

by

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ABSTRACT

The linear complementarity problem is that of finding an $n \times 1$ vector z such that,

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz+q) = 0$$

where M is a given $n \times n$ real matrix and q is a given $n \times 1$ vector. In this paper the class of matrices M for which this problem is solvable by a single linear program is enlarged to include matrices other than those that are Z -matrices or those that have an inverse which is a Z -matrix. (A Z -matrix is real square matrix with nonpositive offdiagonal elements.) Included in this class are other matrices such as nonnegative matrices with a strictly dominant diagonal and matrices that are the sum of a Z -matrix having a nonnegative inverse and the tensor product of any two positive vectors in R^n .

1. INTRODUCTION

We consider the linear complementarity problem of finding a z in R^n such that

$$(1) \quad Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz+q) = 0$$

where M is a given real $n \times n$ matrix and q is a given vector in R^n . Many problems of mathematical programming such as linear programming problems, quadratic programming problems and bimatrix games

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can be reduced to the above problem [4]. In addition some free boundary problems of fluid mechanics can be reduced to the solution of a linear complementarity problem [5,6,7]. The purpose of this paper is to extend the class of the matrices M for which the linear complementarity problem (1) can be solved by solving the single linear program

$$(2) \quad \text{minimize } p^T z \quad \text{subject to } Mz + q \geq 0, \quad z \geq 0$$

for an easily determined p in R^n . In [10] it was shown that for cases including those when M or its inverse is a Z -matrix, that is a real square matrix with nonpositive offdiagonal elements, the linear complementarity problem (1) can be solved by solving the linear program (2) for a certain p . In Section 2 of this paper we sharpen the main result of [10] by giving in Theorem 1 a characterization for the key condition which insures the solvability of the linear complementarity problem (1) by the linear program (2). Theorem 2 is a specific realization of Theorem 1 which has been given previously [10] in a slightly different form.

In Section 3 of the paper we extend further the class of linear complementarity problems solvable by a single linear program by considering an equivalent linear complementarity problem (7) with slack variables and by employing the results of Section 2. We obtain extensions which include cases such as when M is a nonnegative matrix with a strictly dominant diagonal or when M is the sum of a K -matrix, that is a Z -matrix having a nonnegative inverse, and the tensor product of any two positive vectors in R^n . A tabular summary of some of the linear complementarity problems solvable by a linear program is given at the end of the paper.

2. SOLUTION OF LINEAR COMPLEMENTARITY PROBLEMS BY LINEAR PROGRAMMING

In this section we shall characterize classes of matrices for which the linear complementarity problem (1) can be solved by solving the linear program (2). We begin by stating the dual to problem (2)

$$(3) \quad \text{maximize } -q^T y \quad \text{subject to } -M^T y + p \geq 0, \quad y \geq 0$$

and establishing the following key lemma [10] which insures that, under suitable conditions, any solution of the linear program (2) also solves the linear complementarity problem (1).

Lemma 1 If z solves the linear program (2) and if an optimal dual variable y satisfies

$$(4) \quad (I - M^T)y + p > 0$$

where I is the identity matrix, then z solves the linear complementarity problem (1).

Proof: $y^T(Mz+q) + z^T(-M^T y+p) = y^T q + z^T p = 0$.

Since $y \geq 0$, $Mz + q \geq 0$, $z \geq 0$ and $-M^T y + p \geq 0$ it follows that

$$y_i(Mz+q)_i = 0, z_i(-M^T y+p)_i = 0 \quad i = 1, \dots, n$$

where subscripted quantities denote the i th element of a vector. But $y_i + (-M^T y+p)_i > 0$, $i = 1, \dots, n$, hence either $y_i > 0$ or $(-M^T y+p)_i > 0$, $i = 1, \dots, n$, and consequently $(Mz+q)_i = 0$ or $z_i = 0$, $i = 1, \dots, n$. \square

We give now a necessary and sufficient condition for the satisfaction of the key inequality (4) of Lemma 1.

Theorem 1 Let the set $\{z | Mz+q \geq 0, z \geq 0\}$ be nonempty. A necessary and sufficient condition that the linear program (2) have a solution z with each optimal dual variable y satisfying (4) is that M , q and p satisfy

$$(5) \quad \begin{aligned} MZ_1 &= Z_2 + qc^T \\ M^T X &\leq pc^T \\ p^T Z_1 &> q^T X \\ X &\geq 0, c \geq 0, (Z_1, Z_2) \in Z \end{aligned}$$

and

$$(6) \quad p = r + M^T s, r \geq 0, s \geq 0$$

where X, Z_1, Z_2 are $n \times n$ matrices, c, r, s are vectors in R^n , and Z is the set of square matrices with nonpositive offdiagonal elements. If conditions (5) and (6) are satisfied then there exists at least one solution of the linear program (2), and each such solution solves the linear complementarity problem.

Proof: The existence of $(r, s) \geq 0$ satisfying (6) is a necessary and sufficient condition for dual feasibility, which in turn is a necessary and sufficient condition that the feasible linear program (2) possess a solution. That each optimal dual variable y must satisfy (4), is equivalent to the system

$$\begin{aligned} Mz + q\zeta &\geq 0, z \geq 0, -M^T y + p\zeta \geq 0, y \geq 0, p^T z + q^T y \leq 0, \zeta > 0 \\ (M^T - I)_i y - p_i \zeta &= 0 \end{aligned}$$

not having a solution (z, y, ζ) in R^{2n+1} for each $i = 1, \dots, n$, where $(M^T - I)_i$ denotes the i th row of $M^T - I$. By Motzkin's transposition theorem [9] this in turn is equivalent to the existence of n -vectors

c and d , and $n \times n$ matrices X, Y, U, V and D , where D is diagonal, satisfying

$$M^T X + U - pc^T = 0$$

$$-MY + V - qc^T + (M-I)D = 0$$

$$q^T X + p^T Y + d^T - p^T D = 0$$

$$(X, Y, U, V) \geq 0, c \geq 0, d > 0$$

By defining $Z_1 = D - Y$, $Z_2 = D - V$ these conditions become conditions (5).

The last statement of the theorem follows from Lemma 1. \square

By taking $X = 0$, and $c = 0$ in (5) we obtain the following theorem which is equivalent to Theorem 1 of [10].

Theorem 2. Let the set $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty, and let M and p satisfy

$$MZ_1 = Z_2, p^T Z_1 > 0, (Z_1, Z_2) \in Z$$

$$p = r + M^T s, r \geq 0, s \geq 0$$

Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2).

Useful special cases are obtained by setting $Z_1 = I$, $p = e$ and $Z_2 = I$, $p = M^T e$, where e is any positive vector and in particular it may be a vector of ones. In the first case we have that $M = Z_2 \in Z$, $p = e$, and in the second case that $M^{-1} = Z_1 \in Z$, $p = M^T e$. Other methods for solving (1) for Z -matrices and other related matrices are given in [1,2,3,6,11,12,13,14].

In order to enlarge further the class of matrices for which the linear complementarity problem can be solved by a linear program we consider a complementarity problem with slack variables which is equivalent to problem 1.

3. SOLUTION OF SLACK LINEAR COMPLEMENTARITY PROBLEMS BY LINEAR PROGRAMMING

We consider the following linear complementarity problem with a slack variable z_0 in R^m

$$(7) \quad \begin{bmatrix} w \\ w_0 \end{bmatrix} = \begin{bmatrix} M & A \\ 0 & B \end{bmatrix} \begin{bmatrix} z \\ z_0 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} \geq 0, \begin{bmatrix} z \\ z_0 \end{bmatrix} \geq 0, z^T w + z_0^T w_0 = 0$$

where A is an $n \times m$ matrix and B is an $m \times m$ matrix.

Lemma 2. Let B be a strictly copositive or conegative matrix, that is $x^T M x \neq 0$ whenever $0 \leq x \neq 0$. Then z solves the linear complementarity problem (1) if and only if $(z, z_0=0)$ solves the slack linear complementarity problem (7).

Proof: Obviously if z solves (1) then $(z, z_0=0)$ solves (7). If (z, z_0) solves (7) then since $0 = z_0^T w_0 = z_0^T B z_0$, $z_0 \geq 0$, and B is strictly copositive or conegative, then $z_0 = 0$ and z solves (1). \square

By combining this lemma with Theorem 2 we can extend the class of matrices for which a linear program solves the linear complementarity problem.

Theorem 3. Let the set $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty, and suppose there exist $Z_1, Z_2, Z_3, A, G, H, p$ and p_0 satisfying

$$(8) \quad MZ_1 = Z_2 + AG, \quad MH \geq AZ_3 \quad (Z_1, Z_2, Z_3) \in Z, \quad (G, H) \geq 0$$

$$(9) \quad (p^T \quad p_0^T) \begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix} > 0 \quad (p, p_0) \geq 0$$

where the dimensionalities of Z_1, Z_2, Z_3, A, G, H , and p_0 are respectively, $n \times n$, $n \times n$, $m \times m$, $n \times m$, $m \times n$, $n \times m$ and $m \times 1$. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2).

Proof: Set $B = I$ in problem (7) and apply Theorem 2 to it. In particular we have from (8) and (9) that

$$\begin{bmatrix} M & A \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix} = \begin{bmatrix} Z_2 & -\bar{H} \\ -G & Z_3 \end{bmatrix}$$

$$(p^T \quad p_0^T) \begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix} > 0$$

where \bar{H} is an $n \times m$ nonnegative matrix and

$$\begin{pmatrix} p \\ p_0 \end{pmatrix} = \begin{pmatrix} r \\ r_0 \end{pmatrix} \geq 0 \quad p_0 \in R^m, \quad r_0 \in R^m$$

Hence by Theorem 2 the slack linear complementarity problem (7) has a solution which can be obtained by solving the linear program

$$\text{minimize } p^T z + p_0^T z_0 \quad \text{subject to } Mz + Az_0 + q \geq 0, \quad (z, z_0) \geq 0$$

But since each solution of this linear program solves (7) it follows that $z_0 = 0$ at each solution of this linear program and hence we can set $z_0 = 0$ which reduces this linear program to (2). \square

We observe that a sufficient condition for the inequality (9) to

to hold is that $\begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix}^{-1} \geq 0$. In fact this condition is also

necessary for (9) to hold because the nonnegativity of the inverse \bar{Z}^{-1} of a Z-matrix \bar{Z} is equivalent to the existence of $p \geq 0$ such that $p^T \bar{Z} > 0$ [8, Theorem 4,3]. Z-matrices with nonnegative inverses are called K-matrices [8] and sometimes M-matrices. The set of all K-matrices is denoted by K. By making use of these facts we can obtain the following consequence of Theorem 3.

Theorem 4. Let the set $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty, and let M satisfy

$$(10) \quad M = (Z_2 + AG)Z_1^{-1}, \quad MH \geq AZ_3, \quad (Z_1, Z_2, Z_3) \in Z, \quad (G, H) \geq 0$$

$$(11) \quad Z_1^{-1} \geq 0, \quad (Z_3 - GZ_1^{-1}H)^{-1} \geq 0$$

Then there exists $(p, p_0) \in R^{n+m}$ satisfying (9) and the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2).

Proof: We will show that the conditions of Theorem 3 hold and hence (1) has a solution and is solvable by the linear program (2). We have that

$$\begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix}^{-1} = \begin{bmatrix} Z_1^{-1}(I + HC^{-1}GZ_1^{-1}) & Z_1^{-1}HC^{-1} \\ C^{-1}GZ_1^{-1} & C^{-1} \end{bmatrix}$$

where $C = Z_3 - GZ_1^{-1}H$. It follows from $Z_1^{-1} \geq 0, C^{-1} \geq 0, H \geq 0$ and

$$G \geq 0 \quad \text{that} \quad \begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix}^{-1} \geq 0 \quad \text{and that} \quad (p^T \ p_0^T) = e^T \begin{bmatrix} Z_1 & -H \\ -G & Z_3 \end{bmatrix}^{-1} \geq 0,$$

where e is any positive vector in R^{n+m} , satisfies (9). Conditions (8) follows from (10). \square

By setting $Z_1 = I$ in the above theorem and defining $Z_4 = Z_3 - GH$ we obtain the following theorem.

Theorem 5. Let the set $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty, and let M satisfy

$$(12) \quad M = Z_2 + AG, \quad MH \geq AZ_3, \quad GH = Z_3 - Z_4 \geq 0$$

$$(13) \quad (Z_2, Z_3) \in Z, \quad Z_4 \in K, \quad (G, H) \geq 0$$

then there exists $(p, p_0) \in R^{n+m}$ satisfying (9) with $Z_1 = I$ and the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2).

Note that since $Z_3 = Z_4 + GH \geq Z_4$ and Z_4 is a K-matrix, it follows by Theorem 4,6 of [8] that Z_3 is also a K-matrix.

We conclude by giving some specific realizations of Theorem 5.

Theorem 6. Let the set $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty and let M satisfy any of the conditions below. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear problem (2) with the p indicated below:

(a) $M = Z_2 + ab^T$, $Z_2 \in K$, $a \in R^n$, $b \in R^n$, $0 \neq a \geq 0$, $b > 0$, $p = b$.

(b) $M = Z_2 + A(Z_3 - Z_4)$, $(Z_2, Z_3) \in Z$, $Z_4 \in K$, $Z_3 \geq Z_4$, $Z_2 \geq AZ_4$,
 $p_0^T Z_4 > 0$, $p_0 > 0$, $p^T = p_0^T (Z_3 - \frac{1}{2}Z_4)$

(c) $M = 2Z_2 - Z_4$, $Z_2 \in Z$, $Z_4 \in K$, $Z_2 \geq Z_4$, $p_0^T Z_4 > 0$, $p_0 > 0$, $p^T = p_0^T M$

(d) $M \geq 0$, $M_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^n M_{ij}$, $j = 1, \dots, n$, $p^T = e^T M$, $e^T = (1, \dots, 1) \in R^n$.

(e) $M \geq 0$, $M_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^n M_{ij}$, $i = 1, \dots, n$, $p^T = p_0^T M$ where $p_0^T Z_4 > 0$,

$p_0 > 0$ and $Z_4 = -M + 2(\text{diagonal of } M)$

Proof: (a) Since $Z_2 \in K$, there exists an h in R^n , $h > 0$, such that $Z_2 h > 0$ [8, Theorem 4,3]. Set in Theorem 5 above: $A = a$,

$G = b^T$, $H = h$, $Z_4 = \frac{1}{2} \min_j \frac{(Z_2 h)_j}{a_j} > 0$ and $Z_3 = b^T h + Z_4$. Note that

here Z_3 and Z_4 are real numbers. We now have that

$$MH - AZ_3 = (Z_2 + ab^T)h - a(b^T h + Z_4) = Z_2 h - aZ_4 > 0$$

To satisfy inequality (9), which in this case is $p^T > p_0^T b^T$ and $p_0^T Z_3 > p^T h$, set $p = b$ and take p_0 satisfying $1 > p_0 > \frac{b^T h}{Z_3}$.

Application of Theorem 5 gives the desired result.

(b) Set in Theorem 5, $H = I$. Conditions (12) and (13) are satisfied. Inequality (9) requires that

$$(p^T \ p_0^T) \begin{bmatrix} I & -I \\ -(Z_3 - Z_4) & Z_3 \end{bmatrix} > 0$$

That is we require that

$$p_0^T Z_3 > p^T > p_0^T (Z_3 - Z_4)$$

Now we have that $p_0^T z_4 > 0$, $p_0 > 0$, $z_3 - z_4 \geq 0$ and hence

$$p_0^T z_3 > p_0^T (z_3 - z_4) \geq 0$$

But since $p^T = p_0^T (z_3 - \frac{1}{2} z_4) > 0$ is the average of the first two terms in the above inequalities, it follows that the desired inequality

$$p_0^T z_3 > p^T = p_0^T (z_3 - \frac{1}{2} z_4) > p_0^T (z_3 - z_4)$$

holds.

(c) Set $A = I$ and $z_2 = z_3$ in part (b) of this theorem, and take $p^T = p_0^T M$ instead of $p^T = \frac{1}{2} p_0^T M$ since this change does affect the solution of (2).

(d) Take part (c) of this theorem and set

$$(z_2)_{ij} = 0, i \neq j, (z_2)_{jj} = M_{jj}, i, j = 1, \dots, n$$

$$(z_4)_{ij} = -M_{ij}, i \neq j, (z_4)_{jj} = M_{jj}, i, j = 1, \dots, n$$

The matrix z_4 is a K-matrix because, for $j = 1, \dots, n$, $M_{jj} - \sum_{i=1, i \neq j}^n M_{ij} > 0$.

Hence $p_0^T = e^T = (1, \dots, 1) \in R^n$ satisfies $p_0^T z_4 > 0$. Take $p^T = p_0^T M = e^T M$.

(e) We again apply part (c) of this theorem and define

$$(z_2)_{ij} = 0, i \neq j, (z_2)_{ii} = M_{ii}, i, j = 1, \dots, n$$

The matrix z_4 is a K-matrix because, for $i = 1, \dots, n$, $M_{ii} - \sum_{j=1, j \neq i}^n M_{ij} > 0$.

Hence there exists a $p_0 > 0$ such that $p_0^T z_4 > 0$. Take $p^T = p_0^T M$. \square

Note that in both cases (d) and (e) above, that is when M is a nonnegative strictly diagonally dominant matrix, $p^T = p_0^T M$, where $p_0 > 0$ is determined from the matrix z_4 obtained from M by reversing the sign of the offdiagonal elements of M and requiring that $p_0^T z_4 > 0$.

We close with a summary given in Table 1 below which gives the required assumptions on M and the corresponding vector p used in the linear program to obtain a solution of the linear complementarity problem. It is hoped that further research will substantially enlarge this table.

TABLE 1

Linear Complementarity Problems Solvable
by Linear Programming

<u>Matrix M of (1)</u>	<u>Conditions on M</u>	<u>Vector p of (2)</u>	<u>Conditions on p</u>
$M = Z_2 Z_1^{-1}$	$Z_1 \in K, Z_2 \in Z$	p	$p \geq 0, p^T Z_1 > 0$
$M = Z_2 Z_1^{-1}$	$Z_1 \in Z, Z_2 \in K$	$p = M^T s$	$s \geq 0, s^T Z_2 > 0$
M	$M \in Z$	p	$p > 0$
M	$M^{-1} \in Z$	$p = M^T e$	$e > 0$
$M = Z_2 + ab^T$	$Z_2 \in K$ $0 \neq a \geq 0, b > 0$	$p = b$	
$M = 2Z_2 - Z_4$	$Z_2 \in Z, Z_4 \in K$ $Z_2 \geq Z_4$	$p = M^T p_0$	$p_0 > 0, p_0^T Z_4 > 0$
M	$M \geq 0$ $M_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^n M_{ij}, j=1, \dots, n$	$p = M^T e$	$e^T = (1, \dots, 1)$
M	$M \geq 0$ $M_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^n M_{ij}, i=1, \dots, n$	$p = M^T p_0$	$p_0 > 0, p_0^T Z_4 > 0$ $Z_4 = -M + 2 \text{diag } M$

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