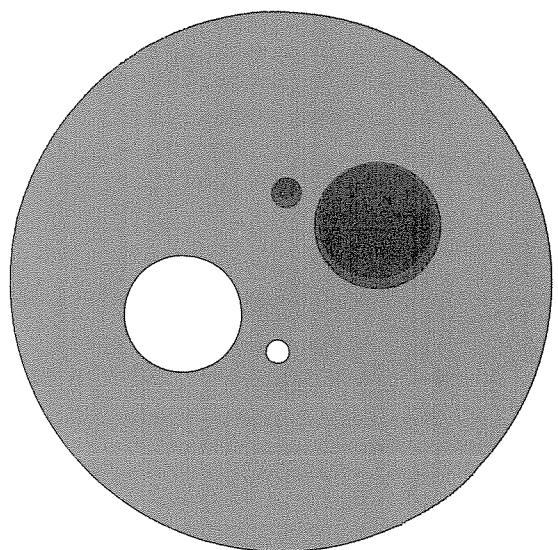


COMPUTER SCIENCES DEPARTMENT

University of Wisconsin -
Madison

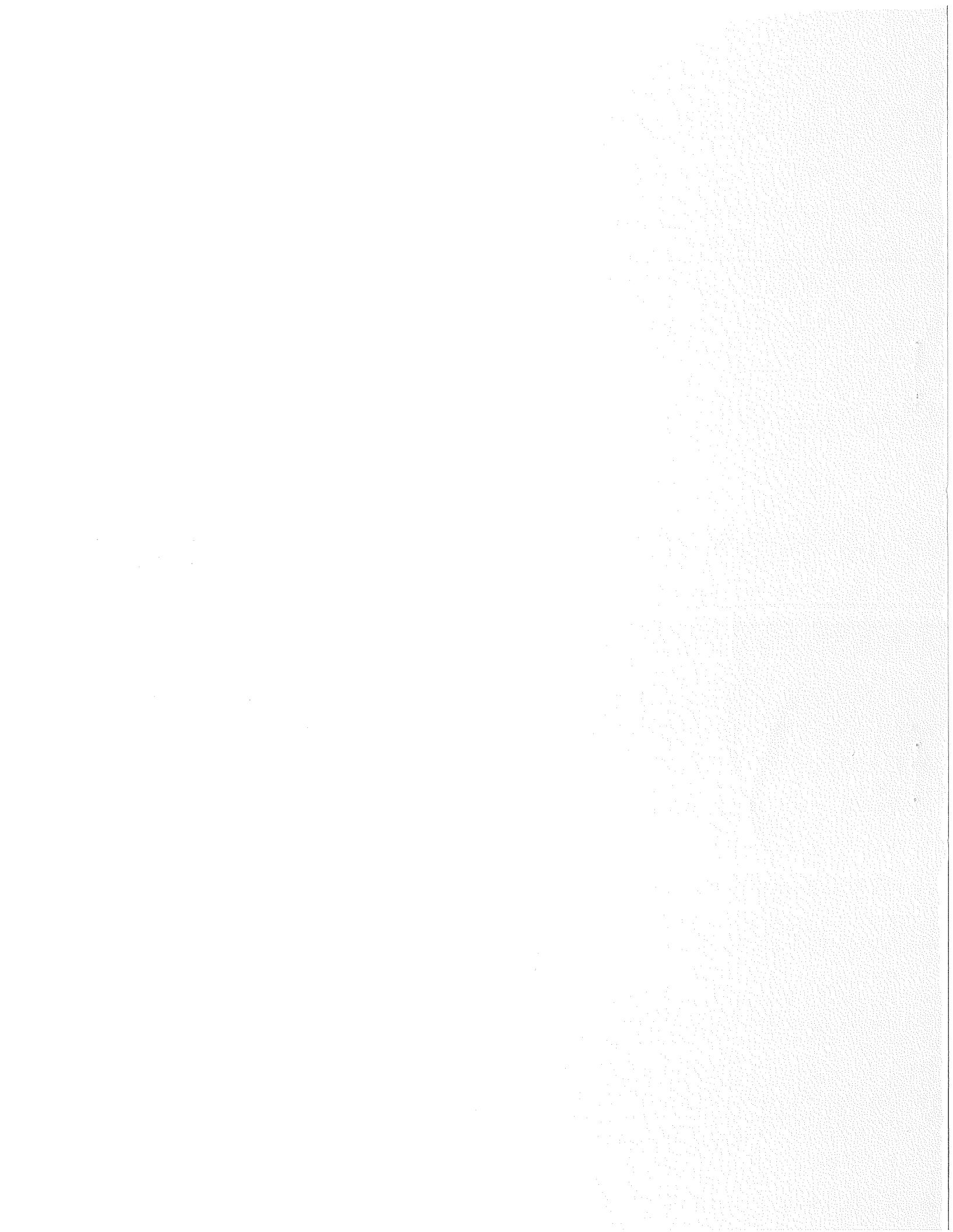


THE NON-MONOTONICITY OF SOLUTIONS IN SWIRLING FLOW

J. B. McLeod
S. V. Parter

Computer Sciences Technical Report No. 250

April, 1975



The University of Wisconsin
Computer Sciences Department
1210 West Dayton Street
Madison, Wisconsin 53706
U.S.A.

THE NON-MONOTONICITY OF SOLUTIONS IN SWIRLING FLOW

by

J. B. McLeod
S. V. Parter

#250

निर्वाचन विधायक सभा
संसदीय विधायक सभा
विधायक सभा
विधायक सभा
विधायक सभा

विधायक सभा

विधायक सभा विधायक सभा विधायक सभा विधायक सभा

विधायक सभा

विधायक सभा
विधायक सभा

THE NON-MONOTONICITY OF SOLUTIONS IN SWIRLING FLOW

J. B. McLeod and S. V. Parter

ABSTRACT

This paper studies the boundary-value problem arising from the behaviour of a fluid occupying the region $0 \leq x \leq 1$ between two rotating discs, rotating about a common axis perpendicular to their planes, when the discs are rotating in the same sense with speeds $0 \leq \Omega_0 < \Omega_1$. The equations which describe the axially symmetric similarity solutions of this problem are

$$\varepsilon H^{IV} + HH''' + GG' = 0,$$

$$\varepsilon G'' + HG' - H'G = 0,$$

with the boundary conditions

$$H(0) = H'(0) = H(l) = H'(l) = 0,$$

$$G(0) = \Omega_0 / \Omega_1, \quad G(l) = 1,$$

where $\varepsilon = \nu / 2\Omega_1$ and ν is the kinematic viscosity.

The major result is: There is an ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$ there does not exist a solution $\langle H(x, \varepsilon), G(x, \varepsilon) \rangle$ with $G'(x, \varepsilon) \geq 0$.

AMS(MOS) Classification: 34B15, 34E15, 35Q10, 76U05.

Key words: Nonexistence theorems, ordinary differential equations, rotating fluids, similarity solutions, asymptotic behavior, monotone solutions

*Also issued as Technical Report #1551 of the Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706

Sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462, and by the U.S. Office of Naval Research under Contract N00014-67-A-0128-0004.

THE NON-MONOTONICITY OF SOLUTIONS IN SWIRLING FLOW

J. B. McLeod and S. V. Parter

1. Introduction

It was von Kármán [1] who first realized that the fluid motion above an infinite rotating disc which is rotating about an axis perpendicular to its plane can under suitable circumstances be reduced to the study of a pair of non-linear ordinary differential equations, and Batchelor [2] extended this discussion to motion between two rotating discs rotating about a common axis perpendicular to their planes. In the two-disc case, let the discs be placed at $x = 0$ and $x = 1$ and rotating about the x -axis with angular velocities Ω_0 and Ω_1 respectively. Let q_r, q_θ and q_x denote the velocity components in cylindrical polar coordinates (r, θ, x) , and, following Batchelor, write

$$q_\theta = \frac{1}{2}r Q(x), \quad q_x = h(x), \quad q_r = -\frac{1}{2}r h'(x),$$

where the prime denotes differentiation with respect to x . The continuity equation is satisfied exactly and the equations of motion become

$$\begin{aligned} \nu h^{IV} - hh''' - QQ' &= 0, \\ \nu Q'' - hQ' + hQ &= 0, \end{aligned}$$

where ν is the kinematic viscosity. The associated boundary conditions are

$$h(0) = h'(0) = h(1) = h'(1) = 0,$$

$$Q(0) = 2\Omega_0, \quad Q(1) = 2\Omega_1.$$

The fundamental analytical question to be answered for this boundary-value problem is that of existence of solutions, and such answers as at present exist are to be found in three papers. The first, by Hastings [3], proves existence provided that the angular velocities of both discs are

Sponsored by the United States Army under Contract No. DA-31-127-ARO-D-462, and by the U.S. Office of Naval Research under Contract N00014-67-A-0128-0004.

sufficiently small. Elcrat [4] has also used what is essentially a perturbation approach, but it is carried through with sufficient precision that definite numerical estimates can be given of the extent of the allowable perturbation. Further,

Elcrat perturbs both about the rest state, with $\Omega_0 = \Omega_1 = 0$, and about the rigid body rotation, with $\Omega_0 = \Omega_1$. The third paper [5] looks at the case where $\Omega_0 = -\Omega_1$,

and the object is to prove the existence and discuss the behaviour of solutions anti-symmetric about $x = \frac{1}{2}$, so that h and Q are odd functions of $x - \frac{1}{2}$. The discussion of behaviour of solutions becomes most interesting when the rates of rotation of the discs are large, and for this reason we adopt a slightly different normalization in further consideration of the equations.

Since from the point of view of existence theory the case $\Omega_0 = \Omega_1$ is trivial, and since the replacement of Q by $-Q$ does not alter the differential equations but only the boundary conditions $Q(0) = 2\Omega_0, Q(1) = 2\Omega_1$, we may suppose without loss of generality that $Q(1) > 0, Q(0) < Q(1)$, $Q(0)$ being possibly negative. We may therefore set

$$\epsilon = \nu/2\Omega_1 > 0, \quad H(x) = -h(x)/2\Omega_1, \quad G(x) = Q(x)/2\Omega_1,$$

and the equations of motion become, for $0 \leq x \leq 1$,

$$\epsilon H^{IV} + HH''' + GG' = 0, \quad (1.1)$$

$$\epsilon G'' + HG' - H'G = 0, \quad (1.2)$$

with

$$\begin{aligned} H(0) &= H'(0) = H(1) = H'(1) = 0, \\ G(0) &= \Omega_0/\Omega_1 < 1, \quad G(1) = 1. \end{aligned} \quad (1.3)$$

The case of large rotations is therefore the case $\epsilon \downarrow 0$.

The existence of solutions of (1.1)-(1.3) is inextricably bound up with the behaviour of solutions. For existence must depend on finding some a priori bounds, and these will come from a sufficient knowledge of the qualitative behaviour. In particular both [4] and [5] depend heavily on the property $G' \geq 0$ (using the normalization in (1.1)-(1.3)). Thus not only do the solutions whose existence is proved possess the property that $G' \geq 0$, but since, with $G(0) < G(1)$, it is reasonable to look for solutions with the property $G' \geq 0$, the whole discussion of existence is carried out on a set of functions G which possess this property, and much use is made of consequences that can be deduced from it.

It would be reasonable to expect (and certainly very helpful if it were true) that there always exists a solution of (1.1)-(1.3) with $G' \geq 0$ (if $G(0) < G(1)$), or at least that this is so (as indeed Batchelor claims in [2]) when $G(0) \geq 0$, for then the property $G' \geq 0$ would imply $G \geq 0$, and the fact that G does not change sign is itself a useful source of bounds, as noted first in [6]. Unfortunately, however, it is the achieved object of the present paper to prove the following result.

Theorem: If $\varepsilon > 0$ is sufficiently small, and $G(0) \geq 0$, then there cannot exist a solution to (1.1)-(1.3) for which $G' \geq 0$.

It is an interesting question (about which we hesitate to make even a conjecture) to what extent the restriction $G(0) \geq 0$ is essential to the truth of the theorem. Certainly some restriction on $G(0)$ must be made since [5] shows that, when $G(0) = -1$, there does exist a solution with $G' \geq 0$ for arbitrarily small ε . Further, our analysis for the case $G(0) = 0$ is very much harder than that for $G(0) > 0$, which may suggest that the case $G(0) = 0$ is

a boundary beyond which the theorem does not go. But it is also plausible to conjecture that the case of anti-symmetry ($G(0) = -1$) is exceptional and that the theorem fails only then.

The implication of the theorem for existence theory is that a proof of the existence of a solution of (1.1)-(1.3) for general positive ε will be that much harder to find now that one is forced to deal with solutions for which G' oscillates, but there is of course no suggestion that a solution does not exist. Indeed, both on physical and numerical grounds, the existence of a solution to (1.1)-(1.3) would not seem to be in doubt. For numerical work, the reader is referred to [7]-[10], while the paper by Stewartson [11] is an interesting commentary on the original paper by Batchelor [2]. The paper by Tam [12] contains a quite formal discussion, in the case $G(0) = -1$, of the behaviour of solutions of (1.1)-(1.3) as $\varepsilon \downarrow 0$ using the method of matched asymptotic expansions, which is also the motivation of the present paper; Tam's conclusion is that there are many solutions for small ε .

Although progress towards an existence theory is thus remarkably fragmentary, the situation for one disc, when the fluid occupies the whole space above a single rotating disc, is happier. The reason for this is that (1.1) is immediately integrable to give

$$(1.4) \quad \varepsilon H''' + HH' + \frac{1}{2}(G^2 - H'^2) = \text{constant},$$

and the physical situation at infinity when we have only the one disc demands the prescribing of the constant of integration in (1.4) and, correspondingly, the dropping of the prescription of $H(\infty)$. The problem thus becomes

$$\varepsilon H''' + HH'' + \frac{1}{2}(G^2 - H'^2) = \frac{1}{2}\omega_\infty^2, \\ \varepsilon G' + HG' = H'G,$$

$H(0) = H'(0) = 0$, $G(0) = \omega_0$, $H^{(\infty)} = 0$, $G(\infty) = \omega_\infty$, and this is a fifth-order system with five boundary conditions, while the problem for two discs is a sixth-order system with six boundary conditions or (if one uses (1.4)) a fifth-order system with the constant of integration in (1.4), as well as H and G , to be determined from the six boundary conditions.

It is the indeterminacy of this constant of integration which is the essential difficulty in the two-disc problem, for any a priori bounds on solutions must clearly depend upon this constant, and it is therefore necessary to have bounds on the constant itself in terms of the given parameters in the problem, i.e. ε , Ω_0 and Ω_1 . To obtain bounds for the solution in terms of the constant is relatively easy, being largely a repetition of the problem for the single disc. To obtain bounds for the constant in terms of ε , Ω_0 and Ω_1 seems, on the other hand, to be very difficult. The known results for the one-disc problem are contained in [6], [13]-[19] and references therein. A general survey of the field in its present state is to be found in [20].

Before we proceed to the proof of the theorem stated above, it may be as well to indicate heuristically why it should be true, since the heuristic ideas form the basis for the rigorous proof. If we suppose for contradiction that, for arbitrarily small ε , there does exist a solution of (1.1)-(1.3) with $G(0) \geq 0$ and $G' \geq 0$, then it is possible to deduce from the equations that the restriction $G' \geq 0$ places restrictions also on the number of changes of sign possible for H and its derivatives, and that in particular we must have

$H'''(0) > 0$, $H'''(1) < 0$. Substitution of this in (1.4) implies that $0 < \mu < \frac{1}{2}$,

where μ is the constant of integration, and

(1.5)
$$H'''(0) = (\mu - \frac{1}{2}G^2(0))/\varepsilon, \quad H'''(1) = (\mu - \frac{1}{2})/\varepsilon,$$
 so that, unless $G(0) = 1$, which is excluded, at least one of $H'''(0)$, $H'''(1)$ is precisely of order ε^{-1} as $\varepsilon \downarrow 0$. (The value of μ may of course depend on ε .) That at least one of $H'''(0)$, $H'''(1)$ is large can be interpreted as meaning that a boundary layer forms at one disc or the other, possibly both.

If, sufficiently for our present purposes, we suppose that a boundary layer forms at the disc $x = 1$, so that $\mu - \frac{1}{2} \mapsto 0$ as $\varepsilon \downarrow 0$, then we can obtain estimates of the behaviour of H and G in the boundary layer, and scale the problem appropriately there, in fact by writing

$$(1.6) \quad 1 - x = \varepsilon^{\frac{1}{2}}\xi, \quad -\varepsilon^{-\frac{1}{2}}H(x) = \phi(\xi), \quad G(x) = \psi(\xi).$$

The equations (1.4) and (1.2) take the form

$$(1.7) \quad \phi''' + \phi\phi'' + \frac{1}{2}(\psi^2 - \phi'^2) = \mu,$$

$$(1.8) \quad \psi'' + \phi\psi' = \phi'\psi,$$

and an investigation of these indicates that in the boundary layer the fluid behaves like a suitably scaled version of some solution of the von Kármán one-disc problem associated with (1.7) and (1.8) (with $\mu = \mu_0$) and the boundary conditions

$$\begin{aligned} \phi(0) &= \phi'(0) = 0, \quad \psi(0) = 1, \\ \phi'(\infty) &= 0, \quad \psi(\infty) = \sqrt{2\mu_0}, \end{aligned}$$

where $\mu_0 = \lim_{\varepsilon \downarrow 0} \mu$. (There is of course no guarantee that $\lim_{\varepsilon \downarrow 0} \mu$ exists, but if it does not, we can restrict ourselves to a subsequence of values of ε for which the limit does exist.) If $\mu_0 \neq 0$, then the asymptotic analysis in [14]

shows that any solution of this one-disc problem necessarily shows an asymptotic behaviour as $\xi \rightarrow \infty$ for which ϕ' , ψ' have an infinite sequence of changes of sign, and this of course provides the necessary contradiction to the assumption that $G' \geq 0$.

The remaining possibility is that $\mu_0 = 0$, for then (as in [13]) there certainly does exist a solution of the corresponding von Kármán problem for which

$$\phi \geq 0, \quad \phi' \geq 0, \quad \psi > 0, \quad \psi' < 0$$

and which is therefore consistent with $G \geq 0$. As $\xi \rightarrow \infty$, $\phi \rightarrow c$, for some non-zero constant c , and this implies that at the edge of (and so outside) the boundary layer $-H$ is precisely of order $\epsilon^{\frac{1}{2}}$ as $\epsilon \downarrow 0$. In fact, since we are outside the boundary layer, it is not surprising that we can show that the x -derivatives of H and G are of the same order as H itself, so that the terms ϵH^{iv} and $\epsilon G''$ can be neglected in (1.1) and (1.2) and we can integrate the simplified forms of (1.1) and (1.2) explicitly to give the behaviour outside the boundary layer.

We now look at the consequences of these estimates in the neighbourhood of $x = 0$ and find that the appropriate scaling there is now

$$(1.9) \quad x = \epsilon^{\frac{1}{4}} \eta, \quad \epsilon^{-3/4} H(x) = f(\eta), \quad \epsilon^{-\frac{1}{2}} G(x) = g(\eta),$$

when the equations (1.4) and (1.2) take the form

$$\begin{aligned} f''' + ff'' + \frac{1}{2}(g^2 - f'^2) &= \mu/\epsilon, \\ g'' + fg' &= f'g. \end{aligned}$$

An argument similar to that employed on (1.7)-(1.8) enables us to conclude here that we must have $\mu = o(\epsilon)$, and if we carry this and consequent

estimates back to the region between any possible boundary layers, we find that the size of H is not consistent with that previously determined.

The arrangement of the paper is as follows. Section 2 obtains various properties of a solution of (1.1)-(1.3) which can be deduced from the assumption that $G' \geq 0$. In section 3, we make the transformation (1.6) and use the information obtained in section 2 to argue (as we have already indicated heuristically) that either $\mu \rightarrow 0$ as $\epsilon \downarrow 0$ or there is a contradiction to $G' \geq 0$.

Since $G(0) > 0$ implies from (1.5) that $\mu \geq \frac{1}{2}G^2(0) \nrightarrow 0$, we have the result at this stage (Theorem 3.1) that, if $G(0) > 0$, the theorem stated above is true.

The remainder of the paper is devoted to dealing with the more difficult case that $G(0) \approx 0$. Since we may now assume that $\mu \rightarrow 0$, there must be a boundary layer at $x = 1$, and section 4 discusses estimates there and consequential estimates in the region outside any possible boundary layers. Finally, section 5 looks at the behaviour near $x = 0$ under the scaling (1.9), shows that $\mu = o(\epsilon)$ and obtains the final contradiction.

2. The "shape" of the solutions

Throughout this section $\langle H(x, \epsilon), G(x, \epsilon) \rangle$ will denote a solution of

$$(2.1) \quad \epsilon H^{IV} + HH''' + GG' = 0, \quad 0 \leq x \leq 1,$$

$$(2.2) \quad \epsilon G'' + H G' - H' G = 0, \quad 0 \leq x \leq 1,$$

$$(2.3) \quad H(0, \epsilon) = H(1, \epsilon) = H'(0, \epsilon) = H'(1, \epsilon) = 0,$$

$$(2.4) \quad 0 \leq G(0, \epsilon) < G(1, \epsilon) = 1,$$

which satisfies

$$G'(x, \epsilon) \geq 0, \quad 0 \leq x \leq 1.$$

Lemma 2.1. The function $H(x, \varepsilon)$ satisfies

$$(2.5) \quad H(x, \varepsilon) \leq 0.$$

Moreover, there are four distinguished points

$$(2.6a) \quad \begin{aligned} x_1 &= x_1(\varepsilon), & x_3 &= x_3(\varepsilon), \\ x_{2,1} &= x_{2,1}(\varepsilon), & x_{2,2} &= x_{2,2}(\varepsilon), \end{aligned}$$

with

$$(2.6b) \quad 0 < x_{2,1} < x_1 < x_{2,2} < 1, \quad 0 < x_{2,1} < x_3 < x_{2,2} < 1,$$

such that

- a) $H'(x, \varepsilon) < 0$ for $0 < x < x_1$,
- b) $H'(x, \varepsilon) > 0$ for $x_1 < x < 1$,
- c) $H''(x, \varepsilon) < 0$ for $0 \leq x < x_{2,1}$ or $x_{2,2} < x \leq 1$,
- d) $H''(x, \varepsilon) > 0$ for $x_{2,1} < x < x_{2,2}$,
- e) $H'''(x, \varepsilon) > 0$ for $0 \leq x < x_3$,
- f) $H''(x, \varepsilon) < 0$ for $x_3 < x \leq 1$,
- g) $H^{IV}(x, \varepsilon) < 0$ for $x_3 < x \leq 1$.

Further, there exists a distinguished point

$$y_1 = y_1(\varepsilon)$$

which satisfies

$$(2.6c) \quad x_{2,1} < y_1 \leq x_1,$$

and

- h) $G''(x, \varepsilon) < 0$ for $0 < x < y_1$,
- while
- i) $G''(x, \varepsilon) > 0$ for $y_1 < x < 1$.

Proof. From the boundary conditions (2.3) and Rolle's theorem we see that

a) there is at least one point, say x_1 , with $0 < x_1 < 1$ and

$$(2.7a) \quad H'(x_1, \varepsilon) = 0,$$

b) there are at least two points, say $x_{2,1}, x_{2,2}$, with

$$0 < x_{2,1} < x_1 < x_{2,2} < 1$$

and

$$(2.7b) \quad H''(x_2, k, \varepsilon) = 0, \quad k = 1, 2,$$

such that

- (2.7c) $x_{2,1} < x_3 < x_{2,2}$
 - and
 - $H'''(x_3, \varepsilon) = 0$.
- Moreover, if there is a unique point $x_3 \in (0, 1)$ at which (2.7c) holds, then $x_{2,1}, x_{2,2}$ and x_1 are also unique and $H''(0, \varepsilon) \neq 0, H''(1, \varepsilon) \neq 0$.
- On the other hand, we may rewrite (2.1) as

$$(2.8a) \quad [H''' \exp(\varepsilon^{-1} u(x, \varepsilon))]' = -\varepsilon^{-1} G(x, \varepsilon) G'(x, \varepsilon) \exp(\varepsilon^{-1} u(x, \varepsilon))$$

where

$$(2.8b) \quad u(x, \varepsilon) = \int_0^x H(t, \varepsilon) dt.$$

Hence $H''' \exp(\varepsilon^{-1} u)$ is nonincreasing, and if H''' had two zeros it would be identically zero between the two. In view of the analyticity of the solutions of (2.1), (2.2) we would obtain

$$H \equiv G' \equiv 0,$$

which contradicts the boundary conditions (2.4).

Thus we have established (e), (f), and (a)-(d) are immediate consequences. Further, if H has a zero in $(0, 1)$, then there must be at least two

zeros of H' , which is impossible. Hence H is of constant sign, and that sign is clearly negative. Conclusion (g) follows from these results and (2.1).

To obtain the results on G'' we differentiate (2.2) and obtain

$$\varepsilon G''' + HG'' = H''G,$$

$$[G'' \exp(\varepsilon^{-1}u)]' = \varepsilon^{-1}H''G \exp(\varepsilon^{-1}u).$$

Therefore $G'' \exp(\varepsilon^{-1}u)$ is first decreasing ($0 < x < x_{2,1}$), then increasing ($x_{2,1} < x < x_{2,2}$), and finally decreasing. Since $G''(0, \varepsilon) = G''(1, \varepsilon) = 0$, we see that G'' is first negative and then positive. Once more the analyticity of the solution $\langle H, G \rangle$ shows that G'' has a unique zero, which we call y_1 . Finally, the results above show that $G''(x_1, \varepsilon) \geq 0$, so that $y_1 \leq x_1$.

Lemma 2.2. Integrating (2.1) we find that

$$(2.9) \quad \varepsilon H''' + HH'' + \frac{1}{2}(G^2 - H'^2) = \mu = \mu(\varepsilon)$$

where μ is a constant. Then

$$(2.10) \quad \frac{1}{2}G^2(0, \varepsilon) < \mu < \frac{1}{2}.$$

Moreover

$$(2.10a) \quad \varepsilon |H'''(0, \varepsilon)| + \varepsilon |H''(1, \varepsilon)| = \frac{1}{2}(1 - G^2(0, \varepsilon)),$$

$$(2.10b) \quad H''' = O(\varepsilon^{-1}), \quad H' = O(1),$$

$$(2.10c) \quad H'' = O(\varepsilon^{-\frac{1}{2}}).$$

Proof. From Lemma 2.1, (e), (f) we obtain

$$\varepsilon |H'''(0, \varepsilon)| + \frac{1}{2}G^2(0, \varepsilon) = \mu = -\varepsilon |H''(1, \varepsilon)| + \frac{1}{2},$$

which gives (2.10) and (2.10a). Moreover, we have

$$H'''(0, \varepsilon) = O(\varepsilon^{-1}), \quad H''(1, \varepsilon) = O(\varepsilon^{-1}),$$

so that, from Lemma 2.1(g),

$$H'''(x, \varepsilon) = O(\varepsilon^{-1}), \quad x_3 \leq x \leq 1.$$

Integrating (2.8a) from $x < x_3$ to x_3 and recalling that $H \leq 0$, so that $u \leq 0$, $u' \leq 0$, we obtain, for $0 \leq x \leq x_3$,

$$0 < H'''(x, \varepsilon) \leq \frac{1}{2\varepsilon} \{G^2(x_3, \varepsilon) - G^2(x, \varepsilon)\} \leq \frac{1}{2\varepsilon}.$$

Thus $H''' = O(\varepsilon^{-1})$ on the entire interval.

The function H' has a positive maximum and negative minimum (at $x_{2,2}$ and $x_{2,1}$ respectively). Hence, in order to bound H' it is sufficient to obtain bounds at these points. However, at either of these points we have

$$\varepsilon H''' + \frac{1}{2}(G^2 - (H')^2) = \mu = O(1).$$

Thus we have completed the proof of (2.10b).

Finally (2.10c) follows from the elementary estimate ($\varepsilon < \frac{1}{2}$)

$$\|H''\|_\infty \leq \frac{2}{\varepsilon^{\frac{1}{2}}} \|H'\|_\infty + \frac{\varepsilon^{\frac{1}{2}}}{2} \|H'''\|_\infty.$$

Lemma 2.3.

$$G'(x, \varepsilon) = O(\varepsilon^{-\frac{1}{2}}).$$

Proof. Since G'' is first negative and then positive it will be sufficient to prove this result for $x = 0, 1$.

If $1 - \varepsilon^{\frac{1}{2}} \leq x \leq 1$ then using (2.10b) we have

$$H(x, \varepsilon) = - \int_x^1 H'(t, \varepsilon) dt = O(\varepsilon^{\frac{1}{2}}).$$

Hence

$$\varepsilon G'' = H'G - HG' = O(1) + O(\varepsilon^{\frac{1}{2}}G').$$

Integrating over $[x, 1]$ we obtain ($1 - \varepsilon^{\frac{1}{2}} \leq x \leq 1$)

$$\begin{aligned} G'(1, \varepsilon) - G'(x, \varepsilon) &= O(\varepsilon^{-\frac{1}{2}}) + O(\varepsilon^{-\frac{1}{2}}[1 - G(x, \varepsilon)]) \\ &= O(\varepsilon^{-\frac{1}{2}}), \end{aligned}$$

and integration over $[1 - \varepsilon^{\frac{1}{2}}, 1]$ gives

$$\varepsilon^{\frac{1}{2}} G'(1, \varepsilon) - \{1 - G(1 - \varepsilon^{\frac{1}{2}}, \varepsilon)\} = C(1),$$

from which the result follows at $x = 1$.

The argument at $x = 0$ is parallel.

3. Asymptotic behaviour near $x = 1$

Throughout this section we continue to suppose that $\langle H, G \rangle$ is a solution of (2.1)-(2.4) which satisfies $G' \geq 0$.

Consider the transformation

$$(3.1) \quad 1 - x = \varepsilon^{\frac{1}{2}} \xi, \quad H(x, \varepsilon) = -\varepsilon^{\frac{1}{2}} \phi(\xi, \varepsilon), \quad G(x, \varepsilon) = \psi(\xi, \varepsilon).$$

Then the equations (2.9) and (2.2) become

$$(3.2) \quad \phi''' + \phi\phi'' + \frac{1}{2}(\psi^2 - \phi'^2) = \mu = \mu(\varepsilon), \quad 0 \leq \xi \leq \varepsilon^{-\frac{1}{2}},$$

$$(3.3) \quad \psi'' + \phi\psi' = \phi'\psi, \quad 0 \leq \xi \leq \varepsilon^{-\frac{1}{2}},$$

where now denotes differentiation with respect to ξ . The functions ϕ, ψ satisfy the boundary conditions (amongst others)

$$(3.4) \quad \phi(0, \varepsilon) = \phi'(0, \varepsilon) = 0, \quad \psi(0, \varepsilon) = G(1, \varepsilon) = 1.$$

Lemma 3.1. Uniformly in ε and $\xi \in [0, \varepsilon^{-\frac{1}{2}}]$ we have

$$\phi'(\xi, \varepsilon) = O(1),$$

$$\psi'(\xi, \varepsilon) = O(1).$$

$$\phi''(\xi, \varepsilon) = O(1),$$

$$\psi''(\xi, \varepsilon) = O(1).$$

Proof. This is direct computation using Lemmas 2.2, 2.3.

Corollary 3.1. Suppose there is a sequence $\varepsilon_n \downarrow 0$ and a corresponding sequence $\langle H(x, \varepsilon_n), G(x, \varepsilon_n) \rangle$ of solutions of (2.1)-(2.4) which satisfy $G' \geq 0$. Then there is a subsequence, which we again denote by ε_n , such that

of $H(x, \varepsilon), G(x, \varepsilon)$.
Proof. The lemma follows immediately from the continuous dependence of the solutions of the differential equation on the initial conditions and the "shapes" of $H(x, \varepsilon), G(x, \varepsilon)$.

$$\begin{aligned} \phi''(0, \varepsilon_n) &\rightarrow \alpha, \quad \psi'(0, \varepsilon_n) \rightarrow \beta, \\ \phi'''(0, \varepsilon_n) &= \mu(\varepsilon_n) - \frac{1}{2} \rightarrow \mu_0 - \frac{1}{2} \end{aligned}$$

with α, β, μ_0 finite and $\alpha \geq 0, \beta \leq 0$,

$$(3.5) \quad \frac{1}{2} G^2(0, \varepsilon_n) \leq \mu_0 \leq \frac{1}{2}.$$

Lemma 3.2. With the notation of Corollary 3.1, let $\phi_0(\xi), \psi_0(\xi)$ be the solutions of (3.2), (3.3), with $\mu = \mu_0$, which satisfy the boundary conditions (3.4) and

$$(3.6) \quad \phi''(0) = \alpha, \quad \psi'(0) = \beta.$$

Then on any finite ξ -interval $[0, k]$ the functions $\phi(\xi, \varepsilon_n), \psi(\xi, \varepsilon_n)$ and their derivatives tend uniformly to $\phi_0(\xi), \psi_0(\xi)$ and their derivatives as $\varepsilon_n \downarrow 0$. Moreover,

$$(3.7a) \quad \psi'_0 \leq 0,$$

$$(3.7b) \quad \phi''_0, \quad \psi''_0 \text{ each change sign at most once, unless they are}$$

$$(3.7c) \quad \text{identically zero,}$$

Lemma 3.3. ϕ'_0 changes sign at most once, unless it is identically zero, ϕ''_0 changes sign at most twice (and at most once if $\alpha = 0$), unless it is identically zero,

$$(3.7d) \quad \phi''_0 \text{ changes sign at most twice (and at most once if } \alpha = 0\text{),}$$

$$(3.7e) \quad |\phi'_0| \leq M,$$

$$(3.7f) \quad |\psi_0| \leq M,$$

for some constant M .

Lemma 3.3. Let $\phi_0(\xi)$, $\psi_0(\xi)$ be as above. Then there are finite values

ℓ, m such that

$$(3.8a) \quad \phi'_0(\xi) \rightarrow \ell \quad \text{as} \quad \xi \rightarrow \infty,$$

$$(3.8b) \quad \psi_0(\xi) \rightarrow m \quad \text{as} \quad \xi \rightarrow \infty.$$

Moreover

$$(3.8c) \quad \phi''_0(\xi) \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty,$$

$$(3.8d) \quad \psi'_0(\xi) \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty.$$

Proof. Since ϕ''_0 and ψ'_0 are ultimately of one sign, we obtain (3.8a) and

(3.8b) from (3.7e) and (3.7f). Since ϕ'''_0 and ψ''_0 are ultimately of one sign

and ϕ''_0 and ψ'_0 are bounded as a consequence of Lemma 3.1, we see that

$\phi'''_0(\xi)$, $\psi'_0(\xi)$ tend toward limits as $\xi \rightarrow \infty$, and (3.8a) and (3.8b) show that

these limits must be zero.

$$\text{Lemma 3.4.} \quad \ell = 0, \quad m = \sqrt{2\mu_0}.$$

Proof. From (3.2),

$$\phi'''_0 + \phi'_0 \phi''_0 \rightarrow L = \mu_0 - \frac{1}{2}(m^2 - \ell^2) \quad \text{as} \quad \xi \rightarrow \infty.$$

Suppose for contradiction that $L \neq 0$. Then for all ξ sufficiently large we must have either $|\phi'''_0| > \frac{1}{4}|L|$ or $|\phi'_0 \phi''_0| > \frac{1}{4}|L|$. The first alternative can hold only in a set of finite measure, or we would have a contradiction to

$\phi''_0 \rightarrow 0$; the second alternative must thus hold elsewhere, and since $\phi'_0 \rightarrow \ell$ means that $\phi_0 = O(\xi)$, the inequality $|\phi'_0 \phi''_0| > \frac{1}{4}|L|$ implies $|\phi''_0| > K/\xi$ for some positive constant K and contradicts $\phi'_0 \rightarrow \ell$. Thus $L = 0$, and working similarly with (3.3), we conclude that also $m = 0$, from which the final result follows immediately.

Lemma 3.5. Either $\mu_0 = \frac{1}{2}$ or $\mu_0 = 0$.

Proof. $\phi_0(\xi)$, $\psi_0(\xi)$ are solutions of (3.2), (3.3) with $\mu = \mu_0$ and the boundary conditions $\phi'_0(\infty) = 0$, $\psi_0(\infty) = \sqrt{2\mu_0}$. The asymptotic behaviour of such solutions as $\xi \rightarrow \infty$ is discussed in [14], where it is shown that

ϕ'_0 , ψ'_0 must oscillate about zero unless $\mu_0 = 0$ or ϕ_0, ψ_0 are constants.

Since ϕ'_0 , ψ'_0 are not oscillating in our case, we must have $\mu_0 = 0$ or

ψ_0 constant, and ψ_0 constant implies that $\psi_0^{(\infty)} = \psi_0(0)$, i.e. $\mu_0 = \frac{1}{2}$.

Lemma 3.6. Either $\mu_0 = \frac{1}{2} G^2(0, \varepsilon)$ or $\mu_0 = 0$.

Proof. This result comes from repeating the argument which led to Lemma 3.5, but now in the neighborhood of $x = 0$ in place of $x = 1$. Thus we make the transformation

$$x = \varepsilon^{\frac{1}{2}} X, \quad H(x, \varepsilon) = \varepsilon^{\frac{1}{2}} \Phi(X, \varepsilon), \quad G(x, \varepsilon) = \Psi(X, \varepsilon),$$

conclude (as in Lemma 3.1) that Φ' , Φ'' , Ψ' are all bounded in both X and ε

(' denoting differentiation with respect to X), construct solutions $\Phi_0(X)$,

$$\Psi_0(X) \quad \Phi''_0 + \Phi_0 \Phi''_0 + \frac{1}{2}(\Psi_0^2 - \Phi_0'^2) = \mu_0,$$

and deduce (as in Lemma 3.3) that

$$\Phi_0'(X) \rightarrow L \quad \text{as} \quad X \rightarrow \infty, \quad \Psi_0'(X) \rightarrow M \quad \text{as} \quad X \rightarrow \infty.$$

The only difference is one of sign: we now have for example, $\Psi_0' \geq 0$,

whereas previously $\Psi_0' \leq 0$.

As in Lemma 3.4, we can conclude that $L = 0$, $M = \sqrt{2\mu_0}$, and as in Lemma 3.5 that $\mu_0 = 0$ or Ψ_0 is constant, the last case giving

$$\mu_0 = \frac{1}{2} G^2(0, \varepsilon).$$

Theorem 3.1. If $G(0, \varepsilon) > 0$, then there is an $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leq \varepsilon_0$, there does not exist a solution $\langle H(x, \varepsilon), G(x, \varepsilon) \rangle$ of (2.1)-(2.4) which satisfies $G' \geq 0$.

Proof. If the theorem were not true, we should have a sequence $\varepsilon_n \downarrow 0$ as in Corollary 3.1, and $\mu(\varepsilon_n) \rightarrow \mu_0$. From (3.5) we now have $\mu_0 \neq 0$, and since $G(0, \varepsilon) < 1$, Lemmas 3.5, 3.6 are mutually contradictory.

Theorem 3.2. If $G(0, \varepsilon) = 0$, and if there does exist a sequence $\varepsilon_n \downarrow 0$ such that the corresponding solutions $\langle H(x, \varepsilon_n), G(x, \varepsilon_n) \rangle$ of (2.1)-(2.4) satisfy $G' \geq 0$, then $\mu(\varepsilon_n) \rightarrow 0$.

This is an immediate consequence of Lemma 3.6.

4. The case $G(0, \varepsilon) = 0$: estimates in the interior of $(0, 1)$

In view of Theorem 3.1 we can now confine ourselves to the case $G(0, \varepsilon) = 0$; and in view of Theorem 3.2 we can also assume that $\mu_0 = 0$. Thus throughout this section $\langle H(x, \varepsilon), G(x, \varepsilon) \rangle$ is a solution of (2.1)-(2.4) which satisfies $G' \geq 0$ and $G(0, \varepsilon) = 0$. We set up, as in Corollary 3.1, a sequence $\varepsilon_n \downarrow 0$, and the behaviour in the boundary layer at $x = 1$ of the corresponding solutions $\langle H(x, \varepsilon_n), G(x, \varepsilon_n) \rangle$ is given in Lemmas 3.2-3.4, remembering now that $\mu_0 = 0$. Our object in the present section is to use our knowledge of this behaviour to give estimates for $H(x, \varepsilon_n)$, $G(x, \varepsilon_n)$ and their derivatives when x is in the interior of $(0, 1)$, and the final results are given in Theorems 4.1-4.5.

Throughout the remainder of the paper the letter K will be used to denote various positive constants not necessarily the same at each appearance, but always independent of any of the variables under discussion.

Lemma 4.1. If ϕ_0 is the solution introduced in Lemma 3.2, we have

Proof. The functions ϕ_0, ψ_0 satisfy (3.2), (3.3) with $\mu = \mu_0 = 0$ and (by virtue of Lemma 3.4)

$$\phi_0'(\infty) = 0, \quad \psi_0(\infty) = 0.$$

Since ϕ_0'' , ϕ_0''' are ultimately of constant sign, and $\phi_0''(\infty) = 0$ (Lemma 3.3), we have ultimately

$$\begin{array}{ll} \text{either} & \text{(a)} \\ & \phi_0'' \geq 0, \quad \phi_0''' \leq 0 \\ \text{or} & \text{(b)} \\ & \phi_0'' \leq 0, \quad \phi_0''' \geq 0. \end{array}$$

Note that ψ_0 is not constant, and so we cannot have ϕ_0'' or ϕ_0''' identically zero. Since $\phi_0''(0) \geq 0$ and $\phi_0'''(0) < 0$ (Corollary 3.1), and since ϕ_0''' has at most one change of sign (Lemma 3.2), we see that in case (a) ϕ_0''' has no change of sign at all, and so ϕ_0'' has no change of sign, which contradicts $\phi_0'(0) = \phi_0'(\infty) = 0$.

In case (b), with $\phi_0''(0) > 0$, ϕ_0''' (which has at most two changes of sign by Lemma 3.2) must have precisely one change of sign, so that $\phi_0'(0) = \phi_0''(0) = 0$ implies $\phi_0''' \geq 0$, as required. If $\phi_0''(0) = 0$, then ϕ_0''' (which has now at most one change of sign by Lemma 3.2) has precisely one change of sign since $\phi_0'(0) = \phi_0''(0) = 0$, and this again implies $\phi_0''' \geq 0$.

Corollary 4.1. ϕ_0'', ϕ_0''' have each precisely one change of sign.

Proof. These results emerge as part of the proof of Lemma 4.1.

Lemma 4.2. If, as in section 2, x_1, x_3 denote the zeros of H' , H''' , and $x_{2,1}, x_{2,2}$ the zeros of H'' , then

$$1 - x_1(\varepsilon_n) > \varepsilon_n^{\frac{1}{2}}, \quad 1 - x_3(\varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}}, \quad 1 - x_{2,2}(\varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}}.$$

(We say $A > B$ if $B = o(A)$,

and $A \asymp B$ if both $B = O(A)$ and $A = O(B)$.)

Proof. The change of sign of ϕ_0''' occurs at ξ_3 , say, and so, if ε_n is sufficiently small, $H'''(1 - \varepsilon_n^{-\frac{1}{2}}\xi_3, \varepsilon_n)$ changes sign in a given neighborhood $(\xi_3 - \delta, \xi_3 + \delta)$. This, translated into terms of x , says that

$$1 - x_3(\varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}},$$

and the other parts of the lemma are proved similarly.

Lemma 4.3.

$$\sup_{0 \leq x \leq 1} \{-H(x, \varepsilon_n)\} \asymp \varepsilon_n^{\frac{1}{2}}.$$

Proof. Since $1 - x_3(\varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}}$, we have

$$1 - x_3(\varepsilon_n)/\varepsilon_n^{\frac{1}{2}} \rightarrow \xi_3$$

and

$$(4.1) \quad -\varepsilon_n^{-\frac{1}{2}} H(x_3(\varepsilon_n), \varepsilon_n) \rightarrow \phi_0(\xi_3) \asymp 1.$$

We have therefore certainly established

$$(4.2) \quad \sup_{0 \leq x \leq 1} \{-H(x, \varepsilon_n)\} \geq K\varepsilon_n^{\frac{1}{2}},$$

for some positive constant K , and to complete the proof we need to establish

the opposite inequality to (4.2).

We can write (2.2) in the form

$$\{G'(x) \exp(\int_1^x \varepsilon^{-1} H(t) dt)\}' = \varepsilon^{-1} H' G \exp(\int_1^x \varepsilon^{-1} H(t) dt),$$

from which it follows that $\{\dots\}$ is positive decreasing as x decreases from

1 to x_1 , the zero of H' . Hence, for $x \geq x_1$,

$$G'(x) \exp(\int_1^x \varepsilon^{-1} H(t) dt) \leq G'(1) = O(\varepsilon^{-\frac{1}{2}}),$$

$$(4.3) \quad G'(x) = O\{\varepsilon^{-\frac{1}{2}} \exp(\int_x^1 \varepsilon^{-1} H(t) dt)\}.$$

We can argue similarly with the differentiated form of (2.2),

$$\varepsilon G''' + H G'' = H'' G,$$

to show that, if $x_1 \leq x \leq x_2, 2$,

$$G''(x) \exp(\int_{x_2, 2}^x \varepsilon^{-1} H(t) dt) \leq G''(x_2, 2) = O(\varepsilon^{-1}),$$

$$G''(x) = O\{\varepsilon^{-1} \exp(\int_x^{x_2, 2} \varepsilon^{-1} H(t) dt)\}.$$

We can replace the upper limit of integration in the last integral by 1,

and since $1 - x_2, 2(\varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}}$ and $H(t, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}})$ for $x_2, 2(\varepsilon_n) \leq t \leq 1$, we will continue to have the estimate

$$G''(x, \varepsilon_n) = O\{\varepsilon_n^{-1} \exp(\int_x^1 \varepsilon_n^{-1} H(t) dt)\},$$

In the first place for $x_1(\varepsilon_n) \leq x \leq x_2, 2(\varepsilon_n)$, but then for all $x \geq x_1(\varepsilon_n)$ since in $x_2, 2(\varepsilon_n) \leq x \leq 1$ the estimate merely gives the result, already known from section 2, that $G'' = O(\varepsilon^{-1})$.

From (4.1) we have

$$|H(x_3(\varepsilon_n), \varepsilon_n)| \geq 2K_0 \varepsilon_n^{\frac{1}{2}}$$

If ε_n is sufficiently small, for some positive constant K_0 , and since $H' \geq 0$ in $[x_1, 1]$, we in fact have

$$|H(x, \varepsilon_n)| \geq 2K_0 \varepsilon_n^{\frac{1}{2}} \text{ for } x \in [x_1(\varepsilon_n), x_3(\varepsilon_n)].$$

Thus, for such x , from (4.3) we have

$$\begin{aligned} G'(x, \varepsilon_n) &\leq K\varepsilon_n^{-\frac{1}{2}} \exp\{\varepsilon_n^{-1} \int_x^1 H(t) dt\} \\ &= K\varepsilon_n^{-\frac{1}{2}} \exp\{\varepsilon_n^{-1} \int_{x_3}^1 H(t) dt + \varepsilon_n^{-1} \int_{x_3}^x H(t) dt\} \\ &\leq K\varepsilon_n^{-\frac{1}{2}} \exp\{-2K_0 \varepsilon_n^{\frac{1}{2}}(x_3 - x)\} \\ (4.4) \quad &\leq K\varepsilon_n^{-\frac{1}{2}} \exp\{-2K_0 \varepsilon_n^{\frac{1}{2}}(1-x)\}, \end{aligned}$$

the replacement of x_3 by 1 in the last formula line corresponding merely to an adjustment of the constant K , and again the estimate holds in the

First place for x in $[x_1(\varepsilon_n), x_3(\varepsilon_n)]$, but in fact for x in $[x_1(\varepsilon_n), 1]$.

Similarly, for x in $[x_1(\varepsilon_n), 1]$,

$$(4.4a) \quad G''(x, \varepsilon_n) \leq K \varepsilon_n^{-1} \exp \{-2K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\},$$

and we can obtain an estimate for H''' by writing, in $[x_1(\varepsilon_n), x_3(\varepsilon_n)]$,

$$\begin{aligned} & (\varepsilon_n H'' \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\})' \\ &= (-HH''' - GG' - K_0 \varepsilon_n^{\frac{1}{2}} H''') \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\} \\ &\geq (K_0 \varepsilon_n^{\frac{1}{2}} H''' - GG') \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\} \\ &\geq -GG' \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\} \\ &\geq -K \varepsilon_n^{-\frac{1}{2}} \exp \{-K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\}, \end{aligned}$$

using (4.4) and the fact that $G \leq 1$. Integration over $[x, x_3(\varepsilon_n)]$ gives,

$$(4.4b) \quad |H'''(x, \varepsilon_n)| \leq K \varepsilon_n^{-1} \exp \{-K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\},$$

and, as usual, this estimate holds in fact for x in $[x_1(\varepsilon_n), 1]$.

Returning now to (2.2) in $[x_1, 1]$, we have

$$\varepsilon G'' = H'G - HG'$$

with $H'G, -HG'$ of the same sign, so that

$$HG \leq \varepsilon G''.$$

We must thus have

either (a) $G(x_0, \varepsilon_n) \leq \{\varepsilon_n G''(x_0, \varepsilon_n)\}^{\frac{1}{2}}$ for some x_0 in $[x_1(\varepsilon_n), \frac{1}{2}\{x_1(\varepsilon_n) + x_3(\varepsilon_n)\}]$, where x_0 may depend on n ,

or (b) $H(x, \varepsilon_n) \leq \{\varepsilon_n G''(x, \varepsilon_n)\}^{\frac{1}{2}}$ throughout $[x_1(\varepsilon_n), \frac{1}{2}\{x_1(\varepsilon_n) + x_3(\varepsilon_n)\}]$.

In alternative (a), we have, for $x \geq x_0$,

$$\begin{aligned} G(x, \varepsilon_n) - G(x_0, \varepsilon_n) &= \int_{x_0}^x G'(t, \varepsilon_n) dt \\ &\leq K \exp \{-2K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\}, \end{aligned}$$

from (4.4), and thus, from (a) and (4.4a),

$$(4.5) \quad G(x, \varepsilon_n) \leq K \exp \{-K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\}.$$

Now, in $[x_0, x_2(\varepsilon_n)]$,

$$(\varepsilon_n H'' \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\})'$$

$$\begin{aligned} &= (-HH'' + \mu - \frac{1}{2}G^2 + \frac{1}{2}H'^2 - K_0 \varepsilon_n^{\frac{1}{2}} H'') \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\} \\ &\geq -\frac{1}{2}G^2 \exp \{K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\} \\ &\geq -K \exp \{-K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\}. \end{aligned}$$

from (4.5), and an integration over $[x, x_2(\varepsilon_n)]$ gives, for $x \geq x_0$,

$$(4.6) \quad |H''(x, \varepsilon_n)| \leq K \varepsilon_n^{-\frac{1}{2}} \exp \{-K_0 \varepsilon_n^{-\frac{1}{2}} (1-x)\}.$$

We may suppose that (b) holds for x in $[x_1(\varepsilon_n), x_0]$, since otherwise we could have chosen x_0 further to the left, and so, integrating (4.6) over $[x_0, x]$, we obtain

this holding in fact for x in $[x_1(\varepsilon_n), 1]$. A final integration produces the opposite inequality to (4.2).

In alternative (b), we have (4.6a) holding throughout

[$x_1(\varepsilon_n)$, $\frac{1}{2}\{x_1(\varepsilon_n) + x_3(\varepsilon_n)\}]$, and a standard inequality, along with (4.4b), assures us that (4.6) holds throughout the same interval. If we integrate the estimate (4.4b), we now see that (4.6) in fact holds throughout $[x_1(\varepsilon_n), 1]$, and the argument is then completed as in alternative (a).

the right-hand side being the area of the triangle with base x_1 and height $\{-H'(x_{2,1})\}$ which sits under the curve $-H'(x)$ for $0 \leq x \leq x_1$; and also, since

$$\underline{\text{Lemma 4.4.}} \quad H''(x_1(\varepsilon_n)) \geq K\varepsilon_n^{\frac{1}{2}}.$$

Proof. Suppose for contradiction that the lemma is not true. Then we must

have a subsequence of ε_n , which we can relabel ε_n , such that

$$H''(x_1(\varepsilon_n)) = o(\varepsilon_n^{\frac{1}{2}}),$$

and the shape of H'' then assures us that

$$H''(x, \varepsilon_n) = o(\varepsilon_n^{\frac{1}{2}}) \quad \text{for } x_{2,1}(\varepsilon_n) \leq x \leq x_1(\varepsilon_n).$$

Thus by integration

$$H'(x_{2,1}(\varepsilon_n)) = o(\varepsilon_n^{\frac{1}{2}}),$$

and since $x_{2,1}$ is the negative minimum of H' ,

$$H'(x, \varepsilon_n) = o(\varepsilon_n^{\frac{1}{2}}) \quad \text{for } 0 \leq x \leq x_1(\varepsilon_n),$$

which on integration certainly contradicts Lemma 4.3.

$$\underline{\text{Lemma 4.5.}} \quad 1 - x_1(\varepsilon_n) \leq K\varepsilon_n^{\frac{1}{2}} |\log(\varepsilon_n)|.$$

Proof. Lemma 4.4 and (4.6) together give

$$\varepsilon_n^{\frac{1}{2}} \leq K\varepsilon_n^{-\frac{1}{2}} \exp\{-K_0\varepsilon_n^{-\frac{1}{2}}(1-x_1(\varepsilon_n))\},$$

from which the result is immediate.

$$\underline{\text{Theorem 4.1.}} \quad \text{For } x \text{ in any fixed interval } [K_1, K_2], \quad 0 < K_1 < K_2 < 1, \quad \text{we have}$$

$$(4.7) \quad -H'(x, \varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}},$$

Proof. $H'(x, \varepsilon_n)$ is convex in $[0, x_1(\varepsilon_n)]$, at least if ε_n is sufficiently

small, for $H''' \geq 0$ in $[0, x_3]$ and $x_3(\varepsilon_n) > x_1(\varepsilon_n)$. Thus

$$(4.9) \quad -H(x_1) = \int_0^{x_1} |H'(t)| dt \geq \frac{1}{2}x_1 \{-H'(x_{2,1})\},$$

which contradicts (4.7). The result (4.12) follows by substitution in (2.9).

$$\underline{\text{Lemma 4.5.}} \quad H''(x_{2,1}(\varepsilon_n)) = \int_0^{x_1} |H'(t)| dt \leq x_1 \{-H'(x_{2,1})\}.$$

The inequalities (4.9) and (4.10), together with Lemma 4.3, give

$$-H'(x_{2,1}(\varepsilon_n)) \asymp \varepsilon_n^{\frac{1}{2}},$$

and the convexity of H' then leads to (4.7).

The estimate (4.8) follows by an integration.

Theorem 4.2. For $x \in [K_1, K_2]$,

$$(4.11) \quad H''(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}),$$

$$(4.12) \quad \varepsilon_n H'''(x, \varepsilon_n) + \frac{1}{2} G^2(x, \varepsilon_n) = \mu + O(\varepsilon_n).$$

Proof. Once ε_n is sufficiently small, we know that $H'(x, \varepsilon_n)$ is monotonic

increasing for $x \in [K_1, K_2]$. Thus it suffices to verify (4.11) at $x = K_1, K_2$. We carry out the analysis at $x = K_2$, the case $x = K_1$ being essentially the same.

We may suppose $x_{2,1}(\varepsilon_n) \rightarrow \bar{x}_{2,1}$. Since we may enlarge the interval if necessary, we may assume $x_{2,1} \neq K_2$. If $K_2 < \bar{x}_{2,1}$, then, for ε_n sufficiently small, the maximum of $|H'(x, \varepsilon_n)|$ for x in $[K_1, K_2]$ is assumed at $x = K_1$, and we need not consider this case. If $\bar{x}_{2,1} < K_2$, and if (4.11) is not true, then as $\varepsilon_n \downarrow 0$ through some subsequence (relabelled ε_n) we must have

$$H''(K_2, \varepsilon_n) > \varepsilon_n^{\frac{1}{2}},$$

and so

$$H''(x, \varepsilon_n) > \varepsilon_n^{\frac{1}{2}} \quad \text{for } K_2 \leq x \leq \frac{1}{2}(K_2+1),$$

Theorem 4.3. $\mu = O(\varepsilon_n)$.

Proof. Suppose for contradiction that, as $\varepsilon_n \downarrow 0$ through some subsequence

(relabelled ε_n),

$$\mu > \varepsilon_n .$$

If through any subsequence

$$(4.13) \quad 2\mu - G^2(x, \varepsilon_n) \geq K\mu \quad \text{at } x = K_3 ,$$

for some fixed K_3 in (K_1, K_2) and some fixed positive K , then the monotonicity of G implies that the same estimate holds throughout $[K_1, K_3]$, and (4.12) leads to $\varepsilon_n H''(x, \varepsilon_n) \geq K\mu$ throughout $[K_1, K_3]$, which on integration contradicts (4.11).

The only alternative to (4.13) is that

$$2\mu - G^2(K_3, \varepsilon_n) = o(\mu) \quad \text{for any fixed } K_3 \text{ in } (K_1, K_2) ,$$

since, from (4.12), $H'' > 0$ in $[K_1, K_2]$ implies that

$$2\mu - G^2 \geq O(\varepsilon_n) = o(\mu) .$$

Thus, using the monotonicity of G , we conclude that

$$G(x, \varepsilon_n) = (2\mu)^{\frac{1}{2}} \{1 + o(1)\} \quad \text{throughout } [K_3, K_2] .$$

We may certainly suppose that $y_1(\varepsilon_n) \rightarrow \bar{y}$, y_1 the zero of G'' , and we can choose K_3 so that $K_3 \neq \bar{y}$. If $K_3 < \bar{y}$ (and the argument for $K_3 > \bar{y}$ is similar), then for ε_n sufficiently small G' is decreasing in $[K_3, \frac{1}{2}(K_3 + \bar{y})]$, and if we define

$$K_4 = \min \left\{ \frac{3}{4}K_3 + \frac{1}{4}\bar{y}, \frac{3}{4}K_3 + \frac{1}{4}K_2 \right\} ,$$

$$K_5 = \min \left\{ \frac{1}{2}(K_3 + \bar{y}), \frac{1}{2}(K_3 + K_2) \right\} ,$$

then

$$G(K_4) - G(K_3) = \int_{K_3}^{K_4} G'(t)dt \geq (K_4 - K_3) G(K_4) ,$$

so that

$$G'(K_4) = o(\mu^{\frac{1}{2}}) ,$$

and so

$$G'(x) = o(\mu^{\frac{1}{2}}) = o(G(x)) \quad \text{throughout } [K_4, K_5] .$$

For this range of x , substitution in (2.2) yields

$$\varepsilon_n G'' \sim H'G > \varepsilon_n ,$$

so that $G'' > 1$, which contradicts $G \leq 1$ and so proves the theorem.

Lemma 4.6. Let $\delta > 0$ and $\varphi(x, \delta) \in C^2[a, b]$ with, for $x \in [a, b]$,

$$\delta \varphi'' + N(x, \delta) \varphi' = A(x, \delta) ,$$

$$|\varphi(x, \delta)| \leq B ,$$

$$N(x, \delta) \leq -p < 0 .$$

where Then there is a constant C , depending only on the length $b-a$, such that

$$|\varphi'(x, \delta)| \leq \frac{C}{b-x} (B + \frac{\|A\|_\infty}{p}) \quad \text{for } a \leq x \leq b .$$

Proof. This is a particular case of Theorem 2.7 of [21]. The proof there is for the case $N \geq 0$, but our case is obtained from that by interchanging the roles of a, b .

Theorem 4.4. For $x \in [K_1, K_2]$,

$$G(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}), \quad G'(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}), \quad G''(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}), \quad G'''(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}) ,$$

$$H''(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}), \quad H'(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}}) .$$

Proof. Equation (4.12) and Theorem 4.3 give

$$\varepsilon_n H'''(x, \varepsilon_n) + \frac{1}{2}G^2(x, \varepsilon_n) = O(\varepsilon_n) ,$$

and since $H''' > 0$ in $[K_1, K_2]$ for ε_n sufficiently small, we have the

requisite estimate on G .

The estimate for G' now follows from Lemma 4.6 and the equation

$$\varepsilon_n^{\frac{1}{2}} G'' + (\varepsilon_n^{-\frac{1}{2}} H) G' = \varepsilon_n^{-\frac{1}{2}} H' G,$$

and similarly for the others.

Theorem 4.5. With $[K_1, K_2]$ any fixed interval, $0 < K_1 < K_2 < 1$, let

$$k(\varepsilon_n) = G(K_1, \varepsilon_n)/H(K_1, \varepsilon_n).$$

Then, throughout $[K_1, K_2]$,

$$(4.14a) \quad G = kH + O(\varepsilon_n),$$

$$(4.14b) \quad G' = kH' + O(\varepsilon_n),$$

$$(4.14c) \quad G'' = kH'' + O(\varepsilon_n),$$

and

$$(4.15) \quad 0 < -k(\varepsilon_n) = O(1).$$

Proof. The estimate (4.15) comes from the definition of k , the known signs

of G, H , and the estimates in Theorems 4.1, 4.4. Equation (4.14a) arises by

rewriting (2.2) as

$$\varepsilon_n \frac{G''}{H^2} + \left(\frac{G'}{H} \right)' = 0,$$

so that

$$\varepsilon_n H(x) \int_{K_1}^x \frac{G''(t)}{H^2(t)} dt + G(x) = kH(x),$$

which leads to the required result. Then (4.14a) substituted into (2.2) in the

form

$$H'G - HG' = \varepsilon_n G'' = O(\varepsilon_n^{3/2}),$$

gives

$$HG' = H\{kH + O(\varepsilon_n)\} + O(\varepsilon_n^{3/2}),$$

and so (4.14b). Finally, differentiating (2.2) we obtain

$$H''G - HG'' = \varepsilon_n G''' = O(\varepsilon_n^{3/2}),$$

which leads to (4.14c).

5. The case $G(0, \varepsilon) = 0$ continued: estimates near $x = 0$

The situation is that outlined at the opening of section 4. Thus $G(0, \varepsilon) = 0$, and $\mu_0 = 0$. We want to use the estimates collected in section

4 to give information about the behaviour near $x = 0$.

Lemma 5.1.

$$H'''(0, \varepsilon_n) = O(1), \quad H''(0, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{4}}), \quad G'(0, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{4}}).$$

Proof. The estimate on H''' comes at once from (2.9) and the estimate on μ contained in Theorem 4.3. Indeed, for $x \leq \min\{\frac{1}{2}, x_{2,1}(\varepsilon_n)\}$, we have

$$\varepsilon_n H''' + HH'' = O(\varepsilon_n^{\frac{1}{4}}),$$

and since $H''' > 0$, $HH'' \geq 0$, we have

$$(5.1) \quad H''' = O(1) \text{ for } x \leq \min\{\frac{1}{2}, x_{2,1}(\varepsilon_n)\}.$$

From this, the estimate on $H''(0, \varepsilon_n)$ follows directly, since $H''(0, \varepsilon_n) > \varepsilon_n^{\frac{1}{4}}$ implies, with (5.1), that $x_{2,1}(\varepsilon_n) > \varepsilon_n^{\frac{1}{4}}$, that

$$H''(x, \varepsilon_n) > \varepsilon_n^{\frac{1}{4}} \text{ for } 0 \leq x \leq \varepsilon_n^{\frac{1}{4}},$$

and that

$$H'(\varepsilon_n^{\frac{1}{4}}, \varepsilon_n) > \varepsilon_n^{\frac{1}{2}},$$

which is a contradiction.

Finally, to obtain the estimate on G' , we have, for $x \leq y_1(\varepsilon_n)$, y_1 the zero of G'' , that

$$|\varepsilon_n G''| \leq |H'G| \leq |x H' G'(0)|,$$

and so

$$|G'| \leq K \varepsilon_n^{-\frac{1}{2}} \times G'(0),$$

$$|G'(x) - G'(0)| \leq \frac{1}{2} K \varepsilon_n^{-\frac{1}{2}} x^2 G'(0),$$

which implies that $G'(x)$ is of the same order as $G'(0)$ for $x \leq \min(K^{-\frac{1}{2}} \varepsilon_n^{\frac{1}{2}}, Y_1(\varepsilon_n))$. Indeed, since $G'(x)$ is increasing for $x \geq Y_1$, we can say that $G'(x)$ is at least of order $G'(0)$ for $x \leq K^{-\frac{1}{2}} \varepsilon_n^{\frac{1}{4}}$. Thus $G'(0) = O(\varepsilon_n^{\frac{1}{4}})$, since the contrary would on integration contradict the estimate on G in Theorem 4.4.

Now consider the transformation

$$x = \varepsilon^{\frac{1}{4}} \eta, \quad H(x, \varepsilon) = \varepsilon^{3/4} f(\eta, \varepsilon), \quad G(x, \varepsilon) = \varepsilon^{\frac{1}{2}} g(\eta, \varepsilon).$$

Then the equations (2.9) and (2.2) become

$$(5.2) \quad f''' + ff' + \frac{1}{2}(g^2 - f^2) = \mu/\varepsilon, \quad 0 \leq \eta \leq \varepsilon^{-\frac{1}{4}},$$

$$(5.3) \quad g'' + fg' = f'g, \quad 0 \leq \eta \leq \varepsilon^{-\frac{1}{4}},$$

and f, g satisfy the boundary conditions (amongst others)

$$(5.4) \quad f(0, \varepsilon) = f'(0, \varepsilon) = 0, \quad g(0, \varepsilon) = 0.$$

Lemma 5.2.

$$f''(0, \varepsilon_n) = O(1), \quad g'(0, \varepsilon_n) = O(1).$$

Proof. This is direct computation from Lemma 5.1.

We can now take the usual subsequence (relabelled ε_n) such that $f''(0, \varepsilon_n) \rightarrow a \leq 0$, $g'(0, \varepsilon_n) \rightarrow b \geq 0$, $\mu(\varepsilon_n)/\varepsilon_n = f'''(0, \varepsilon_n) \rightarrow \lambda_0 \geq 0$. If we define $f_0(\eta), g_0(\eta)$ to be the solutions of (5.2), (5.3), with $\mu/\varepsilon = \lambda_0$, which satisfy the boundary conditions (5.4) and $f_0'(0) = a$, $g_0'(0) = b$, we have the following lemma.

Lemma 5.3. With f_0, g_0 defined as above, the functions $f(\eta, \varepsilon_n), g(\eta, \varepsilon_n)$ and their derivatives tend uniformly on any finite η -interval to the functions

$f_0(\eta), g_0(\eta)$ and their derivatives. Moreover,

$$f'_0 \leq 0, \quad g'_0 \geq 0, \quad f'''_0 \geq 0,$$

f'''_0 changes sign at most once unless it is identically zero.

Proof. The lemma follows immediately from the continuous dependence of the solutions of the differential equation on the initial conditions and the shapes of H, G , remembering that $x_1(\varepsilon_n), x_3(\varepsilon_n), x_2, z(\varepsilon_n) \rightarrow 1$ as $\varepsilon_n \downarrow 0$.

Lemma 5.4. $f_0 \equiv 0, g_0 \equiv 0$.

Proof. The change of sign of f'''_0 does not in fact take place, since if it did we would have f'''_0 ultimately positive increasing, which would contradict

$f'_0 \leq 0$. (The case $f'''_0 \equiv 0$ leads at once to $f_0 \equiv 0, g'_0 \equiv 0, g'''_0 \equiv \text{constant}$, and

the boundedness of g_0 , which follows from the estimate on G in Theorem 4.4, means that $g'_0 \equiv 0, g \equiv 0$.) We may thus suppose $f'''_0 \leq 0$, and so $g'''_0 \leq 0$, since $Y_1 > x_{2,1}$. Then from differentiation of (5.3) we have $g'''_0 \leq 0$.

Since $g'''_0(0) = 0$, $g'''_0(\eta)$ tends to a strictly negative limit as $\eta \rightarrow \infty$ (which contradicts $g'_0 \geq 0$ unless $g'''_0 \equiv 0$). Hence $g'''_0 \equiv 0$ and, as before,

$$g_0 \equiv 0.$$

Then $f'''_0 \geq 0$, which implies that f'''_0 is positive increasing and contradicts $f'''_0 \leq 0$ unless $f'''_0 \equiv 0$. Then $f'''_0 \equiv \lambda_0 \geq 0$, and this again is a contradiction unless $\lambda_0 = 0$, $f'''_0 \equiv 0$, $f'_0 \equiv 0$ (the constant being zero because of the boundedness of f_0 that follows from Theorem 4.1), $f'_0 \equiv 0, f_0 \equiv 0$.

Theorem 5.1. If $G(0, \varepsilon) = 0$, then there is an $\varepsilon_0 > 0$ such that, for all ε with $0 < \varepsilon \leq \varepsilon_0$, there does not exist a solution $\langle H(x, \varepsilon), G(x, \varepsilon) \rangle$ of (2.1)-(2.4) which satisfies $G' \geq 0$.

Proof. The proof is a matter of obtaining a final contradiction from the results collected in section 4 and so far in the present section.

It is an immediate consequence of Lemma 5.4 that

$$(5.5) \quad H'(0, \varepsilon_n) = o(\varepsilon_n^{\frac{1}{4}}), \quad G'(0, \varepsilon_n) = o(\varepsilon_n^{\frac{1}{4}}), \quad \mu = o(\varepsilon_n),$$

and this suggests that we investigate a different scaling near $x = 0$. We

therefore consider the transformation

$$x = x_{2,1}(\varepsilon) X, \quad H(x, \varepsilon) = \varepsilon^{\frac{1}{2}} x_{2,1} u(X, \varepsilon), \quad G(x, \varepsilon) = \varepsilon^{\frac{1}{2}} v(X, \varepsilon),$$

and (2.9) and (2.2) become

$$\frac{\varepsilon^{\frac{1}{2}}}{2} u''' + \{uu'' + \frac{1}{2}(v^2 - u'^2)\} = \frac{u}{\varepsilon},$$

$$\frac{\varepsilon^{\frac{1}{2}}}{2} v''' + \{uv' - u'v\} = 0,$$

with the boundary conditions (amongst others)

$$u(0, \varepsilon) = 0, \quad u'(0, \varepsilon) = 0, \quad u''(0, \varepsilon) = 0, \quad v(0, \varepsilon) = 0.$$

The estimates in Theorems 4.1 and 4.4 imply that

$$u'(X, \varepsilon_n) = O(1) \quad \text{for } X \leq x_{2,1}^{-1} K_2,$$

and that

$$u'(1, \varepsilon_n) \asymp 1.$$

Also, integrating the estimate on H' in Theorem 4.1, we have for all x that

$$H(x, \varepsilon_n) = O(\varepsilon_n^{\frac{1}{2}} x) = O(\varepsilon_n^{\frac{1}{2}} x_{2,1} X), \quad \text{so that}$$

$$u(X, \varepsilon_n) = O(1) \quad \text{for } X \leq K.$$

We can now assert that

$$(5.6) \quad u''(X, \varepsilon_n) = O(1) \quad \text{for } \delta \leq X \leq x_{2,1}^{-1} K_2,$$

for any fixed small positive δ , the proof being essentially a repetition of the

first part of the proof of Theorem 4.2, and a similar argument gives

$$v'(X, \varepsilon_n) = O(1) \quad \text{for } \delta \leq X \leq x_{2,1}^{-1} K_2.$$

It follows that we can find a subsequence of ε_n (relabelled ε_n) such that, as $\varepsilon_n \downarrow 0$,

$$u(X, \varepsilon_n) \rightarrow u_0(X), \quad u'(X, \varepsilon_n) \rightarrow u'_0(X), \quad v(X, \varepsilon_n) \rightarrow v_0(X),$$

uniformly in any bounded subinterval of $[\delta, K_2 \lim_{\varepsilon_n \downarrow 0} x_{2,1}^{-1}(\varepsilon_n)]$. From the equation for u , and remembering from (5.5) that $\mu = o(\varepsilon_n)$, we see that

$$(5.7) \quad uu'' + \frac{1}{2}(v^2 - u'^2) \rightarrow 0$$

p.p. on any bounded subinterval of $[\delta, K_2 \lim_{\varepsilon_n \downarrow 0} x_{2,1}^{-1}(\varepsilon_n)]$:

for otherwise $u''' \geq K \varepsilon_n^{-\frac{1}{2}} x_{2,1}^2$ in a set of positive measure, and since $x_{2,1} > \varepsilon_n^{\frac{1}{4}}$, we should contradict (5.6). (The fact that $x_{2,1} > \varepsilon_n^{\frac{1}{4}}$ arises because the contrary would imply that H' , which takes its negative minimum at $x_{2,1}$, would have a minimum $o(\varepsilon_n^{\frac{1}{2}})$ since $H'' = o(\varepsilon_n^{\frac{1}{2}})$ throughout $[0, x_{2,1}]$ and this contradicts Theorem 4.1.)

From (5.7) we can conclude that u'' converges p.p. in the bounded subinterval, except where $u_0 = 0$; since u_0 is monotone, we conclude that either $u_0 = 0$ throughout any bounded subinterval, or $u_0 = 0$ throughout

$[\delta, \delta_1]$, say, which may be void, and u'' converges boundedly p.p. in any bounded subinterval of $[\delta_1, K_2 \lim_{\varepsilon_n \downarrow 0} x_{2,1}^{-1}(\varepsilon_n)]$. It is a standard argument that in the latter case $u'' \rightarrow u''$ and

$$(5.8) \quad u_0 u'' + \frac{1}{2}(v_0^2 - u_0'^2) = 0.$$

A similar argument with the equation for v yields

$$u_0 v'_0 - u'_0 v_0 = 0,$$

from which

$$(5.9) \quad v_0 = \lambda u_0 \quad \text{for some finite constant } \lambda.$$

The possibility that $u_0 = 0$ throughout some non-void $[6, \delta_1]$ can be excluded. For then $u'_0 = 0$, and this implies that $H'(x, \varepsilon_n) = o(\varepsilon_n^{\frac{1}{2}})$ throughout $[\delta x_{2,1}(\varepsilon_n), \delta x_{2,1}(\varepsilon_n)]$. But $H'(x_{2,1}(\varepsilon_n)) \asymp \varepsilon_n^{\frac{1}{2}}$ by Theorem 4.1, and the convexity of H' assures us that, for fixed δ , we also have

$$(5.10) \quad H'(\delta x_{2,1}(\varepsilon_n)) \asymp \varepsilon_n^{\frac{1}{2}}.$$

We may therefore suppose that (5.8), (5.9) hold throughout $(0, K_2 \lim_{\varepsilon_n \downarrow 0} x_{2,1}(\varepsilon_n))$,

using the open interval on the right since it is possible that $x_{2,1}^{-1} \rightarrow \infty$.

Two cases now arise, depending on whether or not $\lambda = 0$.

Case 1: $\lambda \neq 0$. The equation (5.8) can be differentiated to give

$$u_0 u''_0 + v_0 v'_0 = 0,$$

and substituting from (5.9) we obtain

$$u''_0 + \lambda^2 u'_0 = 0.$$

If $x_{2,1} \rightarrow 0$, then this equation holds over $(0, \infty)$, and this contradicts the monotonicity properties of u'_0 unless we suppose $u'_0 \equiv 0$, and this we have already excluded at (5.10).

Hence $x_{2,1} \nrightarrow 0$, and since u_0, v_0 are of opposite sign, we see that u'_0, v''_0 are of opposite sign, unless $u'_0 \equiv 0, v''_0 \equiv 0$. The case $u'_0 \equiv 0$ implies $v'_0 \equiv u'_0$ from (5.8) and also $v_0 \equiv \lambda u_0$ from (5.9). Thus

$$u'_0 \equiv \lambda u_0, \quad 0 \equiv u''_0 \equiv \lambda u'_0,$$

so that $u'_0 \equiv 0$, already excluded.

If on the other hand u''_0, v''_0 are of opposite sign, then $H'(x, \varepsilon_n)$, $G''(x, \varepsilon_n)$ are of opposite sign in any interval $[K_1, K_2]$ if ε_n is sufficiently small, and since H'', G'' are of opposite sign only in the intervals $(x_{2,1}, K_1)$ and $(x_{2,2}, 1)$, we conclude that $x_{2,1} < K_1, Y_1 > K_2$, i.e. (since K_1, K_2 are arbitrary) $x_{2,1} \nrightarrow 0, Y_1 \nrightarrow 1$, and this contradicts the already established fact that $x_{2,1} \nrightarrow 0$. This completes the discussion of the case $\lambda \neq 0$.

Case 2: $\lambda = 0$. We now have $v_0 \equiv 0$, and so, from (5.8) differentiated, $u''_0 \equiv 0, u'_0 \equiv \text{constant}$. If this constant is zero, then from (5.8) we have $u'_0 \equiv 0$, and this is already excluded.

Hence

$$(5.11) \quad u''_0 \equiv \text{constant} \neq 0.$$

If $x_{2,1} \nrightarrow 1$, then the value $X = 1$ is allowable, and since $u''(1, \varepsilon_n) = 0$, we have $u''_0(1) = 0$, contradicting (5.11).

Finally, if $x_{2,1} \nrightarrow 1, X$ and X are effectively the same variable, and we can conclude from (5.11) that in any interval $[K_1, K_2]$ of either X or X , and for ε_n sufficiently small,

$$H'(x, \varepsilon_n) \asymp \varepsilon_n^{\frac{1}{2}}.$$

Further, in $[K_1, K_2]$ we must have $H'' < 0, G'' < 0$ since $x_{2,1} \nrightarrow 1$, and so, from (2.2) differentiated,

$$\begin{aligned} |\varepsilon_n G'''| &> |H''G|, \\ |G'''| &> K \varepsilon_n^{-\frac{1}{2}} G. \end{aligned}$$

At the same time, $G'' < 0$ implies from (2.2) that

$$\begin{aligned} |HG'| &< |H'G|, \\ G' &< KG. \end{aligned} \tag{5.13}$$

Integrating (5.13), we see that if $[K_3, K_4]$ is a subinterval of $[K_1, K_2]$ with $K(K_4 - K_3) = \log 2$, then throughout $[K_3, K_4]$,

$$G(K_3) \leq G(x) \leq 2G(K_3).$$

We can now integrate (5.12), for $x \in [K_3, K_4]$, to obtain

$$|G''(x) - G''(K_3)| \geq K\varepsilon_n^{-\frac{1}{2}} G(K_3)(x - K_3),$$

and since G'' is negative decreasing,

$$|G''(x)| \geq K\varepsilon_n^{-\frac{1}{2}} G(K_3)(x - K_3).$$

Integrating once again, we have

$$|G'(x) - G'(K_3)| \geq \frac{1}{2} K\varepsilon_n^{-\frac{1}{2}} G(K_3)(x - K_3)^2,$$

so that, with $x = K_4$, remembering that G' is positive decreasing,

$$G'(K_3) \geq \frac{1}{2} K\varepsilon_n^{-\frac{1}{2}} G(K_3)(K_4 - K_3)^2,$$

which contradicts (5.13). This completes the proof for the case $\lambda = 0$, and so of Theorem 5.1.

REFERENCES

1. T. von Kármán, Über laminare und turbulente Reibung, *Z. Angew. Math. Mech.* 1 (1921), 232-252.
2. G. K. Batchelor, Note on a class of solutions of the Navier-Stokes equations representing steady rotationally-symmetric flow, *Quart. J. Mech. Appl. Math.* 4 (1951), 29-41.
3. S. P. Hastings, On existence theorems for some problems from boundary layer theory, *Arch. Rational Mech. Anal.* 38 (1970), 308-316.
4. A. R. Elcrat, On the swirling flow between rotating coaxial disks, *J. Diff. Equ.*, to appear.
5. J. B. McLeod and S. V. Parter, On the flow between two counter-rotating infinite plane disks, *Arch. Rational Mech. Anal.* 54 (1974), 301-327.
6. J. B. McLeod, The existence of axially symmetric flow above a rotating disk, *Proc. Roy. Soc. A* 324 (1977), 391-414.
7. G. N. Lance and M. H. Rogers, The axially symmetric flow of a viscous fluid between two infinite rotating disks, *Proc. Roy. Soc. A* 266 (1962), 109-121.
8. C. E. Pearson, Numerical solutions for the time-dependent viscous flow between two rotating coaxial disks, *J. Fluid Mech.* 21 (1965), 623-633.
9. G. L. Mellor, P. J. Chapple and V. K. Stokes, On the flow between a rotating and a stationary disk, *J. Fluid Mech.* 31 (1968), 95-112.
10. D. Greenspan, Numerical studies of flow between rotating coaxial disks, *J. Inst. Math. Appl.* 9 (1972), 370-377.
11. K. Stewartson, On the flow between two rotating coaxial disks, *Proc. Cambridge Philos. Soc.* 49 (1953), 333-341.
12. K. K. Tam, A note on the asymptotic solution of the flow between two oppositely rotating infinite plane disks, *SIAM J. Appl. Math.* 17 (1969), 1305-1310.
13. J. B. McLeod, Von Kármán's swirling flow problem, *Arch. Rational Mech. Anal.* 33 (1969), 91-102.
14. J. B. McLeod, The asymptotic form of solutions of von Kármán's swirling flow problem, *Quart. J. Math. (Oxford)* 2 20 (1969), 483-496.

15. P. Hartman, The swirling flow problem in boundary layer theory,
Arch. Rational Mech. Anal. 42 (1971), 137-156.
16. J. Watson, On the existence of solutions for a class of rotating disc flows and the convergence of a successive approximation scheme,
J. Inst. Math. Appl. 1 (1966), 348-371.
17. P. J. Bushell, On von Kármán's equations of swirling flow, J. London Math. Soc. (2) 4 (1972), 701-710.
18. J. B. McLeod, A note on rotationally symmetric flow above an infinite rotating disc, Mathematika 17 (1970), 243-249.
19. P. Hartman, On the swirling flow problem, Indiana Univ. Math. J. 21 (1972), 849-855.
20. J. B. McLeod, Swirling flow, Springer Lecture Notes 448 (1975), 242-255.

21. F. W. Dorr, S. V. Parter and L. F. Shampine, Applications of the maximum principle to singular perturbation problems, SIAM Review 15 (1973), 43-88.

