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The University of Wisconsin
Computer Sciences Department
1210 West Dayton Street
Madison, Wisconsin 53706

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Colin W. Cryer

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ABSTRACT

A linear multistep method (ρ,σ) is defined to be <u>DA_o-stable</u> if when it is applied to the delay differential equation $\dot{y}(t) = - y(t-\tau)$ the approximate solution $y^h(t_n) \to 0$ as $n \to \infty$ for all $\mu\epsilon(0,\pi/2\tau)$ and all stepsizes h of the form $h = \tau/m$, m a positive integer.

General properties of DA_o-stable methods are derived. These properties are similar to the properties of A-stable and $A(\alpha)$-stable methods; for example, it is proved that a k-step DA_o-stable method of order k must be implicit. As an application it is shown that the trapezoidal method is DA_o-stable.

Finally, the condition that $h = \tau/m$ is dropped and the resulting methods, which we call GDA_o-stable methods, are studied.

1. Introduction

Let

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j, \tag{1.1}$$

be such that $\alpha_k > 0$, σ is not identically zero, and ρ and σ have no zeros in common. If the linear multistep method $\{\rho,\sigma\}$ is applied to the problem

$$\begin{aligned} \dot{x}(t) &= \lambda x(t), \qquad t > 0, \\ x(0) &= 1, \end{aligned} \tag{1.2}$$

then (Henrici [6]),

$$\begin{aligned} \rho(E)\, x^h(t_n) &= -\lambda h \sigma(E) x^h(t_n), \ n \geq 0, \\ x^h(t_0) &= 1 \\ x^h(t_j) &= \psi^h(t_j), \ 0 \leq j < k, \end{aligned} \tag{1.3}$$

where h is the stepsize, x^h is the approximate solution, $t_j = jh$, E is the shift operator, and $\psi^h(t_j)$ is an approximation to $x(t_j)$.

If $\lambda \in H_+ = \{z : \mathrm{Re}(z) > 0\}$ then $x(t) = \exp(-\lambda t) \to 0$ as $t \to \infty$. By requiring that this property of x be inherited, at least in part, by x^h we obtain various classes of highly-stable methods such as the A-stable methods (Dahlquist), the A(α)-stable methods (Widlund), the strongly A-stable methods (Axelsson) and the stiffly-stable methods (Gear). (See Gear [5] and Lapidus and Seinfeld [7]). For the purposes of the present paper the most relevant class of highly-stable methods is one previously studied by the author (Cryer [4]): $\{\rho,\sigma\}$ is A_0-stable iff $x^h(t_n) \to 0$ as $n \to \infty$ when $\lambda \in (0,\infty)$. Equivalently, $\{\rho,\sigma\}$ is A_0-stable iff the roots of the characteristic equation

$$\rho(\zeta) + q\,\sigma(\zeta) = 0, \tag{1.4}$$

lie strictly inside the unit circle in the ζ-plane when $q \in (0,\infty)$.

In the present paper we study the analog of A_0-stable methods for delay differential equations. Let

$$\dot{y}(t) = -\mu y(t - \tau),\ t > 0, \qquad y(t) = \psi(t),\ -\tau \leq t \leq 0, \tag{1.5}$$

where $\tau > 0$ is the delay, where ψ is a given continuous function, and where μ is a real constant. Applying the method $\{\rho,\sigma\}$ to (1.5) and choosing the stepsize h so that

$$\tau = mh \tag{1.6}$$

where $m \in Z_+$, the set of strictly positive integers, we obtain

$$\begin{aligned} E^m \rho(E)\, y^h(t_n) &= -\mu h\, \sigma(E)\, y^h(t_n), \quad n \geq -m, \\ y^h(t_j) &= \psi(t_j), \quad -m \leq j \leq 0, \\ y^h(t_j) &= \psi^h(t_j), \quad 0 < j < k. \end{aligned} \tag{1.7}$$

It follows from the work of Tavernini [10] that $y^h \to y$ as $h \to 0$ iff the method $\{\rho,\sigma\}$ is convergent for ordinary differential equations. (For further references on numerical methods for delay differential equation see Tavernini [11] and Cryer [3].)

It is known (Bellman and Cooke [1, p. 444]) that $y(t) \to 0$ as $t \to \infty$ for all ψ iff $\mu \in (0,\pi/2\tau)$. We will say that $\{\rho,\sigma\}$ is $\underline{DA_0\text{-stable}}$ if $y^h(t_n) \to 0$ as $n \to \infty$ for all $\mu \in (0,\pi/2\tau)$ and all ψ. It is easily seen that $\{\rho,\sigma\}$ is DA_0-stable iff all the zeros of the characteristic equation

$$\phi\,(\zeta;q;m) \equiv \zeta^m \rho(\zeta) + (q/m)\,\sigma(\zeta) = 0, \tag{1.8}$$

lie in D, the open unit circle in the ζ-plane, for all $q \in (0,\pi/2)$ and all $m \in Z_+$.

So far as we are aware, the stability of numerical methods for delay differential equations has been considered previously only by Brayton and Willoughby [2] and Wiederholt [12]. Brayton and Willoughby show by means of an example that when Euler's method is used to solve a neutral differential equation the range of values of h for which the method is stable can differ from the corresponding range of values of h for ordinary differential equations.

In that part of his work which is of concern to us here, Wiederholt determines numerically for $m = 1, 2, 3$ and for specific choices of ρ and σ the set of values of q such that all the roots of (1.8) lie strictly inside D.

The definition of DA_0-stability is rather arbitrary in several respects: (i) the assumption that $\tau = mh$; (ii) the assumption that μ is real and not complex; and (iii) the use of (1.5) instead of more general equations such as

$$\dot{y}(t) = -\mu y(t - \tau) - \nu\, y(t).$$

Concerning these assumptions it may be remarked that the first is convenient but is later removed, the second is necessary because the theory of delay differential equations with complex coefficients has been little studied, while the third is not necessary but is very convenient because (1.5) has a very simple domain of asymptotic stability.

Since each restriction broadens the class of methods, the author believes that every highly-stable multistep method for delay differential equations will be DA_0-stable, so that by studying DA_0-stable methods one can determine properties which are common to all highly-stable methods for delay differential equations.

Following Henrici [6, p. 229] we set

$$z = \frac{\zeta - 1}{\zeta + 1}, \qquad \zeta = \frac{1+z}{1-z}. \tag{1.9}$$

We recall that the mapping (1.9) maps the open unit disk D in the ζ-plane onto the open left half plane H_- in the z-plane. We set

$$r(z) = \left(\frac{1-z}{2}\right)^k \rho\left(\frac{1+z}{1-z}\right) = \sum_{j=0}^{k} a_j z^j, \quad \text{say}, \tag{1.10}$$

$$s(z) = \left(\frac{1-z}{2}\right)^k \sigma\left(\frac{1+z}{1-z}\right) = \sum_{j=0}^{k} b_j z^j, \quad \text{say}, \tag{1.11}$$

and

$$f(z;q;m) = (1-z)^m \left(\frac{1-z}{2}\right)^k \phi\left(\frac{1+z}{1-z};q;m\right) = (1+z)^m\, r(z) + (q/m)(1-z)^m\, s(z)\,. \tag{1.12}$$

We note that

$$\rho(\zeta) = (\zeta+1)^k \, r\left(\frac{\zeta-1}{\zeta+1}\right), \quad \sigma(\zeta) = (\zeta+1)^k \, s\left(\frac{\zeta-1}{\zeta+1}\right), \tag{1.13}$$

and that

$$\phi(\zeta;q;m) = \left(\frac{\zeta+1}{2}\right)^m (\zeta+1)^k \, f\left(\frac{\zeta-1}{\zeta+1};q;m\right). \tag{1.14}$$

The characteristic equation of (1.5) is (Bellman and Cooke [1, p. 54]),

$$ve^{\tau v} + \mu = 0. \tag{1.15}$$

There is an interesting connection between (1.8) and (1.15). If the method $\{\rho,\sigma\}$ is of order p then (Henrici [6, p. 227])

$$\rho(\zeta)/\log\zeta - \sigma(\zeta) = O((\zeta-1)^p). \tag{1.16}$$

Using (1.16) and recalling that $\sigma(1) \neq 0$, equation (1.8) becomes

$$\zeta^m \log\zeta + q/m = O((\zeta-1)^{p+1}). \tag{1.17}$$

Setting $\zeta = \exp(v\tau/m)$ we obtain

$$ve^{v\tau} + q/\tau = m\, O((\zeta-1)^{p+1}). \tag{1.18}$$

In other words, (1.8) is a perturbation of (1.15), the magnitude of the perturbation being determined by the order of the method.

There is, of course, a similar relationship between (1.12) and (1.15). Instead of writing down the general case we will look at the trapezoidal method where the results are particularly instructive. For the trapezoidal method (1.12) takes the form

$$(1+z)^m \, 2z + (q/m)\,(1-z)^m = 0.$$

Setting $z = v\tau/2m$ and rearranging we find that

$$v\left(\frac{1+v\tau/2m}{1-v\tau/2m}\right)^m + q/\tau = 0\,. \tag{1.19}$$

The connection between (1.15) and (1.19) is obvious. In particular, it follows from the Hurwitz theorem (Marden [8, p. 4]) that the roots of (1.19) converge to the roots of (1.15) as $m \to \infty$.

Some general properties of DA_0-stable methods are derived in section 3. These properties are similar to properties of A-stable, $A(\alpha)$-stable, and A_o-stable methods; for example, it is proved that a k-step method of order k must be implicit.

To locate the roots of (1.15) is a non-trivial task so that, in view of the relationship between (1.8) and (1.15) it is to be expected that the location of the roots of ϕ will also be non-trivial. There are many elegant theorems concerning the zeros of polynomials (Marden [8]) but none of them seem to be sufficiently delicate. We have found only one general method for proving that a method is DA_0-stable: to first show that the roots of ϕ lie in the unit disk D for small q and then to show that the roots of ϕ do not lie on the unit circle for any $q \in (0,\pi/2)$. We have used this method to determine whether certain simple methods are DA_0-stable: the tedious preliminary computations are given in section 2 and the results are given in section 3.

In section 4 the condition that $\tau = mh$ is dropped and the resulting methods, which we call GDA_0-stable methods, are examined.

2. Preliminary results

In this section we determine (by very laborious means) whether or not the polynomial

$$P(\zeta) = \zeta^m(\zeta-1) + [q/(m-u)][(1-u) + u\zeta][(1-v) + v\zeta], \tag{2.1}$$

has zeros on the unit circle for various values of $u \in [0,1)$, $v \in [0,1]$, $q \in (0,\pi/2)$, and $m \in Z_+$. These results will be used in the analysis of the characteristic equation (1.8).

Let $\zeta_0 = e^{i\theta_0}$ be a zero of P on the unit circle. That is,

$$\zeta_0^m(\zeta_0-1) = -[q/(m-u)][(1-u) + u\zeta_0][(1-v) + v\zeta_0]. \tag{2.2}$$

Since P has real coefficients we will assume that $0 \le \theta_0 \le \pi$. Clearly $\zeta_0 \neq 1$ so that $\theta_0 > 0$. Also, $(1-u) + u\,\zeta_0 \neq 0$ and $(1-v) + v\zeta_0 \neq 0$.

Now,

$$\begin{aligned}(1-u) + u\zeta_0 &= [(\zeta_0^{\frac{1}{2}} + \zeta_0^{-\frac{1}{2}}) + (2u-1)(\zeta_0^{\frac{1}{2}} - \zeta_0^{-\frac{1}{2}})]\ \zeta_0^{\frac{1}{2}}/2,\\ &= [\cos(\theta_0/2) + i(2u-1)\sin(\theta_0/2)]\ \zeta_0^{\frac{1}{2}}, \\ &= [1 + i(2u-1)\tan(\theta_0/2)]\cos(\theta_0/2)\ \zeta_0^{\frac{1}{2}}.\end{aligned} \tag{2.3}$$

Hence, equating the squares of the absolute values of both sides of (2.2) we obtain

$$4\,S^2 = [q/(m-u)]^2[1-4u(1-u)\,S^2][1-4v(1-v)\,S^2], \tag{2.4}$$

or, equivalently,

$$4\,T^2(1 + T^2) = [q/(m-u)]^2[1 + (2u-1)^2T^2][1 + (2v-1)^2T^2], \tag{2.5}$$

where $S = \sin(\theta_0/2)$ and $T = \tan(\theta_0/2)$, and where (2.5) is applicable only if $\theta_0 < \pi$. Using (2.4) and Jordan's inequality (Mitrinovic [9, p. 33]) we find that

$$(m-u)\theta_0 = 2(m-u)\theta_0/2 \le 2(m-u)(\pi/2)\,S \le \pi q/2 < \pi. \tag{2.6}$$

Rearranging (2.2) we find that

$$\zeta_0^{m+\frac{1}{2}} = -[q/(m-u)][(1-u) + u\zeta_0][(1-v) + v\zeta_0]/[\zeta_0^{\frac{1}{2}} - \zeta_0^{-\frac{1}{2}}], \quad (2.7)$$
$$= i[q/(m-u)][(1-u) + u\zeta_0][(1-v) + v\zeta_0]/2S.$$

Let

$$\psi_1 \equiv \psi_1(u) \equiv \arg[(1-u) + u\zeta_0],$$
$$\psi_2 \equiv \psi_2(v) \equiv \arg[(1-v) + v\zeta_0], \quad (2.8)$$

so that $0 \leq \psi_1, \psi_2 \leq \theta_0 \leq \pi$. Then, from (2.7),

$$(m + \tfrac{1}{2})\theta_0 = \pi/2 + \psi_1 + \psi_2. \quad (2.9)$$

It might appear that (2.9) should involve the addition or subtraction of multiples of 2π, but this is not so since from (2.6),

$$(m + \tfrac{1}{2})\theta_0 = (m-u)\theta_0 + (u + \tfrac{1}{2})\theta_0 < 5\pi/2.$$

Since $0 \leq \psi_1 < \theta_0$ it follows from (2.9) that

$$\pi/2 \leq (m + \tfrac{1}{2})\theta_0 < \pi/2 + 2\theta_0. \quad (2.10)$$

In order to proceed further it is necessary to make certain simplifying assumptions. We begin by considering the case $u = 0$. Then, from (2.6), $\theta_0 < \pi$. Noting (2.3) we see that (2.2) is equivalent to

$$\zeta_0^m = i(q/2m)[\cos(\theta_0/2) + i(2v-1)\sin(\theta_0/2)]/\sin(\theta_0/2). \quad (2.11)$$

Equating the real parts of this equation we find that

$$\cos(m\theta_0) = -(q/2m)(2v-1). \quad (2.12)$$

Equating the squares of the absolute values of both sides of (2.11) we find that

$$T^2 = (q/2m)^2/[1-(q/2m)^2(2v-1)^2]. \quad (2.13)$$

Clearly, solving (2.11) is equivalent to solving (2.12) and (2.13) subject to (2.6).

We next consider the case $m = 1$. Then (2.2) takes the form

$$\zeta_0(\zeta_0-1) = -q[(1-u) + u\zeta_0][(1-v) + v\zeta_0]/(1-u), \tag{2.14}$$

which may be rewritten as

$$\zeta_0^2[(1-u) + q\,u\,v] + \zeta_0[-(1-u) + q\,\{u(1-v) + v(1-u)\}] + + q(1-u)(1-v) = 0 \tag{2.15}$$

If ζ_0 exists we must have that

$$(1-u) + q\,u\,v = q(1-u)(1-v), \tag{2.16}$$

which has the useful equivalent forms

$$q[1-u-v] = (1-u), \tag{2.17}$$

$$(1-u)[q(1-v)-1] = q\,u\,v, \tag{2.17a}$$

$$(q-1)(1-u) = q\,v, \tag{2.17b}$$

$$(q-1)(1-u-v) = v. \tag{2.17c}$$

It follows from (2.17a) that

$$\pi(1-v)/2 \geq q(1-v) \geq 1, \tag{2.18}$$

so that

$$q \geq 1, \tag{2.19}$$

and

$$v \leq (\pi-2)/\pi. \tag{2.20}$$

Finally, we consider the case $m \geq 2$ and $u \in [0,\frac{1}{2}]$. Then $\psi_1 \leq \theta_0/2$ so that it follows from (2.9) that $\theta_0 \leq \pi/2$ and $T = \tan\theta_0/2 \leq 1$. Using (2.3) we see that

$$\tan(\psi_1 - \theta_0/2) = (2u-1)\tan\theta_0/2.$$

Since tan is a convex function,

$$(1-2u)\tan\theta_0/2 \geq \tan[(1-2u)\theta_0/2]$$

so that we can conclude that

$$\psi_1 - \theta_0/2 \leq (2u-1)\theta_0/2.$$

On the other hand, let

$$h(u) = \tan[(u-1)\theta_0/2] - (2u-1)\tan\theta_0/2.$$

Then $h(0) = 0$. Furthermore,

$$\begin{aligned}\dot{h}(u) &= (\theta_0/2)\sec^2[(u-1)\theta_0/2] - 2\tan\theta_0/2,\\ &\leq (\theta_0/2)\sec^2(\theta_0/2) - 2\tan\theta_0/2,\\ &\leq T(1+T^2) - 2T,\\ &\leq 0.\end{aligned}$$

Hence, $h(u) \leq 0$ for $u \in [0,\frac{1}{2}]$ so that

$$\psi_1 - \theta_0/2 \geq (u-1)\theta_0/2.$$

Substituting the above bounds for ψ_1 into (2.9) we obtain

$$\pi/2 + u\theta_0/2 + \psi_2 \leq (m+\tfrac{1}{2})\theta_0 \leq \pi/2 + u\theta_0 + \psi_2. \tag{2.21}$$

Lemma 2.1.

Let $u = 0$, $v \in [0,\frac{1}{2})$, and $m > \pi^2/24|2v-1|$. Then there exists a $q \in (0,\pi/2)$ such that (2.2) has a solution ζ_o.

Proof: Equations (2.12) and (2.13) can be solved separately for θ_0: denote the solutions by $\theta_1(q)$ and $\theta_2(q)$, respectively. By (2.6), $\theta_0 < \pi$ so that without ambiguity

$$m\theta_1(q) = \pi/2 - \text{arc}\sin[(q/2m)|2v-1|],$$

$$m\theta_2(q) \;=\; 2m \text{ arc } \tan\{(q/2m)/[1-(q/2m)^2(2v-1)^2]^{\frac{1}{2}}\}.$$

Thus,

$$m\theta_1(0) - m\theta_2(0) \;=\; \pi/2 > 0.$$

On the other hand, using elementary inequalities,

$$\begin{aligned} m\theta_1(\pi/2)-m\theta_2(\pi/2) &\leq [\pi/2 - (\pi/4m)|2v-1|] - 2m \text{ arc } \tan\{\pi/4m\}, \\ &\leq [\pi/2 - (\pi/4m)|2v-1|] - 2m[(\pi/4m)-(\pi/4m)^3/3], \\ &= -(\pi/4m)[|2v-1| - \pi^2/24m], \\ &< 0, \end{aligned}$$

provided that $24m > \pi^2/|2v-1|$. The lemma follows.

Lemma 2.2.

Let $u = 0$ and $v \in [\frac{1}{2},1]$. Then (2.2) does not have a solution.

Proof: If $m = 1$ then the assertion follows from (2.20).

Now let $m \geq 2$ and set $c = (q/2m)(2v-1)$. From (2.12) we have that, unambiguously,

$$(m\theta_0)^2 \;=\; (\pi/2 + \psi)^2 \;\geq\; (\pi/2 + c)^2$$

where $\sin \psi = c$. On the other hand, it follows from (2.13) that

$$(m\theta_0)^2 \;\leq\; (2mT)^2 \;\leq\; (\pi/2)^2/(1-c^2).$$

However, $c \leq \pi/8$ so that, as is easily checked,

$$(\pi/2)^2/(1-c^2) \;<\; (\pi/2 + c)^2.$$

We have thus arrived at a contradiction and the lemma is proved.

Lemma 2.3.

If $v = 1$ then (2.2) does not have a solution.

Proof: If $m = 1$ the assertion follows from (2.20).

Now let $m \geq 2$ and $u \in [\frac{1}{2},1)$. Set $\bar{m} = m-1$, $\bar{q} = q\bar{m}/(\bar{m} + 1 - u)$, and $\bar{v} = u$. Then

$$P(\zeta) = \zeta\{\zeta^{\bar{m}}(\zeta-1) + (\bar{q}/\bar{m})[(1-\bar{v}) + \bar{v}\zeta]\}.$$

and it follows from Lemma 2.2 that P has no zeros on the unit circle.

Finally let $m \geq 2$ and $u \in [0,\frac{1}{2})$. Since $\psi_2 = \theta_0$, (2.21) is equivalent to

$$\pi/2 \leq (m-u/2-\tfrac{1}{2})\theta_0 \leq \pi/2 + u\theta_0/2,$$

so that $T = \tan \theta_0/2 \geq (\pi/4)/(m-u/2-\frac{1}{2})$. From (2.5) we find that

$$1/T^2 = [4 - [q/(m-u)]^2(2u-1)^2]/[q/(m-u)]^2,$$

$$\geq 16(m-u)^2/\pi^2 - (2u-1)^2.$$

Combining the above inequalities we find that

$$16(m - u/2 - \tfrac{1}{2})^2/\pi^2 \geq 1/T^2 \geq 16(m-u)^2/\pi^2 - (2u-1)^2.$$

Hence

$$16\ (2m - 3u/2 - \tfrac{1}{2})(u/2 - \tfrac{1}{2})/\pi^2 \geq -(2u-1)^2,$$

which implies, since $m \geq 2$, that

$$12(1-u)^2/\pi^2 \leq (1-2u)^2.$$

But $u \leq \frac{1}{2}$ so that $1-u \geq 1-2u$ and we have arrived at a contradiction.

Lemma 2.4.

If $v = \frac{1}{2}$ then (2.2) does not have a solution.

Proof: If m = 1 the assertion follows immediately from (2.20).

Next, let $m \geq 2$ and $u \in [\frac{1}{2},1)$. Then, from (2.3),

$$\tan(\psi_1 - \theta_0/2) = (2u-1)\tan\theta_0/2,$$
$$\geq \tan[(2u-1)\theta_0/2],$$

so that $\psi_1 \geq u\theta_0$. Since $\psi_2 = \theta_0/2$ it follows from (2.9) that

$$(m-u)\theta_0 \geq \pi/2.$$

However, from (2.5),

$$4T^2 < [q/(m-u)]^2$$

with the result that

$$(m-u)\theta_0 \leq 2(m-u)\theta_0/2 < 2(m-u)\tan\theta_0/2,$$
$$< \pi/2,$$

so that we have a contradiction.

Finally, let $m \geq 2$ and $u \in [0,\frac{1}{2}]$. Then it follows from (2.21) that

$$(m-u)\theta_0 = (\pi/2 - \psi)$$

where $\psi \geqslant 0$. Now (2.2) may be rewritten in the form

$$\zeta_o^{m-u} = -q[(\zeta_o+1)/(\zeta_o-1)][(1-u)\zeta_o^{-u} + u\zeta_o^{1-u}]/2(m-u).$$

Equating the real parts of this equation we find that

$$\sin\psi = \cos(m-u)\theta_0 = q\, g(\theta_0)/2(m-u)T,$$

where

$$g(\theta) = (1-u)\sin u\theta - u\sin(1-u)\theta.$$

Since $g(0) = \dot{g}(0) = \ddot{g}(0) = 0$, we have that

$$\sin\psi \le \varepsilon\ \theta_0^3 \max_{0\le\theta\le\theta_0} |\ddot{g}(\theta)|/6T,$$

$$\le \varepsilon\ \theta_0^3 \max_{0\le u\le\frac{1}{2}} |u(1-u)^3|/6T,$$

$$= 9\varepsilon\ (\theta_0^3/256T).$$

where $\varepsilon = (q/2)/(m-u) < (1.6)/3.$

Using (2.5) we see that

$$\theta_0 \le 2T \le 2\varepsilon$$

and that

$$T \geqslant \varepsilon/[1 + T^2]^{\frac{1}{2}} \geqslant \varepsilon/[1 + \varepsilon^2]^{\frac{1}{2}} > \varepsilon/(1 + \varepsilon^2/2) > \varepsilon/(1 + (1.6)^2/18) > 6\varepsilon/7.$$

Substituting these bounds into the previous inequality for $\sin\psi$ it is found that

$$\sin\psi < 9\varepsilon(2\varepsilon)^3\ (7/6\varepsilon)/256 < \varepsilon^3/6.$$

On the other hand,

$$(m-u)\theta_0 = 2(m-u)\theta_0/2,$$

$$= 2(m-u) \text{ arc tan } T,$$

$$\leqslant 2(m-u) \text{ arc tan } \varepsilon,$$

$$\le (\pi/2\varepsilon)(\varepsilon-\varepsilon^3/3 + \varepsilon^5/5),$$

$$= (\pi/2)(1 - \varepsilon^2/3 + \varepsilon^4/5),$$

so that

$$\psi = \pi/2 - (m-u)\theta_0 \ge (\pi/2)(\varepsilon^2/3 - \varepsilon^4/5).$$

Combining the above inequalities and using Jordan's inequality, we find that

$$\varepsilon^3/6 \ge \sin\psi \ge (2/\pi)\psi \ge \varepsilon^2/3 - \varepsilon^4/5,$$

which implies that

$$\varepsilon > 2 - 6\varepsilon^2/5 > \cdot 8.$$

But

$$\begin{aligned} \epsilon &= (q/2(m-u)), \\ &\leq (\pi/2)/2(3/2), \\ &= \pi/6, \\ &< \cdot 6. \end{aligned}$$

We have thus arrived at a contradiction and the lemma is proved.

Lemma 2.5.

For each $m \in Z_+$ there exists $q \in (0,\pi/2)$ such that the equation

$$\overline{P}(\zeta) = \zeta^m(\zeta-1) + q/(m + \tfrac{1}{2}) = 0$$

has a solution $\zeta_0 = e^{i\theta_0}$ on the unit circle.

Proof: Comparing $\overline{P}(\zeta)$ and $P(\zeta)$ for $u = v = 0$ we see that the two polynomials are almost the same, the only difference being that the variable q in $P(\zeta)$ is replaced by $\bar{q} = mq/(m+\frac{1}{2})$ in $\overline{P}(\zeta)$. The present proof is therefore a modification of the proof of Lemma 2.1.

If $m = 1$ then it is easily verified that the lemma holds.

If $m \geq 2$ then it follows, as in (2.6), that $\Theta_0 < \pi$. Set $w = q/(2m+1) = \bar{q}/2m$,

$$m\bar{\theta}_1(q) = \pi/2 - \text{arc sin } w,$$

and

$$m\bar{\theta}_2(q) = 2m \text{ arc tan } \{w/[1-w^2]^{\frac{1}{2}}\}.$$

It suffices to prove that $m\bar{\theta}_1(\pi/2) - m\bar{\theta}_2(\pi/2) < 0$.

Let $\bar{w} = (\pi/2)/(2m+1)$ so that $\bar{w} < 1$. Then

$$m\bar{\theta}_1(\pi/2) - m\bar{\theta}_2(\pi/2)$$

$$\leq (\pi/2 - \bar{w}) - [(\pi/2\bar{w})-1] \text{ arc tan } \{\bar{w}/[1-\bar{w}^2]^{\frac{1}{2}}\},$$

$$= (\pi/2 - \bar{w}) [\bar{w} - \text{arc tan } \{\bar{w}/[1-\bar{w}^2]^{\frac{1}{2}}\}]/\bar{w}.$$

But $\sin^2 \bar{w} < \bar{w}^2$ so that $(1-\bar{w}^2) \sin^2 \bar{w} < (1-\sin^2 \bar{w})\bar{w}^2$ and hence $\tan^2 \bar{w} < \bar{w}^2/(1-\bar{w}^2)$. Therefore $\bar{w} < \text{arc tan } (\bar{w}/[1-\bar{w}^2]^{\frac{1}{2}})$. It follows that $m\bar{\theta}_1(\pi/2) - m\bar{\theta}_2(\pi/2) < 0$ and the lemma is proved.

As the reader will have by now appreciated, the analysis of the roots of $P(\zeta)$ is extremely laborious. The basic reason is that when $q = 0$ $P(\zeta)$ has the zero $\zeta = 1$ so that when m is large $P(\zeta)$ must have a zero close to 1, and it is necessary to determine whether this root lies inside or outside the unit circle.

There are many general theorems giving bounds for the roots of polynomials. None of them seem to be sufficiently sharp for our purposes. Those, such as the theorem of Cauchy (Marden [8, p. 123]), which do not take account of the signs of the coefficients of the polynomial are hopelessly inaccurate. However, several theorems do take account of the signs of the coefficients and yield interesting bounds; among such theorems we draw the reader's attention to the theorems of Lucas (Marden [8, p. 22]), of Obreschkoff and Schoenberg (Marden [8, p. 191]), of Pellet (Marden [8, p. 128]), of van Vleck (Marden [8, p. 153]), as well as the various theorems on lacunary polynomials (Marden [8]).

3. DA_0-stable methods

Theorem 3.1.

If the method $\{\rho,\sigma\}$ is DA_0-stable then it is D-stable (stable in the sense of Dahlquist).

Proof: Assume that $\{\rho,\sigma\}$ is DA_0-stable. It suffices to prove that ρ satisfies the root condition, that is, all the zeros of ρ lie inside or on the unit circle and all the zeros of ρ on the unit circle are simple.

Let ζ_1 be a zero of ρ lying outside the unit circle. Then there exists a circle C with centre ζ_1 which lies outside the unit circle and on which ρ does not vanish. Applying the theorem of Rouché (Marden [8, p. 2]) it is easily seen that there exists a $q_1 \in (0,\pi/2)$ such that $\phi(\zeta;q_1;1)$ has at least one zero inside C, which is impossible.

Now assume that ρ has a zero of multiplicity at least two on the unit circle; without loss of generality we may assume that this zero is $\zeta_1 = 1$, so that $\rho(\zeta) = (\zeta-1)^2\,\gamma(\zeta)$, say, where γ is a polynomial. For $m\varepsilon Z_+$ let

$$C_m = \{\zeta : |\zeta| = 1 + 1/m;\ |\arg \zeta| \leq (k+3)\pi/m\}.$$

If $\zeta \in C_m$ then $|\zeta-1| < (k+4)\pi/m$. Since ρ and σ are relatively prime, $\sigma(1) \neq 0$. Now consider

$$F(\zeta) = m\zeta^m\,\rho(\zeta)/\sigma(\zeta) = m\zeta^m(\zeta-1)^2\,\gamma(\zeta)/\sigma(\zeta).$$

For $\zeta \in C_m$ and m sufficiently large, $|F(\zeta)| > 0$ and

$$\begin{aligned}|F(\zeta)| &\leq m(1+1/m)^m((k+4)\pi/m)^2[1+|\gamma(1)/\sigma(1)|],\\ &\leq ((k+4)\pi)^2 e[1+|\gamma(1)/\sigma(1)|]/m,\\ &< 1.\end{aligned}$$

On the other hand, when m is large then $\arg(\zeta^m)$ increases by $(2k + 6)\pi$ as ζ

traverses C_m while arg $(\rho(\zeta)/\sigma(\zeta))$ changes by at most $(2k + 2)\pi$. It follows that there exist $m_2 \in Z_+$ and $\zeta_2 \in C_{m_2}$ such that $q_2 = - F(\zeta_2) \in (0,1)$ and $\phi(\zeta_2;q_2;m_2) = 0$, which is impossible since $\{\rho,\sigma\}$ is DA_0-stable.

Theorem 3.2

The method $\{\rho,\sigma\}$ is DA_0-stable iff all the zeros of $f(z;q;m)$ lie in H_- for all $q \in (0,\pi/2)$ and $m \in Z_+$.

Proof:

Necessity. Assume that $\{\rho,\sigma\}$ is DA_0-stable and that $f(z_1;q_1;m_1) = 0$ for some $q_1 \in (0,\pi/2)$ and $m_1 \in Z_+$. We wish to show that $z_1 \in H_-$.

If $z_1 \neq 1$ then, from (1.12), $\phi(\zeta_1;q_1;m_1) = 0$ where $\zeta_1 = (1+z_1)/(1-z_1)$; since $\{\rho,\sigma\}$ is DA_0-stable $\zeta_1 \in D$ so that $z_1 \in H_-$. If $z_1 = 1$ then, from (1.12), $r(1) = 0$; but, from (1.10), $r(1) = \alpha_k > 0$. Thus $z_1 = 1$ is impossible.

Sufficiency. Assume that all the zeros of $f(z;q;m)$ lie in H_- for all $q \in (0,\pi/2)$ and all $m \in Z_+$. Let $\phi(\zeta_1;q_1;m_1) = 0$ for some $q_1 \in (0,\pi/2)$ and $m_1 \in Z_+$. We wish to show that $\zeta_1 \in D$.

If $\zeta_1 \neq -1$ then, from (1.14), $f(z_1;q_1;m_1) = 0$ where $z_1 = (\zeta_1-1)/(\zeta_1+1)$; hence, $z_1 \in H_-$ and $\zeta_1 \in D$.

If $\zeta_1 = -1$ then, from (1.8), $(-1)^{m_1} \rho(-1) + (q_1/m_1)\, \sigma(-1) = 0$. Since, from (1.13), $\rho(-1) = (-2)^k a_k$ and $\sigma(-1) = (-2)^k b_k$, it follows that $a_k + (-1)^{m_1}$ $(q_1/m_1)b_k = 0$. If $b_k = 0$ then $a_k = 0$ so that $\rho(-1) = \sigma(-1) = 0$ which contradicts the assumption that ρ and σ are relatively prime. Thus $b_k \neq 0$ and $a_k + (-1)^{m_1} (q/m_1)b_k$ changes sign as q passes through the point q_1. Now $a_k + (-1)^{m_1} (q/m_1)b_k$ is the leading coefficient of $f(z;q;m_1)$. By assumption, all the zeros of f lie in H_- so that (Henrici [6, p. 230]) all the coefficients of $f(z;q;m_1)$ have the same sign. It follows that all the coefficients of $f(z;q;m_1)$ change sign as q passes through the point q_1. That is, $f(z;q_1;m_1) \equiv 0$. From (1.14) we conclude that $\phi(\zeta;q_1;m_1) \equiv 0$ which contradicts the assumption that ρ and σ are relatively prime. Hence $\zeta_1 = -1$ is impossible and the proof of sufficiency is complete.

Theorem 3.3

If the k-step method $\{\rho,\sigma\}$ is DA_0-stable and of order k then it is implicit.

Proof: Assume that $\{\rho,\sigma\}$ is DA_0-stable, of order k, and explicit. By Theorem 3.1 $\{\rho,\sigma\}$ is also D-stable.

Using (1.11) we see that $s(1) = \beta_k = 0$, so that

$$b_k = -\sum_{j=0}^{k-1} b_j \tag{3.1}$$

It is shown by Henrici [6, p. 231] that

$$b_j = \sum_{\ell\geq 0} c_{2\ell}\, a_{j+1-2\ell} \quad \text{if } 0 \leq j < k, \tag{3.2}$$

where the constants $C_{2\ell}$ are the coefficients in the power series expansion of $z/\{\log[(1+z)/(1-z)]\}$, and where $a_j = 0$ if $j < 0$.

Henrici proves that $c_0 = \frac{1}{2}$, and that $c_{2\ell} < 0$ for $\ell > 0$. Henrici also proves that

$$\sum_{j=0}^{\ell} c_{2\ell-2j}/(-1+2j) = 0 \quad \text{if} \quad \ell \geq 1,$$

from which it easily follows that

$$\sum_{j=0}^{\ell} c_{2\ell-2j} > 0 \qquad \text{and} \qquad \left(\sum_{j=0}^{\ell} c_{2\ell-2j}\right) + c_{2\ell} > 0, \quad \text{if} \quad \ell \geq 1. \tag{3.3}$$

Since $\{\rho,\sigma\}$ is D-stable it follows (Henrici [6, p. 230]) that $a_0 = 0$, $a_1 \neq 0$, and that all the coefficients a_j have the same sign. From (1.10),

$$\sum_{j=0}^{k} a_j = r(1) = \alpha_k > 0,$$

so that $a_0 = 0$, $a_1 > 0$, and $a_j \geq 0$ for $2 \leq j \leq k$.

Now consider the polynomial

$$P_m(z) = f(z;3/2;m) = (1+z)^m\, r(z) + (3/2m)\,(1-z)^m\, s(z),$$

$$= \sum_{j=0}^{k+m} d_j^{(m)} z^j, \quad \text{say.}$$

By Theorem 3.2 the zeros of $P_m(z)$ lie in H_- so that (Henrici [6, p. 230]) the coefficients of $P_m(z)$ have the same sign. Using (3.2) we see that

$$d_0^{(m)} = a_0 + (3/2m)\ b_0 = (3/4m)\ a_1 > 0,$$

so that the coefficients of $P_m(z)$ are non-negative. In particular,

$$d_{k+2}^{(2)} = (a_k + 3b_k/4) \geq 0, \tag{3.4}$$

and

$$d_k^{(1)} = (a_k + a_{k-1} + (3/2)(b_k - b_{k-1})) \geq 0. \tag{3.5}$$

We now obtain bounds for b_k and b_{k-1}. It is convenient to set $c_j = 0$ if j is odd. Then, from (3.1) and (3.2)

$$\begin{aligned} - b_k &= \sum_{j=0}^{k-1} b_j, \\ &= \sum_{j=0}^{k-1} \sum_{\ell \geq 0} c_{2\ell}\, a_{j+1-2\ell} \\ &= \sum_{t=0}^{k} a_t \sum_{j=0}^{k-t} c_j \quad . \end{aligned}$$

Remembering that the a_j are non-negative, that $a_1 > 0$, and that $c_0 = \frac{1}{2}$, and using (3.2) and (3.3) we obtain three inequalities:

$$- b_k \geq a_1 \sum_{j=0}^{k-1} c_j > 0, \tag{3.6}$$

$$\begin{aligned} - b_k &\geq \sum_{t=k-1}^{k} a_t \sum_{j=0}^{k-t} c_j , \\ &= (a_k + a_{k-1})/2, \end{aligned} \tag{3.7}$$

and

$$- b_k + b_{k-1} = \sum_{t=0}^{k} a_t \left[c_{k-t} + \sum_{j=0}^{k-t} c_j \right] ,$$

$$\geq \sum_{t=k-1}^{k} a_t \left[c_{k-t} + \sum_{j=o}^{k-t} c_j \right] ,$$

$$= a_k + a_{k-1}/2 \qquad (3.8)$$

Substituting (3.7) into (3.4) we find that

$$\begin{aligned} 0 &\leq a_k + 3b_k/4, \\ &\leq a_k - 3\,(a_k + a_{k-1})/8, \\ &= (5a_k - 3a_{k-1})/8. \end{aligned} \qquad (3.9)$$

Substituting (3.8) into (3.5) we find that

$$\begin{aligned} 0 &\leq a_k + a_{k-1} + 3(b_k - b_{k-1})/2, \\ &\leq a_k + a_{k-1} - 3\,(a_k + a_{k-1}/2)/2, \\ &= (-2a_k + a_{k-1})/4. \end{aligned} \qquad (3.10)$$

Inequalities (3.9) and (3.10) are incompatible unless $a_k = 0$. But if $a_k = 0$ then, using (3.6), we have that $a_k + 3b_k/4 < 0$ which contradicts (3.4).

We have thus shown that the assumption that $\{\rho,\sigma\}$ is DA_0-stable, of order k, and explicit leads to a contradiction, and the theorem is proved.

Theorem 3.4

Let the zeros of $\rho(\zeta)$ other than $\zeta = 1$ lie in D, and let $\{\rho,\sigma\}$ be convergent. Then $\{\rho,\sigma\}$ is DA_0-stable iff for all $q \in (0,\pi/2)$ and all $m \in Z_+$ the polynomial $\phi(\zeta;q;m)$ has no zeros on the unit circle.

Proof: Let $m \in Z_+$ be fixed. Let $\zeta_1(q), \ldots, \zeta_{m+k}(q)$ be the zeros of $\phi(\zeta;q;m)$ with $\zeta_1(0) = 1$ and $|\zeta_j(0)| < 1$ for $2 \leq j \leq m+k$. Each $\zeta_j(q)$ depends continuously upon q so that for some $q_1 \in (0,\pi/2)$ we have that $|\zeta_j(q)| < 1$ for $q \in (0,q_1)$ and $2 \leq j \leq m+k$.

Since $\{\rho,\sigma\}$ is convergent, $\rho(1) = 0$ and $\dot{\rho}(1) = \sigma(1) \neq 0$. Differentiating the equation $\phi(\zeta_1(q);q;m) = 0$ with respect to q we find that $\dot{\zeta}_1(0) = -1/m$.

Hence $|\zeta_1(q)| < 1$ for sufficiently small $q \in (0,\pi/2)$.

The theorem is now an immediate consequence of the continuity of the $\zeta_j(q)$ with respect to q.

Using Lemmas 2.1 and 2.2 together with Theorem 3.4 we obtain

Theorem 3.5

Let $\beta_0 = 1-v$ and $\beta_1 = v$, where $v \in [0,1]$. Then the method

$$\rho(\zeta) = \zeta-1, \qquad \sigma(\zeta) = \beta_1\zeta + \beta_0$$

is DA_0-stable iff $v \in [\frac{1}{2},1]$.

4. GDA_0-stable methods

In the definition of DA_0-stability it was assumed that $\tau = mh$. In many important problems this is a reasonable assumption because the delay is constant. When the delay is variable, or very small, however, we must relax this condition. In the present section we assume that

$$\tau = (m-u)h > 0 \tag{4.1}$$

where $u \in [0,1)$ and $m \in Z_+$. (It would perhaps be more natural to set $\tau = (m+u)h$ but this would involve allowing m to take on the value zero.)

When (4.1) holds and $u > 0$ then the use of the method $\{\rho,\sigma\}$ involves the computation of the approximate solution at non-gridpoints. For a general discussion of this question the reader is referred to Tavernini [10]. Here we will assume that the value of the approximate solution y^h at the non-gridpoint t_{j+u} is obtained from the values at the gridpoints t_{j+1}, t_j, ..., $t_{j-\ell+1}$. That is, we set

$$y^h(t_{j+u}) = E^{-\ell+1}\gamma(E;u)\; y^h(t_j) \tag{4.2}$$

where ℓ is a positive integer and $\gamma(E;u)$ is a polynomial in E of degree at most ℓ with coefficients which depend upon u. The resulting method will be denoted by $\{\rho,\sigma,\gamma\}$; when applied to the problem (1.5) we obtain

$$E^{m+\ell-1}\rho(E)\; y^h(t_n) = -\mu h\; \sigma(E)\; \gamma(E;u)\; y^h(t_n),\; n \geq -m+1, \tag{4.3}$$

with appropriate initialization procedures.

Usually, γ will be an interpolation polynomial. For example, if linear interpolation is used then

$$y^h(t_{j+u}) = (1-u)\; y^h(t_j) + uy^h(t_{j+1})$$

so that $\gamma(E;u) = uE + (1-u)$ and $\ell = 1$. We will not insist that γ be an interpolation polynomial but will assume conditions which are always satisfied by interpolation polynomials, namely that (4.2) is exact if $u = 0$ or if y^h is a constant; that is, we require that

$$\gamma(E;0) \;=\; E^{\ell-1} \qquad (4.4)$$

and

$$\gamma(1;u) \;\equiv\; 1. \qquad (4.5)$$

It may be remarked that it was assumed that $y^h(t_{j+u})$ is computed using values of y^h at gridpoints up to at most t_{j+1} in order to ensure that when $m = 1$ equation (4.3) is an equation for $y^h(t_{n+\ell+k})$ in terms of the values of y^h at gridpoints up to at most $t_{n+\ell+k-1}$. In the present context this assumption is therefore necessary. However, for a specific practical problem it might be known that $m \neq 1$, in which case the assumption could be relaxed.

We will say that the method $\{\rho,\sigma,\gamma\}$ is <u>GDA_0-stable</u> if $y^h(t_n) \to 0$ as $n \to \infty$ for all $\mu \in (0,\pi/2\tau)$ and all initial data ψ. Equivalently, $\{\rho,\sigma,\gamma\}$ is GDA_0-stable iff all the zeros of the characteristic equation

$$\Phi(\zeta;q;m;u) \;=\; \zeta^{m+\ell-1}\,\rho(\zeta) + \frac{q}{m-u}\,\sigma(\zeta)\,\gamma(\zeta;u) \;=\; 0 \qquad (4.6)$$

lie in the open unit circle D for all $q \in (0,\pi/2)$, $m \in Z_+$, and $u \in [0,1)$. Noting (4.4) we see that

$$\Phi(\zeta;q;m;0) \;=\; \zeta^{\ell-1}\,\phi(\zeta;q;m). \qquad (4.7)$$

It has not been assumed that $\rho(\zeta)$ and $\gamma(\zeta;u)$ are relatively prime, so that, without further assumptions, it is possible for Φ to vanish identically for certain values of q, m, and u. It would be possible to avoid this difficulty by assuming that $\rho(\zeta)$ and $\gamma(\zeta;u)$ are relatively prime for all $u \in [0,1)$ but this assumption is unacceptably strong; for example, if linear interpolation were used then $\gamma(\zeta;u) = (1-u) + u\zeta$, so that this assumption would be equivalent to requiring that ρ have no negative real zeros.

Instead, we make the assumption, which is satisfied by all convergent methods, that

$$\rho(1) = 0. \tag{4.8}$$

Since ρ and σ are relatively prime, it follows from (4.5), (4.6), and (4.8), that $\Phi(\zeta;q;m;u)$ cannot vanish identically.

As in the case of DA_0-stable methods it is often convenient to work in the z-plane. We set

$$\begin{aligned} F(z;q;m;u) &= (1-z)^{m-1}\left(\frac{1-z}{2}\right)^{k+\ell} \Phi\left(\frac{1+z}{1-z};q;m;u\right) \\ &= (1+z)^{m-1}\left(\frac{1+z}{2}\right)^{\ell} r(z) + (q/(m-u))(1-z)^{m-1}s(z)g(z;u) \end{aligned} \tag{4.9}$$

where

$$g(z;u) = \left(\frac{1-z}{2}\right)^{\ell} \gamma\left(\frac{1+z}{1-z};u\right). \tag{4.10}$$

Clearly

$$\Phi(\zeta;q;m;u) = \left(\frac{\zeta+1}{2}\right)^{m-1} (\zeta+1)^{k+\ell} F\left(\frac{\zeta-1}{\zeta+1};q;m;u\right) \tag{4.11}$$

and

$$\gamma(\zeta;u) = (\zeta+1)^{\ell} g\left(\frac{\zeta-1}{\zeta+1};u\right). \tag{4.12}$$

If the method $\{\rho,\sigma,\gamma\}$ is GDA_0-stable then it follows from (4.7) that $\{\rho,\sigma\}$ is DA_0-stable. Applying Theorems 3.1 and 3.3 we obtain

Theorem 4.1

If $\{\rho,\sigma,\gamma\}$ is GDA_0-stable then $\{\rho,\sigma\}$ is D-stable.

If the k-step method $\{\rho,\sigma,\gamma\}$ is GDA_0-stable and of order k then it is implicit.

We conjecture that the analog of Theorem 3.2 is not true for GDA_0-stable methods. Our reason for believing this is that unless further restrictions are imposed it is possible for $\rho(\zeta)$ and $\gamma(\zeta;u)$ to have the common zero

$\zeta = -1$ for certain values of u. For example, if $\rho(\zeta) = \zeta^2-1$ and $\gamma(\zeta;u) = (1-u) + u\zeta$ then ρ and γ have the common zero $\zeta = -1$ when $u = \frac{1}{2}$. To illustrate the difficulty let $\rho(\zeta) = (\zeta^2-1)$, $\sigma(\zeta) = 2\zeta^2$, $\ell = 1$, and $\gamma(\zeta;u) = (\zeta+1)/2$. Then

$$\Phi(\zeta;q;m;u) = \zeta^m(\zeta^2-1) + [q/(m-u)]\,\zeta^2\,.(\zeta + 1),$$

so that $\zeta = -1$ is zero of Φ. But, $\Phi = \zeta(\zeta+1)\bar{\Phi}$ where

$$\bar{\Phi}(\zeta;q;m;u) = \zeta^{\bar{m}}(\zeta-1) + [q/(m-u)]\,\zeta,$$

and $\bar{m} = m-1$. If $m = 1$ then $\bar{\Phi}$ has the zero $\zeta = 1/[1+q/(m-u)] < 1$. If $m > 1$ then

$$\bar{\Phi}(\zeta;q;m;u) = \zeta^{\bar{m}}(\zeta-1) + (\bar{q}/\bar{m})([1-\bar{v}] + \bar{v}\zeta),$$

where $\bar{q} = q(m-1)/(m-u) < q$ and $\bar{v} = 1$ so that, using Lemma 2.2 and Theorem 3.4, we can conclude that all the zeros of $\bar{\Phi}$ lie in D. Since

$$\begin{aligned} F(z;q;m;u) &= (1-z)^{m-1}\left(\frac{1-z}{2}\right)^3 \Phi\left(\frac{1+z}{1-z};q;m;u\right), \\ &= (1+z)(1-z)^{\bar{m}-1}\left(\frac{1-z}{2}\right)^2 \bar{\Phi}\left(\frac{1+z}{1-z};q;m;u\right), \end{aligned}$$

it follows that all the zeros of F lie in H_-. It should be noted that the above example does not prove the conjecture since γ satisfies (4.5) but not (4.4).

Avoiding the difficulty by brute force we are led to

Theorem 4.2

Assume that $\rho(\zeta)$ and $\gamma(\zeta;u)$ do not have the common zero $\zeta = -1$ for $u \in [0,1)$.

Then $\{\rho,\sigma,\gamma\}$ is GDA_0-stable iff all the zeros of $F(z;q;m;u)$ lie in H_- for all $q \in (0,\pi/2)$, $m \in Z_+$, and $u \in [0,1)$.

Proof:

Necessity. The arguments are the same as in Theorem 3.2, equation (1.12) being replaced by equation (4.9).

Sufficiency. Assume that all the zeros of $F(z;q;m;u)$ lie in H_-. Let $\Phi(\zeta_1;q_1;m_1;u_1) = 0$. We must show that $\zeta_1 \in D$.

If $\zeta_1 \neq -1$ we proceed as in Theorem 3.2, using (4.11) instead of (1.14).

If $\zeta_1 = -1$ then it follows as in Theorem 3.2 that

$$a_k + (-1)^{m_1-1} [q_1/(m_1-u_1)]\, 2^{\ell} b_k\, c_k = 0$$

where $c_k = (-2)^{-\ell}\gamma(-1;u)$ is the leading coefficient of $g(z;u_1)$ and where $b_k \neq 0$. Thus if $a_k = (-2)^{-k}\rho(-1) = 0$ then $c_k = (-2)^{-\ell}\gamma(-1;u) = 0$ which is impossible; hence $a_k \neq 0$, $b_k \neq 0$, and $c_k \neq 0$. As in Theorem 3.2 we can conclude that $F(z;q_1;m_1;u_1) \equiv 0$. Consequently, $\Phi(\zeta;q_1;m_1;u_1) \equiv 0$; but this is impossible, so that the case $\zeta_1 = -1$ cannot arise and the proof is complete.

Theorem 4.3

Let the zeros of ρ other than $\zeta = 1$ lie in D and let $\{\rho,\sigma\}$ be convergent. Then $\{\rho,\sigma,\gamma\}$ is GDA_0-stable iff for all $q \in (0,\pi/2)$, $m \in Z_+$, and $u \in [0,1)$ the polynomial $\Phi(\zeta;q;m;u)$ has no zeros on the unit circle.

Proof: Let $m_1 \in Z_+$ and $u_1 \in [0,1)$ be fixed and denote the zeros of $\Phi(\zeta;q;m_1;u_1)$ by $\zeta_1(q), \ldots, \zeta_{m+\ell+k-1}(q)$, with $\zeta_1(0) = 1$ and $|\zeta_j(0)| < 1$ for $j > 1$. Differentiating (4.6) with respect to q and noting (4.5) we find that $\dot{\zeta}_1(0) < 0$. The theorem now follows as a result of continuity of the functions $\zeta_j(q)$.

Using Lemmas 2.3 and 2.4 and Theorem 4.3 we obtain

Theorem 4.4

The fully implicit method

$$\rho(\zeta) = \zeta-1, \quad \sigma(\zeta) = \zeta, \quad \gamma(\zeta;u) = (1-u) + u\zeta$$

and the trapezoidal method

$$\rho(\zeta) = \zeta-1, \quad \sigma(\zeta) = (\zeta+1)/2, \quad \gamma(\zeta;u) = (1-u) + u\zeta$$

are GDA_0-stable.

We conclude by considering the "modified trapezoidal method" namely

$$y^h(t_{n+1}) - y^h(t_n) = h\,\dot{y}^h(t_{n+\frac{1}{2}}),$$

where it is understood that $\dot{y}^h(t_{n+\frac{1}{2}})$ is to be computed by linear interpolation. The modified trapezoidal belongs to the family of modified Adams methods considered by Zverkina [13]. Such modified Adams methods are particularly useful for delay differential equations. Somewhat surprisingly the next theorem shows that the modified trapezoidal method is not as stable as the ordinary trapezoidal method.

Theorem 4.5

The modified trapezoidal method is DA_0-stable but not GDA_0-stable (in a generalized sense.)

Proof: When (1.6) holds the modified trapezoidal method as applied to (1.5) takes the form

$$y^h(t_{n+1}) - y^h(t_n) = h\dot{y}^h(t_{n+\frac{1}{2}-m}),$$

$$= -\mu\, h[y^h(t_{n-m}) + y^h(t_{n-m+1})]/2,$$

which is simply the trapezoidal method. Thus, by Theorem 3.5, the method is DA_0-stable.

Now assume that (4.1) holds with $u = \frac{1}{2}$ so that the modified trapezoidal method as applied to (1.5) takes the form

$$y^h(t_{n+1}) - y^h(t_n) = -\mu\, hy^h(t_{n-m+1}).$$

Hence, with an obvious generalization of (4.6),

$$\Phi(\zeta;q;m;\tfrac{1}{2}) = \zeta^{m-1}(\zeta-1) + q/(m-\tfrac{1}{2}),$$
$$= \zeta^{\bar{m}}(\zeta-1) + q/(\bar{m}+\tfrac{1}{2}),$$

where $\bar{m} = m-1$. Applying Lemma 2.5 it follows that the method is not GDA_0-stable in a generalised sense.

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