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# A NEW CLASS OF HIGHLY-STABLE METHODS:

# $A_0$ - STABLE METHODS

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Abstract

A linear multistep method $(\rho,\sigma)$ is defined to be $A_o$-stable if when it is applied to the equation $\dot{x}(t) = -\lambda x(t)$ the approximate solution $x^h(t_n)$ tends to zero as $t_n \to \infty$ for all values of the stepsize h and all $\lambda \varepsilon (o,\infty)$.

Various properties of $A_o$-stable methods are derived. It is shown that most of the properties of $A(\alpha)$ - stable methods are shared by $A_o$-stable methods. It is proved that there exist $A_o$-stable methods of arbitrarily high order.

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## 1. Introduction

Consider the problem

$$\left.\begin{aligned} \dot{x}(t) &= -\lambda x(t),\ t > o, \\ x(o) &= 1, \end{aligned}\right\} \tag{1.1}$$

where $\lambda$ is a complex constant. Let

$$\rho(\zeta) = \sum_{j=o}^{k} \alpha_j \zeta^j , \qquad \sigma(\zeta) = \sum_{j=o}^{k} \beta_j \zeta^j , \tag{1.2}$$

where it is assumed that $\alpha_k > o$, that $\sigma$ is not identically zero, and that $\rho$ and $\sigma$ have no zeros in common. If the linear multistep method $(\rho,\sigma)$ is applied to (1.1) then

$$\rho(E)x^h(t_n) = -\lambda h\, \sigma(E)\, x^h(t_n), \tag{1.3}$$

where h is the stepsize, $t_n = nh$, $x^h$ is the approximate solution, and E is the translation operator.

Let

$$\left.\begin{aligned} H_+ &= \{z : \mathrm{Re}(z) > o\}, \\ H_- &= \{z : \mathrm{Re}(z) < o\}, \\ D &= \{z : |z| < 1\}. \end{aligned}\right\} \tag{1.4}$$

The closure and boundary of a set S are denoted by $\bar{S}$ and $\partial S$ respectively; in particular, $\partial H_-$ is the imaginary axis.

The solution of (1.1) is $x(t) = \exp(-\lambda t)$ so that $x(t) \to o$ as $t \to \infty$ iff $\lambda \varepsilon H_+$. Most classes of highly-stable methods involve the requirement that $x^h(t_n) \to o$ as $n \to \infty$ for all h and for all $\lambda$ in some subset S of $H_+$ : for A-stable methods $S = H_+$ (Dahlquist [5]); for $A(\alpha)$-stable methods $S =$

$\{\lambda: |\arg(\lambda)| < \alpha, \lambda \neq o\}$ (Widlund [19]). We consider here yet another class of methods the $A_o$-stable methods, in which S is the open positive real axis; that is, the method $(\rho,\sigma)$ is $A_o$ -stable iff all the zeros of the characteristic polynomial

$$\phi(\zeta;q) = \rho(\zeta) + q \; \sigma(\zeta) \tag{1.5}$$

lie in D for all $q \in (o,\infty)$. For references concerning other types of highly-stable methods see Bjurel et al [1], Gear [9], and Lapidus and Seinfeld [14].

As is usual we will often work in the z-plane where

$$z = \frac{\zeta + 1}{\zeta - 1} \quad , \quad \zeta = \frac{z + 1}{z - 1} . \tag{1.6}$$

We recall that the mapping (1.6) maps the disk D in the $\zeta$-plane onto the half plane $H_-$ in the z-plane. We set

$$r(z) = \left[\frac{z - 1}{2}\right]^k \rho\left(\frac{z + 1}{z - 1}\right) = \sum_{j=o}^{k} a_j \, z^j \text{ , say,} \tag{1.7}$$

$$s(z) = \left[\frac{z - 1}{2}\right]^k \sigma\left(\frac{z + 1}{z - 1}\right) = \sum_{j=o}^{k} b_j \, z^j \text{ , say,} \tag{1.8}$$

and

$$f(z;q) = r(z) + q \; s(z). \tag{1.9}$$

We note that

$$\rho(\zeta) = (\zeta-1)^k \; r\left(\frac{\zeta + 1}{\zeta - 1}\right) \quad , \tag{1.10}$$

$$\sigma(\zeta) = (\zeta-1)^k \; s\left(\frac{\zeta + 1}{\zeta - 1}\right) \quad , \tag{1.11}$$

and that

$$r(1) = \alpha_k; \ s(1) = \beta_k; \quad \rho(1) = 2^k a_k; \quad \sigma(1) = 2^k b_k. \qquad (1.12)$$

Neither $r$ nor $s$ is identically zero because, by assumption, neither $\rho$ nor $\sigma$ is identically zero. Furthermore, $r$ and $s$ have no zeros in common. For, suppose that $r(z) = s(z) = o$. Then $z \neq 1$ because $r(1) = \alpha_k \neq 0$. Hence, from (1.7) and (1.8), $\rho(\zeta) = \sigma(\zeta) = 0$ where $\zeta = (z+1)/(z-1)$. But this contradicts the assumption that $\rho$ and $\sigma$ have no zeros in common.

As is well-known (Henrici [12]) the method $(\rho,\sigma)$ is D-stable (stable in the sense of Dahlquist) iff $\rho$ satisfies the root condition, that is, (i) all the zeros of $\rho$ lie in D and (ii) any zeros of $\rho$ on $\partial D$ are simple. The method $(\rho,\sigma)$ is consistent if

$$\rho(1) = 0 \ ; \ \dot{\rho}(1) = \sigma(1), \qquad (1.13)$$

and is of order $p$ if

$$\rho(\zeta) - \sigma(\zeta) \log \zeta \sim - c \quad (\zeta - 1)^{p+1}, \qquad (1.14)$$

as $\zeta \to 1$, where $c$ is a non-zero constant and $p$ is a non-negative integer. To be useful a method must be convergent and hence both consistent and D-stable. However, it turns out that many results can be proved without making the assumption that $(\rho,\sigma)$ is convergent, and thus we do not make this assumption unless explicitly stated.

Recalling that a method is A(o)-stable if it is A($\alpha$)-stable for some $\alpha > o$ (Widlund [19]), we see that the class of $A_o$ - stable methods includes as subclasses the A - stable methods, the A($\alpha$) - stable methods, and the A(o) - stable methods, and it seems likely that every useful class of highly-stable multistep methods will be a subclass of the $A_o$ - stable methods. By studying $A_o$ - stable methods we are therefore able to determine properties which are shared by all highly-stable multistep methods.

## 2. Characterization of $A_o$ -stable methods.

Theorem 2.1

The method $(\rho,\sigma)$ is $A_o$-stable iff all the zeros of $f(z;q)$ lie in $H\_$ for all $q \in (o,\infty)$.

Proof:

Necessity: Assume that $(\rho,\sigma)$ is $A_o$-stable and that $f(z;q_o) = 0$ for some $q_o \in (o,\infty)$. We wish to show that $z \in H\_$.

First assume that $z \neq 1$. Then, from (1.7) and (1.8), we see that $\phi(\zeta;q_o)=0$ for $\zeta=(z+1)/(z-1)$. It follows that $\zeta \in D$ and hence that $z = (\zeta+1)/(\zeta-1) \in H\_$.

Now assume that $z = 1$, so that $r(1) + q_o\ s(1)=0$. Using (1.12) it follows that $\alpha_k + q_o\ \beta_k = 0$. For $q \neq q_o$ let

$$p(\zeta) = \phi(\zeta;q)/(\alpha_k + q\ \beta_k),$$

$$= \sum_{j=o}^{k} (\alpha_j + q\beta_j)\ \zeta^j / (\alpha_k + q\beta_k).$$

Since p has leading coefficient one and since the zeros of p lie in D, the coefficients of p are uniformly bounded for all $q \neq q_o$. It follows that $\alpha_j + q_o\ \beta_j = o$ for all j which contradicts the assumption that $\rho$ and $\sigma$ are relatively prime. Thus $z = 1$ is impossible, and the proof of necessity is complete.

Sufficiency: Assume that all the zeros of $f(z;q)$ lie in $H\_$ for all $q \in (o,\infty)$. Let $\phi(\zeta;q_o)=o$ for some $q_o \in (o,\infty)$. We must show that $\zeta \in D$.

First assume that $\zeta \neq 1$. Then from (1.10) and (1.11) it follows that $f(z;q_o) = 0$ for $z = (\zeta + 1)/(\zeta-1)$. Hence $z \in H\_$ and $\zeta = (z + 1)/(z - 1) \in D$.

Now assume that $\zeta = 1$, so that $\rho(1) + q_o\ \sigma(1) = 0$. Using (1.12) it follows that $a_k + q_o\ b_k = 0$. Now

$$f(z;q) = \sum_{j=o}^{k} (a_j + qb_j) z^j .$$

But $b_k \neq o$, for if $b_k = o$ then $a_k = o$ and $\rho(1) = \sigma(1) = o$ which is impossible because $\rho$ and $\sigma$ have no zeros in common. Since $b_k \neq o$, the leading coefficient of f changes sign when $q = q_o$. But all the zeros of f lie in $H_-$ so that for each q the coefficients of f must have the same sign (Marden (16,p.181]). Hence $a_j + q_o b_j = o$ for all j so that $r = -q_o s$. This implies that $\rho = -q_o \sigma$ which is impossible. Thus the case $\zeta = 1$ cannot occur and the proof of sufficiency is complete.

Theorem 2.2

Let the method $(\rho,\sigma)$ be $A_o$ - stable. Then the zeros of $\rho$ and $\sigma$ lie in $\bar{D}$ and the zeros of r and s lie in $\bar{H}_-$. Furthermore, any zeros of $\rho$ and $\sigma$ on $\partial D$ and any zeros of r and s on $\partial H_-$ are at most double zeros.

Proof: Assume that $\rho(\zeta_1) = o$ where $\zeta_1 \notin \bar{D}$ so that $|\zeta_1| > 1$. Then there exists a circle C with centre $\zeta_1$ such that C does not intersect D and $\rho$ does not vanish on C. Choose $q > o$ so that

$$q \max_{\zeta \in C} |\sigma(\zeta)| < \min_{\zeta \in C} |\rho(\zeta)|.$$

Then, by the theorem of Rouché (Marden [16,p.2]), $\phi(\zeta;q) = \rho(\zeta) + q\sigma(\zeta)$ has at least one zero inside C. But $(\rho,\sigma)$ is $A_o$-stable so that the zeros of $\phi$ lie in D. We have thus arrived at a contradiction and it follows that the zeros of $\rho$ must lie in D.

Now let $\zeta_1 \in \partial D$ be a zero of $\rho$ of multiplicity m. Then $\rho(\zeta) \sim c(\zeta - \zeta_1)^m$ for some $c \neq o$ and all $\zeta$ close to $\zeta_1$. Since $\rho$ and $\sigma$ have no zeros in common, $\sigma(\zeta_1) \neq o$. Hence, there exists a small circle C with center $\zeta_1$ such that $\arg[-\rho(\zeta) / \sigma(\zeta)]$ changes by at least $(m-\frac{1}{2})\pi$ as $\zeta$ traverses the open arc $C_1$

consisting of the points of C lying outside D. Thus, if $m > 3$ there exists $\zeta_2 \varepsilon\ C_1$ with $-\rho(\zeta_2)/\sigma(\zeta_2) = q\ \varepsilon\ (o,\infty)$. Since this contradicts the assumption that the method $(\rho,\sigma)$ is $A_o$ - stable it follows that $m \leq 2$.

The assertions of the theorem have thus been proved as regards $\rho$. The assertions concerning $\sigma$, r, and s can be proved in the same way.

Theorem 2.3

The following statements are equivalent:

(i) The method $(\rho,\sigma)$ is $A_o$-stable.

(ii) For $z\ \varepsilon\ H_+$, $r(z)/s(z)$ is regular and does not take values in $(-\infty,o)$. For $z\ \varepsilon\ \partial H_+$, $r(z)\ \overline{s(z)}$ does not take values in $(-\infty,o)$.

(iii) For $\zeta$ in the complement of $\bar{D}$, $\rho(\zeta)/\sigma(\zeta)$ is regular and does not take values in $(-\infty,o)$. For $\zeta\varepsilon\ \partial D$, $\rho(\zeta)\overline{\sigma(\zeta)}$ does not take values in $(-\infty,o)$.

Proof: Only the equivalence of (i) and (ii) will be proved since the equivalence of (i) and (iii) is proved in the same way.

Assume that (i) holds. From Theorem 2.2 we know that s has no zeros in $H_+$ so that r/s is regular for $z\varepsilon H_+$, and it is clear that r/s does not take values in $(-\infty,o)$. Suppose that $r(z)\ \overline{s(z)} = Q\ \varepsilon\ (-\infty,o)$ for some $z\ \varepsilon\ \partial H_+$; then $s(z) \neq o$ so that $r(z) + q\ s(z) = o$ for $q = -Q/|s(z)|^2\ \varepsilon (o,\infty)$, which is impossible. Hence (ii) holds.

Now assume that (ii) holds. Let $r(z) + q\ s(z) = o$ for $q\ \varepsilon (o,\infty)$. Then $s(z) \neq o$ since if $s(z) = o$ then $r(z) = o$ which is impossible. Thus $r(z)/s(z) = -q\ \ \varepsilon\ (-\infty,o)$ so that from the first part of (ii), $z \notin H_+$. On the other hand, $r(z)\ \overline{s(z)} = -q|s(z)|^2\ \ \varepsilon(-\infty,o)$ so that from the second part of (ii), $z \notin \partial H_+$. Hence $z\ \varepsilon\ H_-$ and (i) holds.

The above theorems are similar to, and were motivated by, previous results of Dahlquist [5], Widlund [19], Norsett [17], Liniger [15], and Cryer [4] for for A-stable, A($\alpha$) - stable, and A(o) -stable methods. (see also Cooke [2]) The present theorems are more general than the previous results since they hold

for $A_o$ -stable

methods and were obtained without the assumption that the method $(\rho,\sigma)$ is convergent. Since the reader may suspect that the class of $A_o$-stable methods coincides with the class of $A(o)$-stable methods, we conclude this section with a counter-example:

<u>Lemma 2.4</u>

The method

$$\rho(\zeta) = \zeta^2 - \zeta, \qquad \sigma(\zeta) = (\zeta + 1)^2/4,$$

is convergent and $A_o$-stable but not $A(o)$ - stable.

<u>Proof</u>: Since $\rho(1) = o$ and $\dot\rho(1) = \sigma(1)$, the method is consistent. Since $\rho(\zeta) = \zeta(\zeta-1)$ the method is also D-stable and hence convergent.

Now

$$f(z;q) = \frac{z+1}{2} + q\frac{z^2}{4} = \frac{1}{4}[qz^2 + 2z + 2],$$

so that the zeros of $f(z;q)$ are equal to

$$z_\pm = \frac{1}{q}\left[-1 \pm \sqrt{1-2q}\right].$$

If $q \in (o,\infty)$ then $z_+$ and $z_-$ have negative real part. Thus, by Theorem 2.1, the method $(\rho,\sigma)$ is $A_o$-stable

Now let $q = 2de^{i\alpha}$ where $d > o$ and $o < |\alpha| < \pi/2$. Then $f(z;q)$ has zeros with negative real part if the same is true of the polynomial $p(w) = w^2 + w + de^{i\alpha}$. To locate the zeros of p we first observe that p has no pure imaginary zeros. The "complex Hurwitz determinants" (Marden [16, p.180]) are given by $\Delta_1 = 1$ and

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -d\sin\alpha \\ 1 & d\cos\alpha & 0 \\ 0 & d\sin\alpha & 1 \end{vmatrix} = d\cos\alpha - d^2\sin^2\alpha .$$

For large d, $\Delta_1 > o$ and $\Delta_2 < o$ so that p has one zero with negative real part and one zero with positive real part. In consequence, the method $(\rho,\sigma)$ is not $A(o)$-stable.

## 3. Further properties of $A_o$ - stable methods

Theorem 3.1

If the method $(\rho,\sigma)$ is $A_o$ - stable then $a_j \geq o$ and $b_j \geq o$ for $o \leq j \leq k$. Furthermore, $\sum_{j=o}^{k} a_j > o$ and $\sum_{j=o}^{k} b_j > o$. Finally, $\beta_k > o$ and $b_k > 0$.

Proof: From Theorem 2.2 we know that the zeros of $r(z)$ lie in $H_-$. Therefore the coefficients of r, namely $a_o,\dots,a_k$, are non-negative (Marden [16, p. 181]). Since r is not identically zero not all the $a_j$ are zero so that $\sum_{j=o}^{k} a_j > o$. The assertions concerning the coefficients $b_j$ are proved in the same way. The final assertion of the theorem follows from the fact that

$$\beta_k = s(1) = \sum_{j=o}^{k} b_j .$$

We now assume that the method $(\rho,\sigma)$ is consistent so that from (1.13), $\rho(1) = a_k = o$ and $\sigma(1) = b_k \neq o$. Rewriting (1.14) in terms of z we find (Widlund [19]) that

$$r(z) - s(z) \log [(z+1)/(z-1)] \sim -c\,(2/z)^{p-k+1}, \tag{3.1}$$

as $z \to \infty$. Hence

$$a_m = 2 \sum_{j \geq o} [b_{m+1+2j} / (1+2j)], \quad k-p \leq m \leq k, \tag{3.2}$$

with the convention that $a_m = o$ if $m < o$ and $b_j = o$ if $j > k$. With the aid of (3.2) we obtain

Theorem 3.2

Let the method $(\rho,\sigma)$ be $A_o$-stable. If $k \geqslant 3$ and $p \geqslant 3$ then: $b_j > o$ for $2 \leqslant j \leqslant k$; $a_j > o$ for $\max(o,k-p) \leqslant j \leqslant k-1$; and $a_j > o$ for $2 \leqslant j \leqslant k-1$.

Proof: Assume that $b_{k-1} = 0$. The zeros of $f(z;q)$ lie in $H_-$ so that, by the Routh-Hurwitz criterion (Gantmacher [7,p.194]), the corresponding Hurwitz determinants $\Delta_k$ must be positive. But, using (3.2), we find that

$$\Delta_2 = \begin{vmatrix} a_{k-1} + qb_{k-1}, & a_k + q\,b_k \\ a_{k-3} + qb_{k-3}, & a_{k-2} + qb_{k-2} \end{vmatrix},$$

$$= \begin{vmatrix} 2b_k & , & q\,b_k \\ 2b_{k-2} + \frac{2}{3}b_k + qb_{k-3} & , & q\,b_{k-2} \end{vmatrix},$$

$$= -q\,b_k \left( \frac{2}{3}b_k + q\,b_{k-3} \right) < 0.$$

We have thus reached a contradiction from which it follows that $b_{k-1} > 0$.

Now the zeros of $\mathcal{S}$ lie in $H_-$. Hence, $\mathcal{S}(z) = b_k z^m \mathcal{S}_1(z)\mathcal{S}_2(z)$ where

$S_1(z)$ contains the imaginary zeros of $S$ and $S_2$ contains the zeros of $S$ with strictly negative real part. It is easily seen that the even coefficients of $S_1$ are non-zero and that all the coefficients of $S_2$ are non-zero. Since $b_k \neq o$ and $b_{k-1} \neq o$ it follows that $S_2$ is of degree at least one, so that all the coefficients of $S_1 S_2$ are non-zero. Furthermore, by Theorem 2.2, $S$ has at most a double zero at $z = o$ so that $m \leqslant 2$. Thus, as asserted, $b_j > o$ for $2 \leqslant j \leqslant k$.

That $a_j > o$ for $\max (o,k-p) \leqslant j \leqslant k-1$ follows immediately from (3.2). Since $a_{k-1} > o$ and $a_{k-2} > o$ it follows, as shown above for the coefficients $b_j$, that $a_j > o$ for $2 \leqslant j \leqslant k-1$.

As an immediate application of Theorem 3.2 we have

Theorem 3.3.

There is only one $A_o$ - stable multistep method of order $p \geqslant k + 1$ namely the trapezoidal method with $p = 2$, $k = 1$, $r(z) = 2b_1$, and $s(z) = b_1 z$.

Proof: Let $p \geqslant k + 1$. From (3.2) with $m = -1$ we see that $b_{2j} = o$ for $j \geqslant o$. Since $b_k \neq o$, it follows that $k$ is odd and that $b_{k-1} = o$. Appealing to Theorem 3.2 we conclude that $k = 1$ and that $b_o = o$. Using (3.2) again the theorem follows.

## 4. High order $A_o$-stable methods

From Theorem 3.1 we know that an $A_o$-stable method must be implicit. The best-known implicit multistep methods are the Adams-Moulton methods and we begin by studying these:

Lemma 4.1

The k-step Adams-Moulton methods of order $k \geq 2$ are not $A_o$-stable.

Proof: For the Adams-Moulton method of order $k \geqslant 2$ we have (Henrici [12,p.194]),

$$\rho(\zeta) = \zeta^k, \quad \sigma(\zeta) = \sum_{m=o}^{k} \gamma_m^* \ \zeta^{k-m} \ (\zeta-1)^m ,$$

where

$$\gamma_m^* = (-1)^m \int_{-1}^{o} \binom{-s}{m} \ ds.$$

In particular , $\gamma_o^* = 1$ and

$\gamma_1^* = -\frac{1}{2}$. Since $(-1)^m \binom{-s}{m} < o$ for $s \varepsilon (-1,0)$ and

and $m \geqslant 1$, it follows that $\gamma_m^* < 0$ for $m \geqslant 1$.

Now

$$s(z) = \sum_{m=o}^{k} b_m z^m = 2^{-k} \sum_{m=o}^{k} \gamma_m^* \ (z+1)^{k-m} \ 2^m.$$

Hence, $b_k = 2^{-k} > 0$ and

$$b_o = 2^{-k} \sum_{m=o}^{k} \gamma_m^* \ 2^m = 2^{-k} [\gamma_o^* + 2\gamma_1^* + \sum_{m=2}^{k} \gamma_m^* \ 2^m ] < 0.$$

Appealing to Theorem 3.1 the lemma follows.

Theorem 4.2

The k-step method of order k corresponding to $s(z) = (z + d)^k$ is $A_o$-stable if $d \geq 2^{k+1}$.

Proof: Let $s(z) = (z+d)^k$ where $d \geq 2^{k+1}$, and let $r(z)$ be determined from (3.2). The zeros of $s(z)$ lie in $H_-$ and the zeros of $f(z;q)$ are continuous functions of $q$ so that, using Theorem 2.1, it suffices to prove that $f(z;q)$ has no imaginary zeros for $q \in (o,\infty)$.

Now $b_j = \binom{k}{j} d^{k-j}$ so that, using (3.2),

$$\begin{aligned} f(dw;q) &= qs(dw) + r(dw), \\ &= qd^k(1+w)^k + \sum_{m=o}^{k-1} a_m d^m w^m, \\ &= qd^k(1+w)^k + 2\sum_{m=o}^{k-1} d^m w^m \sum_{j\geq o} b_{m+1+2j} \cdot /(1+2j), \\ &= qd^k(1+w)^k + 2\sum_{m=o}^{k-1} w^m \sum_{j\geq o} \binom{k}{m+1+2j} d^{k-1-2j} /(1+2j), \\ &= d^{k-1} [T_o(w) + T_1(w)], \end{aligned}$$

where

$$\begin{aligned} T_o(w) &= qd(1+w)^k + 2\sum_{m=o}^{k-1} w^m \binom{k}{m+1}, \\ &= qd(1+w)^k + 2\,[(1+w)^k - 1]/w, \\ &= [(qdw + 2)(1+w)^k - 2\,]/w, \end{aligned}$$

and

$$T_1(w) = 2\sum_{m=o}^{k-1} w^m \sum_{j>o} \binom{k}{m+1+2j} d^{-2j} / (1+2j),$$

with the convention that $\binom{k}{\ell} = 0$ if $\ell > k$. It thus suffices to prove that

$|T_1(iy)| < |T_o(iy)|$ for all $q\epsilon(o,\infty)$ and all $y\ \epsilon[o,\infty)$ .

To estimate $T_o(iy)$ it is necessary to consider three cases:

1. If $y\ \epsilon\ [1,\infty\ )$ then

$$|T_o(i,y)| \geqslant [|qdiy + 2| \,.\, |1 + iy|^k - 2] / y,$$
$$\geqslant 2 \ [(1 + y^2)^{k/2} - 1\ ] / y,$$
$$\geqslant 2\ y^{k-1}.$$

2. If $y\ \epsilon\ [\frac{1}{2}k,\ 1]$ then

$$|T_o(iy)| \geqslant [\ |qdiy + 2|.\ |1 + iy|^k - 2] / y,$$
$$\geqslant 2\ [(1 + y^2)^{\ k/2} - 1]\ /y,$$
$$\geqslant 2\ [(1 + y^2/4)^k - 1\ ]\ /y,$$
$$\geqslant ky/2,$$
$$\geqslant 1/4.$$

3. If $y\ \epsilon\ (o,\ \frac{1}{2}k\ ]$ and $w = iy$ then

$$|T_o(w)| = |(qdw + 2)(1 + w)^k - 2| / y,$$
$$= |(qdw + 2)[1 + kw + \sum_{j=2}^{k} w^j \binom{k}{j}] - 2\ | / y,$$
$$= |(qd + 2k)w + qdkw^2 + (qdw + 2) \sum_{j=2}^{k} w^j \binom{k}{j}\ |\ / y,$$
$$\geqslant (qd + 2k) - qdky - (qdy + 2)y \sum_{j=2}^{k} y^{j-2} \binom{k}{j}\ ,$$
$$\geqslant qd/2 + 2k - (qdy + 2)y\binom{k}{2}(1 + y)^{k-2},$$
$$\geqslant qd/2 + 2k - (qdy + 2)\ y(k^2/2)\ 3,$$
$$\geqslant qd/2 + 2k - 3\ (qd/8 + k/2),$$
$$\geqslant k/2,$$

where we have used the fact that if $y \leqslant 1/2k$ then

$$(1 + y)^{k-2} \leqslant (1 + 1/2k)^{k-2} < (1 + 1/(k-2))^{k-2} < e < 3.$$

In the above argument it was assumed that $k > 2$; if $k = 1$ or $k = 2$ the inequality is trivially true. Also, since $T_o(w)$ is a polynomial in w the above bound also holds for $y = o$.

To estimate $T_1(i,y)$ we observe that

$$\sum_{\ell \geqslant o} \binom{k}{\ell} = 2^k \leqq d/2.$$

Using this fact it follows that for $y \in [o, \infty)$,

$$|T_1(iy)| \leqslant 2\max \, [1, y^{k-1}] \sum_{m=o}^{k-1} \sum_{j>o} \binom{k}{m+1+2j} d^{-2j}/(1 + 2j),$$

$$\leqslant \frac{2}{3} \max \, [1, y^{k-1}] \sum_{m=o}^{k-1} 1/2d,$$

$$\leqslant (k\, 2^{-k-1}/3) \max \, [1, y^{k-1}].$$

$$\leqslant \frac{1}{12} \max \, [1, y^{k-1}].$$

Comparing the bounds for $T_o(iy)$ and $T_1(iy)$ we see that $|T_o(iy)| > |T_1(iy)|$ for all $y \in [o, \infty)$, and the proof of the theorem is complete.

<u>Theorem 4.3</u>

The k-step methop of order k corresponding to $s(z) = (z + d)^k$ is not $A_o$ - stable if $k \geqslant 8$ and $d \leqslant \frac{1}{2} k^{\frac{1}{2}} - 1$

Proof: Let $s(z) = (z + d)^k$ and let $r(z)$ be determined by (3.2). Assume that the method $(r,s)$ is $A_o$ - stable so that all the zeros of $r(z)$ lie in $H_-$. Let

$$u(x) = \sum_{j \geq o} a_{2j} \; x^j, \qquad v(x) = \sum_{j \geq o} a_{2j+1} \; x^j.$$

Then

$$r(iy) = u(-y^2) + iy \, v(-y^2).$$

Since $r(z)$ has real coefficients the zeros of $r(z)$ occur in conjugate pairs except for zeros at the origin. Hence

$$r(z) = z^m h(z^2) \; \Upsilon(z)$$

where $m = 0$ or $1$, the zeros of $h(z^2)$ are imaginary, and the zeros of $\Upsilon(z)$ lie in $H_-$. Let

$$\Upsilon(iy) = \tilde{u}(-y^2) + iy \tilde{v}(-y^2).$$

From the Hermite-Biehler theorem (Obreschkoff [18,p.131) it follows that $\tilde{u}(x)$ and $\tilde{v}(x)$ have real non-positive zeros. If $m = o$ then $u(x) = h(x)\tilde{u}(x)$ and $v(x) = h(x)\tilde{v}(x)$, while if $m = 1$, then $u(x) = xh(x)\tilde{v}(x)$ and $v(x) = h(x)\tilde{u}(x)$. Hence $u(x)$ and $v(x)$ have real non-positive zeros.

If $k$ is even set $\mu = (k-2)/2$, $\nu = 1$ and $w(x) = x^\mu v(1/x)$ while if $k$ is odd set $\mu = (k-1)/2$, $\nu = o$, and $w(x) = x^\mu u(1/x)$. Then

$$w(x) = \sum_{j=o}^{\mu} a_{2j+\nu} \; x^{\mu-j}.$$

From Theorem 3.2 we note that all the coefficients of w are non-zero. The sum of the zeros of w(x) is equal to $-a_{\nu+2}/a_\nu$ while the product of the zeros of w(x) is equal to $\pm a_{2\mu+\nu}/a_\nu$. Since the roots of w(x) are all real and of the same sign we may apply the theorem of the arithmetic and geometric means to obtain (Hardy et al [11,p.52]),

$$a_{2\mu+\nu}/a_\nu \leqslant [a_{\nu+2}/\mu a_\nu]^\mu . \tag{4.1}$$

From (3.2) we see that

$$a_{\nu+2} = \sum_{j\geqslant 0} b_{\nu+3+2j}/(1+2j),$$

$$\leqslant \sum_{j\geqslant 0} 3b_{\nu+3+2j}/(3+2j),$$

$$= 3a_\nu .$$

Since $s(z) = (z+d)^k$, we have that $b_j = \binom{k}{j} d^{k-j}$. Hence, $a_{2\mu+\nu} = a_{k-1} = b_k = 1$ and

$$a_\nu = \sum_{j\geqslant 0} b_{\nu+1+2j}/(1+2j),$$

$$\leqslant \sum_{j\geqslant 0} b_j ,$$

$$= (1+d)^k .$$

Substituting these bounds into (4.1) we find that

$$\begin{aligned}(1+d)^k &\geq a_\nu, \\ &\geq a_{2\mu+\nu}\,[\mu a_\nu/a_{\nu+2}]^\mu, \\ &\geq [\mu/3]^\mu, \\ &\geq [(k-2)/6]^{(k-2)/2}, \end{aligned}$$

Hence, for $k \geq 8$,

$$\begin{aligned} 1+d &\geq [(k-2)/6]^{\frac{1}{2}}\;[6/(k-2)]^{1/k}, \\ &\geq [k/9]^{\frac{1}{2}}\;[1/k]^{1/k}, \\ &\geq [k/9]^{\frac{1}{2}}\,[1/3]^{1/3}, \\ &\geq \tfrac{1}{2}\,k^{\frac{1}{2}}, \end{aligned}$$

where we have used the fact that $(k-2)/6 > k/9$ together with the fact (Hardy [10,p.142]) that $k^{1/k} < 3^{1/3}$. The theorem follows.

We conclude with some remarks:

1.The bounds for d in theorems 4.2 and 4.3 were chosen to simplify the analysis and could obviously be strengthened. The important point is that for each k there exists a d such that the method corresponding to $s(z) = (z+d)^k$ is $A_0$-stable, but that as k increases so must d.

2. The algorithm of Routh (Obreschkoff [18,p.107]) was implemented on the ICL 1906A at the University of Oxford and used to detemine numerically the location of the roots of the function $f(z;q)$ corresponding to $s(z) = (z+d)^k$. If it was found that the roots of f lay in $H_-$ for $q=j/(50-j)$, $j=0,1,2,\ldots,50$, then it was assumed that the method was $A_0$-stable. If $k(d)$ denotes the maximum value of k for which the method corresponding to $s(z) = (z+d)^k$ is

is $A_o$ - stable, then it was found that $k(.5) = 4$, $k(1) = 6$, $k(5) = 15$, $k(10) = 23$, $k(20) = 36$. These results must be treated with caution since only 51 values of q were examined. The computations were performed using double precision arithmetic and were apparently quite stable numerically. To compute the constants k(d) exactly it would, however, be necessary to use a symbolic system such as the SAC-1 system of Collins as was done by Cryer [3,4].

3. The choice $s(z) = (z+d)^k$ in theorems 4.2 and 4.3 simplified the analysis but other choices of s(z) could probably be used provided that the coefficients $b_k, b_{k-1}, \ldots, b_o$ increased so rapidly that $r(z) \sim [s(z) - b_o] / z$.

4. In theorem 4.3 we used the fact that if the zeros of a polynomial lie in $H_-$ then the "even" and "odd" parts of the polynomial have real zeros. This fact can be used to derive many interesting relationships among the coefficients of $A_o$-stable methods. For example, it can easily be shown that $(b_j)^2 > b_{j-2}b_{j+2}$, and it can also be shown that the sequence $b_o, b_2, b_4, \ldots$ is either monotone increasing, or monotone decreasing, or initially monotone increasing and then monotone decreasing.

5. When trying to construct k-step $A_o$-stable methods of order k it is convenient to first choose a polynomial s(z) with $b_j > o$ for $2 \leq j \leq k$ and then use (3.2) to find the corresponding r(z). The corresponding f(z;q) has strictly positive coefficients so that according to the Liénard-Chipart refinement of the Routh-Hurwitz theory (Marden [16,p.181], Gantmacher [7,p.221]), the method (r,s) is $A_o$-stable iff either all the even Hurwitz determinants of f are strictly positive for all q>o or all the odd Hurwitx determinants of f are strictly positive for all q > o. As an example, let

$$s(z) = z^4 + z^3 + z^2$$

so that, from (3.2)

$$r(z)/2 = z^3 + z^2 + 4z/3 + 1/3,$$

and

$$f(z;q) = qz^4 + (q+2)\, z^3 + (q+2)z^2 + 8z/3 + 2/3.$$

In this case it is convenient to examine the odd Hurwitz determinants.

Since $\Delta_1 = q > o$ and

$$\Delta_3 = \begin{vmatrix} q+2 & q & o \\ 8/3 & q+2 & q+2 \\ 0 & 2/3 & 8/3 \end{vmatrix} = [18q^2+8q+72]/9 > 0,$$

the method (r,s) is a fourth order $A_o$-stable method. It should be noted that $b_1=b_o = 0$ which shows that Theorem 3.1 cannot be strengthened to require that $b_1 > 0$.

6. Dahlquist [5] has proved that an A-stable linear multistep method has order at most two. Widlund [19] has given A(α)-stable methods of order three and four. Gear [8] has shown that the "backward difference" or "numerical differentiation" methods

$$\sum_{m=1}^{k} \frac{1}{m} \nabla^m x^h(t_{n+k}) = hf(t_{n+k},\ x^h(t_{n+k}))$$

are stiffly stable, but Cryer [3] has proved that these methods are not D-stable if k > 6. Dill and Gear [6] and Jain and Srivastava [13] have used computers to construct stiffly stable methods of order eight and eleven respectively, but were unable to construct higher-order stiffly stable methods. Even though we have shown here that $A_o$-stable methods of arbitrarily high order exist, we conjecture that A(o)-stable linear multistep methods of high order, of order greater than 20 say, do not exist.

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