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BOUNDS FOR ITERATED NORMAL MATRICES*

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ABSTRACT

Given a normal matrix $\underset{\sim}{A}$, asymptotic bounds are obtained for $||\underset{\sim}{A}^m||_\infty$ in terms of the spectral radius of $\underset{\sim}{A}$, the number of eigenvalues of $\underset{\sim}{A}$ with modulus equal to the spectral radius of $\underset{\sim}{A}$, and the order of $\underset{\sim}{A}$. These results are extended to provide bounds for $||\underset{\sim}{A}^m||_\infty$ for all $m \geq 1$.

I. INTRODUCTION

The spectral radius of any $n \times n$ matrix $\underset{\sim}{A}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be defined by

$$\rho(\underset{\sim}{A}) \equiv \max_{1 \le i \le n} |\lambda_i| ,$$

while the ℓ_∞ norm of $\underset{\sim}{A}$ can be defined by

$$||\underset{\sim}{A}||_\infty \equiv \max_{1 \le i \le n} \sum_{j=1}^{n} |A_{ij}| .$$

In general, $||\underset{\sim}{A}||_\infty$ can be arbitrarily greater than $\rho(\underset{\sim}{A})$ as is shown by the matrix

$$\underset{\sim}{B} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

for which $\rho(\underset{\sim}{B}) = 0$, while $||\underset{\sim}{B}||_\infty = |\alpha|$. Defining ν_m, $m \ge 1$, by

$$(1) \qquad \nu_m = \nu_m(\underset{\sim}{A}) = \frac{||\underset{\sim}{A}^m||_\infty}{[\rho(\underset{\sim}{A})]^m} ,$$

it is known that both $\sup \nu_m$ and $\nu = \nu(\underset{\sim}{A}) = \limsup \nu_m$ are finite. In general, no bounds can be placed on ν without specific knowledge of $\underset{\sim}{A}$.

In contrast, a different situation arises for normal matrices-- Those matrices which commute with their adjoint (i.e. $\underset{\sim}{A}^*\underset{\sim}{A} = \underset{\sim}{A}\underset{\sim}{A}^*$). An important property of a normal matrix is that it is unitarily similar to a diagonal matrix. A real symmetric matrix is orthogonally similar to a real diagonal matrix and is the most familiar example of a normal matrix (see [1], Chapter 9). For normal matrices, we will obtain bounds for ν which depend only on the order of $\underset{\sim}{A}$ and the number of eigenvalues of $\underset{\sim}{A}$ with modulus equal to $\rho(\underset{\sim}{A})$. Bounds are also obtained for $||\underset{\sim}{A}^m||_\infty$, $m \geq 1$, which depend only on $\rho(\underset{\sim}{A})$ and the order of $\underset{\sim}{A}$.

This work derived some of its inspiration from Theorem 3.1, p. 65 of [3], which states:

Let $\underset{\sim}{A}$ be an arbitrary $n \times n$ complex matrix such that $\rho(\underset{\sim}{A}) > 0$. Then,

$$||\underset{\sim}{A}^m||_2 \sim \bar{\nu} \binom{m}{p-1} [\rho(\underset{\sim}{A})]^{m-(p-1)}, \quad m \to \infty,$$

where p is the largest order of all diagonal submatrices $\underset{\sim}{J}_r$ of the Jordon normal form of $\underset{\sim}{A}$ with $\rho(\underset{\sim}{J}_r) = \rho(\underset{\sim}{A})$, and $\bar{\nu}$ is a positive constant.

The ℓ_2 norm of $\underset{\sim}{A}$ is defined by $\|\underset{\sim}{A}\|_2 = [\rho(\underset{\sim}{A}^*\underset{\sim}{A})]^{1/2}$ and $\sim$ is defined by

<u>Definition 1</u>. $h(m) \sim g(m)$, $m \to \infty$ if and only if $h(m)/g(m) \to 1$ as $m \to \infty$.

For convenience, we also define $\underset{\sim}{<}$ by

<u>Definition 2</u>. $h(m) \underset{\sim}{<} g(m)$, $m \to \infty$ if and only if there exists a function $\epsilon(m)$ such that

$$h(m) \leq g(m) + \epsilon(m)$$

and $\epsilon(m) \to 0$ as $m \to \infty$.

The proof of this Theorem depends on the correct observation that, if $\underset{\sim}{J} = \underset{\sim}{S}^{-1}\underset{\sim}{A}\underset{\sim}{S}$ is the Jordan normal form of $\underset{\sim}{A}$, then

$$c_m^{-1} \|\underset{\sim}{A}^m\|_2 = c_m^{-1} \|\underset{\sim}{S}\underset{\sim}{J}^m\underset{\sim}{S}^{-1}\|_2 \sim \|\underset{\sim}{S}\underset{\sim}{M}_m\underset{\sim}{S}^{-1}\|_2;$$

where $c_m = \binom{m}{p-1} [\rho(\underset{\sim}{A})]^{m-(p-1)}$ and $\underset{\sim}{M}_m$ has the number $[\lambda_r/\rho(\underset{\sim}{A})]^m$ in positions corresponding to the upper right-hand corners of those Jordan blocks $\underset{\sim}{J}_r$ of $\underset{\sim}{J}$ with $\rho(\underset{\sim}{J}_r) = |\lambda_r| = \rho(\underset{\sim}{A})$ which are of maximal order p; and zeros everywhere else. Varga argues that $\underset{\sim}{M}_m$ is a constant matrix (independent of m) and puts $\bar{\nu} = \|\underset{\sim}{S}\underset{\sim}{M}\underset{\sim}{S}^{-1}\|_2$. However, this is not always the case. For example, if

$$\underset{\sim}{A} = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix},$$

then it is easily verified that $\underset{\sim}{A}^{2m} = \underset{\sim}{I}$, $\underset{\sim}{A}^{2m+1} = \underset{\sim}{A}$, and $c_m = 1$. Thus $\|\underset{\sim}{S}\underset{\sim}{M}_{2m}\underset{\sim}{S}^{-1}\|_2 = 1$ and $\|\underset{\sim}{S}\underset{\sim}{M}_{2m+1}\underset{\sim}{S}^{-1}\|_2 =$

$$= \|A\|_2 = \sqrt{\rho(A^TA)} = \sqrt{129 + \sqrt{16640}} > 16.$$

This phenomenon depends somewhat on the norm employed. For the ℓ_2 norm, which Varga used, $\bar{\nu}$ still exists for normal matrices, even though $\underset{\sim}{M}_m$ may not be constant. In this case, $c_m = [\rho(\underset{\sim}{A})]^m$ and

$$\begin{aligned} c_m^{-1} \|A^m\|_2 &= c_m^{-1} \|\underset{\sim}{S}\underset{\sim}{J}^m S^*\|_2 \\ &= c_m^{-1} \sqrt{\rho((\underset{\sim}{J}^m)^*(\underset{\sim}{J}^m))} \\ &= c_m^{-1} [\rho(\underset{\sim}{A})]^m \\ &= 1, \end{aligned}$$

implying $\bar{\nu} = 1$. However, for the ℓ_∞ norm, $\bar{\nu}$ may not exist, even for normal matrices. The matrix $\underset{\sim}{A} = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$ is such that $\underset{\sim}{A}^{2m} = 5^m \underset{\sim}{I}$ and $\underset{\sim}{A}^{2m+1} = 5^m \underset{\sim}{A}$. Thus $c_m = (\sqrt{5})^m$ and $c_{2m}^{-1} \|A^{2m}\|_\infty = 1$ while $c_{2m+1}^{-1} \|\underset{\sim}{A}^{2m+1}\|_\infty = 3/\sqrt{5}$.

Varga's Theorem is true, however, if $\sim$ is replaced by $\underset{\sim}{<}$, since

$$c_m^{-1} \|\underset{\sim}{A}^m\|_2 \sim \|\underset{\sim}{S}\, \underset{\sim}{M}_m \underset{\sim}{S}^{-1}\|_2$$
$$\leq \|\underset{\sim}{S}\|_2 \, \|\underset{\sim}{S}^{-1}\|_2 \,.$$

II. MAIN RESULTS

We first treat the case when only one of the eigenvalues of $\underset{\sim}{A}$ has modulus equal to $\rho(\underset{\sim}{A})$.

Theorem 1. Let $\underset{\sim}{A}$ be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n| \geq 0$ (i.e. only one eigenvalue of maximum modulus). Then

$$(2) \quad \|\underset{\sim}{A}^m\|_\infty \sim \nu[\rho(\underset{\sim}{A})]^m, \quad m \to \infty$$

where ν is a positive constant and

$$(3) \quad \nu \leq \frac{\sqrt{n}+1}{2} .$$

Proof. Since $\underset{\sim}{A}$ is a normal matrix, there exists a unitary matrix $\underset{\sim}{S}$ such that (see [1], p. 273)

$$(4) \quad \underset{\sim}{A} = \underset{\sim}{S} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \underset{\sim}{S}^* .$$

Thus

$$(5) \quad \|\underset{\sim}{A}^m\|_\infty = \| \underset{\sim}{S} \begin{pmatrix} 1 & & & 0 \\ & (\frac{\lambda_2}{\lambda_1})^m & & \\ & & \ddots & \\ 0 & & & (\frac{\lambda_n}{\lambda_1})^m \end{pmatrix} \underset{\sim}{S}^* \|_\infty [\rho(\underset{\sim}{A})]^m .$$

Since $|\lambda_1| > |\lambda_j|$, $j = 2, \ldots, n$, we have

$$\|\underset{\sim}{A}^m\|_\infty \sim \| \underset{\sim}{S} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \underset{\sim}{S}^* \|_\infty [\rho(\underset{\sim}{A})]^m, \; m \to \infty .$$

Defining ν by

$$\nu \equiv \left\| \underset{\sim}{S} \begin{pmatrix} 1 & 0 \dots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix} \underset{\sim}{S}^* \right\|_\infty ,$$

we see that

$$\nu = \max_{1 \le i \le n} \sum_{j=1}^{n} |\underset{\sim}{S}_{i1}| \, |\underset{\sim}{S}_{j1}| .$$

Since $\underset{\sim}{S}\,\underset{\sim}{S}^* = \underset{\sim}{S}^*\underset{\sim}{S} = \underset{\sim}{I}$, we have $\sum_{j=1}^{n} |\underset{\sim}{S}_{ji}|^2 = 1$.

Thus we can bound ν by the maximum of the problem

(Pl) $\quad \underset{\alpha_i \in R, i=1,\dots,n}{\text{maximize}} \quad f_1(\alpha_1,\dots,\alpha_n) = \sum_{i=1}^{n} \alpha_1 \alpha_i$;

subject to the constraints $g_1(\alpha_1,\dots,\alpha_n) = \sum_{i=1}^{n} \alpha_i^2 = 1$,

and $\alpha_i \ge 0, \; i=1,\dots,n$.

Now if $(\beta_1,\dots,\beta_n)$ solves (Pl), then we must have $\beta_2 = \beta_3 = \dots = \beta_n$. If not consider $(\hat{\beta}_1,\hat{\beta}_2,\dots,\hat{\beta}_n)$ defined by

$$\hat{\beta}_1 = \beta_1, \; \hat{\beta}_2 = \hat{\beta}_3 = \dots = \hat{\beta}_n = \frac{\sqrt{1-\beta_1}}{\sqrt{n-1}} .$$

$(\hat{\beta}_1,\hat{\beta}_2,\dots,\hat{\beta}_n)$ satisfies the constraints and thus is a feasible point. Using Holder's inequality, we have

$$\sum_{i=2}^{n} \beta_i < \sqrt{n-1} \sqrt{\sum_{i=2}^{n} \beta_i^2}$$

$$= (n-1) \frac{\sqrt{1-\beta_1^2}}{\sqrt{n-1}}$$

$$= \sum_{i=2}^{n} \hat{\beta}_i$$

Thus $(\sum_{i=2}^{n} \beta_i) < (\sum_{i=2}^{n} \hat{\beta}_i)$, and since any solution to (P1) would have $\alpha_1 > 0$, we have $f_1(\beta_1, \ldots, \beta_n) < f(\hat{\beta}_1, \ldots, \hat{\beta}_n)$. Hence the objective function, f_1, is increased, contradicting the assumption that $(\beta_1, \ldots, \beta_n)$ is a solution to (P1).

It follows that the maximum of (P1) will also be the maximum of the problem

(P2) maximize $f_2(x,y) = x^2 + (n-1)xy;$

subject to $g_2(x,y) = x^2 + (n-1)y^2 = 1,$

$x \geq 0,$

and

$y \geq 0.$

Using the change of variables $x = \sin\theta$, $y = \cos\theta/\sqrt{n-1}$, we see that (P2) is solved by maximizing

(7) $$f_2(\sin\theta, \cos\theta/\sqrt{n-1}) = \sin^2\theta + \sqrt{n-1} \sin\theta \cos\theta$$

$$= \frac{1}{2} (1-\cos 2\theta + \sqrt{n-1} \sin 2\theta)$$

over $\theta \in [0,\frac{\pi}{2}]$. For $\theta = 0$ and $\theta = \frac{\pi}{2}$, f_2 is 0 and 1 respectively. It is easily verified that $\frac{d}{d\theta}\, f_2(\sin\theta, \cos\theta/\sqrt{n-1}\,) = 0$ when

$$\text{(8)} \qquad \tan 2\theta = -\sqrt{n-1} \quad .$$

Since we require $\theta \in [0,\frac{\pi}{2}]$, (8) implies $2\theta \in [\frac{\pi}{2},\pi]$ and we obtain $\sin 2\theta = \frac{\sqrt{n-1}}{\sqrt{n}}$ and $\cos 2\theta = -\frac{1}{\sqrt{n}}$. It is easily verified that this yields

$$f_2(\sin\theta,\cos\theta) = \frac{1}{2}(1 - \cos 2\theta + \sqrt{n-1}\,\sin 2\theta)$$

$$= \frac{1+\sqrt{n}}{2} \quad ,$$

and thus maximizes (7), completing the proof of Theorem 1.

Equality can occur in (3). J. H. Halton (private communication) has point out that if

$$\underset{\sim}{S} = \begin{pmatrix} \sin\theta & \cos\theta & 0 & 0 & \cdots & 0 \\ \frac{\cos\theta}{\sqrt{n-1}} & \frac{-\sin\theta}{\sqrt{n-1}} & (n-2)\alpha_1 & 0 & \ddots & \vdots \\ \frac{\cos\theta}{\sqrt{n-1}} & \frac{-\sin\theta}{\sqrt{n-1}} & -\alpha_1 & (n-3)\alpha_2 & \ddots & 0 \\ & & & -\alpha_2 & \ddots & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & & \\ \frac{\cos\theta}{\sqrt{n-1}} & \frac{-\sin\theta}{\sqrt{n-1}} & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-2} \end{pmatrix}$$

where $\alpha_i = \frac{1}{\sqrt{(n-i)(n-i-1)}}$, then for $\theta = \frac{\tan^{-1}(-\sqrt{n-1})}{2}$,

$\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, (see (7), (8))

$$\| S \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} S^T \|_\infty = \frac{\sqrt{n}+1}{2} .$$

Bounds for $\|A^m\|_\infty$ can also be given as a consequence of Theorem 1.

<u>Corollary 1.2</u>. If A is on $n \times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then for $m \geq 1$

$$\|A^m\|_\infty \leq (\frac{\sqrt{n}+1}{2}) \; (\sum_{j=1}^{n} |\lambda_i|^m) .$$

Proof: Since $A = S \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} S^*$ for some unitary matrix S ,

we have

$$\|A^m\|_\infty \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} \sum_{u=1}^{n} |S_{iu} \lambda_u^m \bar{S}_{uj}|$$

$$\leq \max_{1 \leq i \leq n} \sum_{u=1}^{n} |\lambda_u|^m (\sum_{j=1}^{n} |S_{iu}| \, |S_{ju}|).$$

Since $(\sqrt{n}+1)/2$ is the maximum of (P1),

$$\|A^m\|_\infty \leq \frac{\sqrt{n}+1}{2} (\sum_{i=1}^{n} |\lambda_i|^m) .$$

We now consider the general case when there are $k \geq 2$ eigenvalues of $\underset{\sim}{A}$ such that $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k| > |\lambda_{k+1}| \geq \lambda_{k+2}| \geq \cdots \geq |\lambda_n| \geq 0$. As in the proof of Theorem 1, (4) and (5) will still hold for some unitary matrix $\underset{\sim}{S}$. We will not make use of the fact that $\sum_{j=1}^{n} S_{ij}\bar{S}_{\ell j} = 0$ if $\ell \neq i$ and this will lead to grosser bounds than if we did; but we thereby greatly simplify the analysis. Now,

$$\|\underset{\sim}{A}^m\|_\infty \sim \nu_m [\rho(\underset{\sim}{A})]^m$$

where

$$\nu_m = \| \underset{\sim}{S} \begin{pmatrix} 1 & & & & & & \\ & (w_2)^m & & & & 0 & \\ & & \ddots & & & & \\ & & & (w_k)^m & & & \\ & 0 & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \underset{\sim}{S}^* \|_\infty$$

for some $|w_i| = 1$, $i = 2, \ldots, k$. Hence,

$$\|\underset{\sim}{A}^m\|_\infty \underset{\sim}{<} \nu[\rho(\underset{\sim}{A})]^m, \; m \to \infty$$

where $\nu = \limsup \nu_m$.

We can bound each ν_m (and hence ν) by the maximum of the problem

(P3) maximize $\sum_{j=1}^{n} |S_{11}|\ |S_{j1}| + \cdots + \sum_{j=1}^{n} |S_{1k}|\ |S_{jk}|$;

subject to $\sum_{i=1}^{n} |S_{i1}|^2 = 1,$

$$\vdots$$

$$\sum_{i=1}^{n} |S_{ik}|^2 = 1,$$

and $|S_k|^2 + |S_{12}|^2 + \cdots + |S_{1k}|^2 \leq 1$.

Lemma 1. For some constants $b_1, b_2, \ldots, b_k$, (P3) is solved when $|S_{21}| = |S_{31}| = \cdots = |S_{n1}| = b_1, \cdots, |S_{2k}| = |S_{3k}| = \cdots = |S_{nk}| = b_k$.

Proof. Let $|\tilde{S}_{ij}|$, $1 \leq i, j \leq n$, be a solution to (P3). If for some $1 \leq \ell \leq k$ we do not have $|\tilde{S}_{2\ell}| = |\tilde{S}_{3\ell}| = \cdots = |\tilde{S}_{n\ell}|$; then define $|\hat{S}_{ij}|$, $1 \leq i, j \leq n$ by $|\hat{S}_{2\ell}| = \hat{S}_{\ \ell}| = \cdots = |\hat{S}_{n\ell}| = \dfrac{\sqrt{1-|\tilde{S}_{1\ell}|^2}}{\sqrt{n-1}}$,

$|\hat{S}_{1\ell}| = |\tilde{S}_{1\ell}|$, and $|\hat{S}_{ij}| = |\tilde{S}_{ij}|$ for $j \neq \ell$.

Proceeding as in (6), we will have

$$\sum_{i=1}^{n} |\tilde{S}_{i\ell}| < \sum_{i=1}^{n} |\hat{S}_{i\ell}|$$

and so

$$\sum_{i=1}^{n} |\tilde{S}_{1\ell}|\ |\tilde{S}_{i\ell}| \leq \sum_{i=1}^{n} |\hat{S}_{1\ell}|\ |\hat{S}_{i\ell}| \tag{8}$$

with equality in (8) only if $|\hat{S}_{1\ell}| = 0$. Thus we have that either the $\hat{S}_{ij}$ also solve (P3) in the case of equality in (8), or a contradiction to the assumption that the $\tilde{S}_{ij}$ solve (P3). In either case, the lemma is proved.

Lemma 2. At the solution to (P3), all $|S_{ij}|$ such that $0 < |S_{ij}| < \frac{\sqrt{2}}{2}$, $j = 1,\cdots,k$, are equal to $\sqrt{\beta/p}$; where $p \geq 1$ is the number of $|S_{11}|, |S_{12}|,\cdots,|S_{1k}|$ in the interval $(0,\frac{\sqrt{2}}{2})$, and

$$\beta = 1 - \sum_{\ell \in R} |S_{1\ell}|^2$$

where $R = \{\ell : |S_{1\ell}| \geq \sqrt{2}/2\}$.

Proof. Without loss of generality, we assume that any solution to (P3) has $|S_{11}|, |S_{12}|,\cdots,|S_{1p}|$ in the interval $(0,\frac{\sqrt{2}}{2})$, and for $k \geq p \geq 1$. The lemma will be proved by showing that for fixed $S_{1,p+1},\cdots,S_{1n}$, (P3) is maximized when $|S_{11}| = |S_{12}| = \cdots = |S_{1p}| = \sqrt{\frac{\beta}{p}}$.

Holding $S_{1,p+1},\cdots,S_{1n}$ fixed and maximizing over $|S_{ij}|$ for $1 \leq i \leq \quad , 1 \leq i \leq p$ is equivalent, by Lemma 1, to the maximization problem

$$\text{(P4)} \quad \text{maximize} \quad |S_{11}|^2 + \cdots + |S_{1p}|^2 + (n-1)[|S_{11}|b_1 + \cdots + |S_{1p}|b_p];$$

$$\text{subject to} \quad \begin{array}{ccc} |S_{11}|^2 + (n-1)b_1^2 & = & 1, \\ \vdots & \vdots & \vdots \\ |S_{1p}|^2 + (n-1)b_p^2 & = & 1, \end{array}$$

$$\text{and} \quad |S_{11}|^2 + \cdots + |S_{1p}|^2 \leq \beta .$$

Using the substitutions

$$\text{(9)} \qquad \begin{aligned} \sin\theta_1 &= |S_{11}|, & \cos\theta_1 &= \sqrt{n-1}\ b_1, \\ &\vdots & &\vdots \\ \sin\theta_p &= |S_{1p}|, & \cos\theta_p &= \sqrt{n-1}\ b_p, \end{aligned}$$

$\theta_i \in (0,\frac{\pi}{4})$ $i = 1,\cdots,p,$ the problem becomes

$$\text{(P5)} \qquad \underset{\substack{\theta_i \in (0,\frac{\pi}{2}) \\ i=1,\cdots,p}}{\text{maximize}} \quad [\sin^2\theta_1 + \cdots + \sin^2\theta_p] + \frac{\sqrt{N-1}}{2}[\sin 2\theta_1 + \cdots + \sin 2\theta_p];$$

$$\text{subject to} \quad \sin^2\theta_1 + \cdots + \sin^2\theta_p \le \beta .$$

The first part of the objective function is bounded by β . The rest of the proof will consist of maximizing the second part of the objective function; and observing that when the second part is maximized, the first part is equal to β .

Maximization of the second part of the objective function is equivalent to the minimization problem

$$\text{(P6)} \qquad \underset{\substack{\theta_i \in (0,\frac{\pi}{4}) \\ i=1,\cdots,p}}{\text{minimize}} \quad f_6(\theta_1,\cdots,\theta_p) = -\frac{\sqrt{n-1}}{2}[\sin 2\theta_1 + \cdots + \sin 2\theta_p]$$

$$\text{subject to} \quad g_6(\theta_1,\cdots,\theta_p) = \sin^2\theta_1 + \cdots + \sin^2\theta_p - \beta \le 0.$$

The Kuhn-Tucker problem associated with (P6) is (see [2], p. 94) to find a $\bar{u},\bar{\theta}_1,\bar{\theta}_2,\cdots,\bar{\theta}_p$ such that

$$-\sqrt{n-1}\begin{pmatrix}\cos 2\bar{\theta}_1\\ \vdots\\ \cos 2\bar{\theta}_p\end{pmatrix}+\bar{u}\begin{pmatrix}\sin 2\bar{\theta}_1\\ \vdots\\ \sin 2\bar{\theta}_p\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix},$$

$$\sin^2\bar{\theta}_1+\cdots+\sin^2\bar{\theta}_p-\beta\le 0,$$

$$\bar{u}[\sin^2\bar{\theta}_1+\cdots+\sin^2\bar{\theta}_p-\beta]=0,$$

and $\bar{u}\ge 0.$

The first condition implies

$$\bar{u}=\sqrt{n-1}\ \cot 2\bar{\theta}_i\quad i=1,\cdots,p.$$

Thus we must have $\bar{\theta}_i=\cdots=\bar{\theta}_p$. By taking $\sin\bar{\theta}_1=\sin\bar{\theta}_2=\cdots=\sin\bar{\theta}_p=\sqrt{\beta/p}$, the other conditions are satisfied. Now the Hessians of f_6 and g_6 are

$$\nabla^2 f_6(\theta_1,\cdots,\theta_p)=\begin{pmatrix}2\sqrt{n-1}\ \sin 2\theta_1 & & 0\\ & \ddots & \\ 0 & & 2\sqrt{n-1}\ \sin 2\theta_p\end{pmatrix}$$

and

$$\nabla^2 g_6(\theta_1,\cdots,\theta_p)=\begin{pmatrix}2\cos 2\theta_1 & & 0\\ & \ddots & \\ 0 & & 2\cos 2\theta_p\end{pmatrix}.$$

By Theorem 2, p. 89 of [2], f_6 and g_6 are convex on the set $0<\theta_i<\frac{\pi}{4}$, $i=1,\cdots,p$. Thus Theorem 1, p. 94 of [2] implies that the solution to the Kuhn-Tucker problem is also a solution to (P6). Thus $|s_{11}| = |s_{12}| = \cdots = |s_{1p}| = \sqrt{\beta/p}$ at the solution to (P3), proving the lemma.

We are now ready to prove

<u>Theorem 2</u>. Let $\tilde{A}$ be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k| > |\lambda_{k+1}| \geq |\lambda_{k+2}| \geq \cdots \geq |\lambda_n| \geq 0$$

for $k \geq 2$. Then

$$\| \tilde{A}^m \|_\infty \lesssim \nu [\rho(\tilde{A})]^m$$

where

$$\nu \leq 1 + \sqrt{n-1}\sqrt{k-1} \ .$$

<u>Proof</u>. At most two of $|s_{11}|, |s_{12}|, \ldots, |s_{1k}|$ are greater than or equal to $\frac{\sqrt{2}}{2}$. Thus we need show only three special cases--when either zero, one, or two of $|s_{11}|, |s_{12}|, \ldots, |s_{1k}|$ are greater than or equal to $\frac{\sqrt{2}}{2}$.

Case 1. If none of $|s_{11}|, |s_{12}|, \ldots, |s_{1k}|$ are greater than or equal to $\frac{\sqrt{2}}{2}$, then let p be the number of nonzero s_{1i}'s. By Lemma 2, the solution of (P3) has either $|s_{ij}| = \sqrt{1/p}$ or $|s_{1j}| = 0$, $j = 1, \ldots, k$. Thus the maximum is (see (P4), (9))

$$1 + (n-1)\,p\sqrt{\frac{1}{p}}\ \sqrt{\frac{p-1}{p}}\ \sqrt{\frac{1}{n-1}}$$

$$= 1 + \sqrt{n-1}\ \sqrt{p-1}\ .$$

It is obvious from this expression that (P3) is solved when $p = k$. Thus in this case

$$\nu \le 1 + \sqrt{n-1}\ \sqrt{k-1}\ .$$

Case 2. If exactly one of $|S_{11}|, S_{12}|, \ldots, |S_{1k}|$ is greater than or equal to $\sqrt{2}/2$, say $|S_{1q}| = \frac{\sqrt{2+\epsilon}}{2}$ for $0 \le \epsilon \le 2$, then by Lemma 2 for $1 \le p \le k-1$, we have that the maximum is (see (P4), 9)

$$1 + (n-1)\left[\frac{\sqrt{2+\epsilon}}{2}\ \frac{\sqrt{2-\epsilon}}{2}\ \sqrt{\frac{1}{n-1}} + p\sqrt{\frac{2-\epsilon}{4p}}\ \sqrt{\frac{4p-2+\epsilon}{4p}}\ \sqrt{\frac{1}{n-1}}\right]$$

$$= 1 + \sqrt{n-1}\left[\frac{\sqrt{2+\epsilon}\ \sqrt{2-\epsilon}}{4} + \frac{\sqrt{1-\epsilon/2}\ \sqrt{2p-1+\epsilon/2}}{2}\right].$$

Therefore

$$\nu \le 1 + \sqrt{n-1}\left[\frac{\sqrt{2+\epsilon}\ \sqrt{2-\epsilon}}{4} + \frac{\sqrt{1-\epsilon/2}\ \sqrt{2p-1+\epsilon/2}}{2}\right]$$

$$\le 1 + \sqrt{n-1}\left[\frac{1+\sqrt{2p-1}}{2}\right]$$

$$\le 1 + \sqrt{n-1}\left[\frac{1+\sqrt{2k-3}}{2}\right]$$

$$\le 1 + \sqrt{n-1}\ \sqrt{k-1}\ ,$$

and so in this case

$$\nu \le 1 + \sqrt{n-1}\sqrt{k-1}\ .$$

Case 3. If exactly two of $|S_{11}|, |S_{12}|, \ldots, |S_{1k}|$ are greater than or equal to $\sqrt{2}/2$, then each must be equal to $\sqrt{2}/2$ and the maximum is $1 + \sqrt{n-1}$. Thus, since $k \geq 2$,

$$\nu \leq 1 + \sqrt{n-1}\ \sqrt{k-1},$$

and the last case is proved.

Corollary 2.1. If $\underset{\sim}{A}$ is an $n \times n$ normal matrix and $m \geq 1$, then

$$\|\underset{\sim}{A}^m\|_\infty \leq n[\rho(\underset{\sim}{A})]^m.$$

Proof. This follows from the fact that the solution to (P3) in the case $k = n$ bounds the absolute row sums of $\dfrac{1}{[\rho(\underset{\sim}{A})]^m}\ \underset{\sim}{A}^m$.

III. CONCLUDING REMARKS

We have shown that for a normal matrix $\underset{\sim}{A}$,

$$\|\underset{\sim}{A}^m\|_\infty \lesssim \nu[\rho(\underset{\sim}{A})]^m .$$

Bounds have been obtained for ν which depend on the order of $\underset{\sim}{A}$ and on the number of eigenvalues of $\underset{\sim}{A}$ with modulus equal to $\rho(\underset{\sim}{A})$. If there is only one such eigenvalue, the bound can be attained.

In Corollary 1.2, we have a bound for $\|\underset{\sim}{A}^m\|_\infty$ which depends on the eigenvalues of $\underset{\sim}{A}$ and the order of $\underset{\sim}{A}$. In Corollary 2.1, the bound for $\|\underset{\sim}{A}^m\|_\infty$ depends only on $\rho(\underset{\sim}{A})$ and the order of $\underset{\sim}{A}$.

All the bounds have been stated using the ℓ_∞ norm. However, the results hold equally for the ℓ_1 norm since $\|\underset{\sim}{A}\|_1 = \|\underset{\sim}{A}^*\|_\infty$ and $\underset{\sim}{A}^*$ normal implies $\underset{\sim}{A}$ is normal.

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16. Abstracts

Asymptotac bounds are obtained for powers of a normal matrix in terms of the spectral radius, the number of eigenvalues with modulus equal to the spectral radius, and the order of the matrix. These results are extended to provide bounds for all powers if a normal matrix.

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