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UNCONSTRAINED LAGRANGIANS IN NONLINEAR PROGRAMMING

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Abstract

The main purpose of this work is to associate a wide class of Lagrangian functions with a nonconvex, inequality and equality constrained optimization problem in such a way that unconstrained stationary points of each Lagrangian are related to Kuhn-Tucker points or local or global solutions of the optimization problem. As a consequence of this we are able to obtain duality results and two computational algorithms for solving the optimization problem. One algorithm is a Newton algorithm which has a local superlinear or quadratic rate of convergence. The other method is a locally linearly convergent method for finding stationary points of the Lagrangian and is an extension of the method of multipliers of Hestenes and Powell to inequalities.

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1. INTRODUCTION

In 1970 Rockafellar [21] introduced a Lagrangian for inequality constrained convex programming problems for which an unconstrained saddlepoint corresponded to a solution of the convex programming problem. Moreover this Lagrangian was once differentiable everywhere if the objective and constraint functions of the convex programming problem were also differentiable everywhere. In 1971 Arrow, Gould and Howe [1] considered a general class of Lagrangians (including Rockafellar's) for nonconvex programming problems and established local saddlepoint properties for this class of Lagrangians. For their class of Lagrangians however, the saddlepoint was in general nonnegatively constrained just as it is in the classical Kuhn-Tucker [11] Lagrangian for nonlinear programming. The local saddlepoint property was obtained by the presence of a convexifying parameter in their Lagrangian which made the Hessian of the Lagrangian positive definite for large enough, but finite, values of the parameter. This elegant idea of local convexification was first introduced by Arrow and Solow in 1958 [2] in connection with equality constrained problems and was later independently reconsidered in a different algorithmic context by Hestenes [8,9] and Powell [19] in 1969 and by Haarhoff and Buys [7] in 1970. Miele, Moseley and Cragg [14,15] have conducted numerical experiments on these ideas for equality constrained problems. More recently Rockafellar [22] gave an illuminating derivation of his Lagrangian for inequality constrained problems from the Arrow-Solow Lagrangian for equality constrained problems by the use of slack variables.

A primary purpose of this work is to relate Kuhn-Tucker points of non-convex, inequality and equality constrained nonlinear programming problems to unconstrained stationary points of a wide class of Lagrangian functions. Such a relation is important because it can bring to bear all the algorithms and results of nonlinear equations theory [17, 18] on nonlinear programming. As a consequence of this relationship we present in this work local and global duality results (section 3), a new superlinearly or quadratically convergent algorithm (algorithm 4.1 and theorem 4.3), and a linearly convergent extension to inequality constraints and to more general Lagrangians of the method of multipliers (algorithm 4.4 and theorem 4.5).

The difference between our approach and that of Rockafellar [21, 22, 23] is that Rockafellar's results are valid only for convex problems, whereas in our approach convexity plays only a minor role in some of the peripheral results. In [24] Rockafellar extends the results for his specific Lagrangian to nonconvex optimization problems and relates global solutions of the optimization problem to global saddlepoints of his Lagrangian. Our results are principally aimed at relating local stationary points of the two problems and are established for a general class of Lagrangians. Also Rockafellar's Lagrangian is differentiable only once globally, whereas ours are twice differentiable globally. This is an important distinction in the application of Newton type algorithms which require twice differentiability. The difference between our approach and that of Arrow, Gould and Howe [1] is that for their general result the Lagrangian saddlepoint is constrained by nonnegativity constraints whereas the stationary points of our Lagrangians are completely unconstrained. Their Lagrangians are not twice differentiable globally, whereas ours are. Also, the conditions imposed on our Lagrangians are different from their conditions. In addition we give a new general formulation for unconstrained Lagrangians together with new concrete realizations (section 5).

We shall be concerned throughout this paper with the following problem

1.1 $$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad g_i(x) \leqq 0 \qquad i = 1,\dots,m \\ &\qquad\qquad\quad g_i(x) = 0 \qquad i = m+1,\dots,k \end{aligned}$$

where $f:R^n \to R$, $g_i:R^n \to R$, $i=1,\dots,k$, and R^n is the n-dimensional real Euclidean space. We shall associate with this problem a differentiable Lagrangian function $L:R^n \times R^k \to R$ in such a way that Kuhn-Tucker points or local solutions of 1.1 are related to stationary points of L, that is $(\bar{x},\bar{y}) \in R^n \times R^k$ satisfying

1.2 $$\nabla_1 L(\bar{x},\bar{y}) = 0 \text{ and } \nabla_2 L(\bar{x},\bar{y}) = 0$$

where

$$\nabla_1 L(x,y) = \left[\frac{\partial L(x,y)}{\partial x_1},\dots,\frac{\partial L(x,y)}{\partial x_n}\right], \quad \nabla_2 L(x,y) = \left[\frac{\partial L(x,y)}{\partial y_1},\dots,\frac{\partial L(x,y)}{\partial y_k}\right]$$ [1)]

This is done in section 2 of the paper where, in addition, sufficient conditions, different from those of [1], for the $n \times n$ Hessian of L with respect to x to be positive definite are given. This result is important in establishing the local duality results of section 3 and the convergence of the algorithms of section 4. In section 3 we establish duality results between problem 1.1 and an equality constrained dual problem, problem 3.1. We establish a weak duality theorem 3.3 in the presence of convexity, and a duality theorem 3.4 and a converse duality theorem 3.6 in which convexity plays a secondary role. In particular we relate, among other things, points satisfying Kuhn-Tucker conditions and second order optimality conditions without any convexity assumptions. In section 4 we present two computational algorithms for the

[1)] All vectors are either row or column vectors depending on the context. A prime will denote the transpose and will be used only in denoting the tensor product of two vectors or the transpose of a matrix.

solution of 1.1 based upon the solution of 1.2. Algorithm 4.1 is essentially a Newton method applied to 1.2 and for which we establish under suitable conditions a superlinear or quadratic rate of convergence. Algorithm 4.4 is an extension of the method of multipliers [8,9,19,7] to inequalities and for which we give a local linear convergence proof. Finally in section 5 we discuss how to generate a wide class of Lagrangians for problem 1.1 such that unconstrained stationary points of the Lagrangians are related to solutions of 1.1. We give a general formulation for such Lagrangians in theorem 5.3 as well as specific formulations such as 2.10 and 5.21. The appendix contains proofs of all the results of the paper.

2. EQUIVALENCE OF KUHN TUCKER POINTS AND UNCONSTRAINED STATIONARY POINTS

A primary objective of this work is to relate, under the weakest possible conditions, points that satisfy the Kuhn-Tucker conditions for problem 1.1 to stationary points of an appropriately defined Lagrangian $L:R^n \times R^k \to R$. For that purpose we begin by defining such a Lagrangian as follows

2.1 $$L(x,y) = f(x) + \sum_{i=1}^{m} \lambda(g_i(x), y_i) + \sum_{i=m+1}^{k} \varphi(g_i(x), y_i)$$

where $\lambda:R^2 \to R$ and $\varphi:R^2 \to R$. This general type of Lagrangian formulation was studied by Arrow, Gould and Howe [1] from the point of view of obtaining <u>constrained</u> local saddlepoints associated with 1.1.

We shall begin by establishing the following equivalence theorem.

2.2 <u>Equivalence theorem</u> Let f and g_i, $i=1,\ldots,k$, be differentiable at $\bar{x}$.

(a) If $\bar{x}$ is a local or global solution of 1.1 such that a constraint qualification [11,12 pp. 171-173] is satisfied at $\bar{x}$, or if $\bar{x}$ and some $\bar{u} \in R^k$ satisfy the Kuhn-Tucker conditions

2.3 $$\begin{cases} \nabla f(\bar{x}) + \sum_{i=1}^{k} \bar{u}_i \nabla g_i(\bar{x}) = 0 \\ \bar{u}_i g_i(\bar{x}) = 0,\ g_i(\bar{x}) \leqq 0,\ \bar{u}_i \geqq 0,\ i=1,\ldots,m \\ g_i(\bar{x}) = 0,\ i=m+1,\ldots,k \end{cases}$$

where $\nabla f(\bar{x}) = \left[\frac{\partial f(\bar{x})}{\partial x_1}, \ldots, \frac{\partial f(\bar{x})}{\partial x_n}\right]$ and similarly for $\nabla g_i(\bar{x})$, then $\bar{x}$

and some $\bar{y} \in R^k$ form a stationary point of L as defined by 2.1, that is

2.4 $$\nabla_1 L(\bar{x},\bar{y}) = 0 \text{ and } \nabla_2 L(\bar{x},\bar{y}) = 0$$

provided that the functions λ and φ of 2.1 are differentiable and satisfy

(i) $\lambda_1(0,\eta) = \mu$, $\lambda_2(0,\eta) = 0$ have a solution η for each $\mu \geqq 0$

$\lambda_1(\xi,\eta) = 0$, $\lambda_2(\xi,\eta) = 0$ have a solution η for each $\xi < 0$

(i') $\varphi_1(0,\eta) = \mu$, $\varphi_2(0,\eta) = 0$ have a solution η for each μ

where the notation $\lambda_1(\xi,\eta)$ and $\lambda_2(\xi,\eta)$ is defined by

$$\lambda_1(\xi,\eta) = \frac{\partial\lambda(\xi,\eta)}{\partial\xi}, \quad \lambda_2(\xi,\eta) = \frac{\partial\lambda(\xi,\eta)}{\partial\eta}$$

(b) Conversely, if $(\bar{x},\bar{y}) \in R^n \times R^k$ is a stationary point of L , that is $(\bar{x},\bar{y})$ satisfies 2.4, then $\bar{x}$ and $\bar{u} \in R^k$ defined by

$$\bar{u}_i = \lambda_1(g_i(\bar{x}),\bar{y}_i) \quad i = 1,\ldots,m$$
$$= \varphi_1(g_i(\bar{x}),\bar{y}_i) \quad i = m+1,\ldots,k$$

satisfy the Kuhn-Tucker conditions 2.3, provided that the λ and φ functions of 2.1 are differentiable and satisfy

(ii) $\lambda_1(0,\eta) \geqq 0$, $\lambda_1(\xi,0) = 0$ for $\xi < 0$

$\lambda_2(\xi,\eta) = 0$ implies $\xi \leqq 0$ and $\xi\eta = 0$

(ii') $\varphi_2(\xi,\eta) = 0$ implies $\xi = 0$

If in addition f is convex or pseudoconvex at $\bar{x}$, g_i, $i = 1,\ldots,m$, are convex or quasiconvex at $\bar{x}$, and g_i , $i = m+1,\ldots,k$ are affine or simultaneously quasiconvex and quasiconcave at $\bar{x}$, then $\bar{x}$ is also a global solution of 1.1. ▨

All proofs are given in the appendix.

It is convenient to establish now a result due to Arrow-Gould-Howe [1] under conditions different from those imposed by them on the λ and φ functions of 2.1. This result, which is the positive definiteness of the Hassian of L at $(\bar{x},\bar{y})$ with respect to x, will play an important role in deriving the local duality results of section 3 and in the computational algorithms of section 4.

2.5 Theorem (Positive-definiteness of $\nabla_{11}L(\bar{x},\bar{y})$) Let the assumptions of part a of theorem 2.2 hold and let in addition λ and φ be twice differentiable and satisfy

(iii) $\lambda_1(\xi,\eta) = 0$, $\xi < 0$ imply that $\lambda_{11}(\xi,\eta) = 0$

(iv) $\lambda_{11}(0,\eta) = \lambda_{11}(\alpha,0,\eta) \to \infty$ as $\alpha \to \infty$ for each fixed η for which $\lambda_1(0,\eta) > 0$

(iv') $\varphi_{11}(0,\eta) = \varphi_{11}(\alpha,0,\eta) \to \infty$ as $\alpha \to \infty$ for each fixed η, for which $\varphi_1(0,\eta) \neq 0$

Let $f,g_i,i=1,\ldots,k$ be twice differentiable at $\bar{x}$, let strict complementarity hold with respect to inequality and equality constraints that is

$$2.6 \qquad g_i(\bar{x}) = 0,\ i \in \{1,\ldots,k\} \text{ implies } \bar{u}_i \neq 0$$

and let the second order sufficiency conditions [5, p. 30] for an isolated local minimum of problem 1.1 hold at $(\bar{x},\bar{u})$ that is

2.7 $x\nabla_{11}L^0(\bar{x},\bar{u})x > 0$ for each $x \in R^n$, $x \neq 0$, such that $\nabla g_i(\bar{x})x = 0$,

for all $i \in \{i \mid g_i(\bar{x}) = 0, i=1,\ldots,k\}$

where

2.8 $$L^0(x,u) = f(x) + \sum_{i=1}^{k} u_i g_i(x)$$

is the standard Lagrangian and $\nabla_{11}L^0(x,u)$ denotes the $n \times n$ Hessian matrix of L^0 with respect to its first argument x. Then, for sufficiently large α, $\nabla_{11}L(\bar{x},\bar{y})$, where L is defined by 2.1, is positive definite and hence $L(x,\bar{y})$ has an isolated unconstrained local minimum at $\bar{x}$, that is $L(\bar{x},\bar{y}) < L(x,\bar{y})$ for all x is some open neighborhood of $\bar{x}$. ▯

2.9 Remark The strict complementarity conditon 2.6 can be completely eliminated or modified if we make corresponding changes in conditions iv and iv'. In particular theorem 2.5 is still valid if we

(a) Remove condition 2.6 and condition $\lambda_1(0,\eta) > 0$ from iv and remove condition $\varphi_1(0,\eta) \neq 0$ from iv'

(b) Replace 2.6 by

$$g_i(\bar{x}) = 0,\ i = \{1,\ldots,m\} \text{ implies } \bar{u}_i > 0$$

and remove condition $\varphi_1(0,\eta) \neq 0$ from iv'

(c) Replace 2.6 by

$$g_i(\bar{x}) = 0\ \ i = \{m+1,\ldots,k\} \text{ implies } \bar{u}_i \neq 0$$

and remove condition $\lambda_1(0,\eta) > 0$ from iv.

The significance of the results of this section lie in the fact that under suitable conditions the solution of a nonlinear programming problem can be reduced to the solution of a system of nonlinear equations: $\nabla_1 L(x,y) = 0$ and $\nabla_2 L(x,y) = 0$. In section 4 we shall describe two computational algorithms and establish quadratic or superlinear convergence of one algorithm and linear convergence of the other. In section 5 of this paper we shall show how to generate different Lagrangians all of which have all the required properties derived in this paper. Suffice it here to give such a typical Lagrangian for problem 1.1. For $\alpha \in R$, $\alpha > 0$ define

2.10 $$L(x,y) = f(x) + \frac{1}{4\alpha}\sum_{i=1}^{m}((\alpha g_i(x) + y_i)_+^4 - y_i^4) + \frac{1}{2\alpha}\sum_{i=m+1}^{k}((\alpha g_i(x) + y_i)^2 - y_i^2)$$

$$= f(x) + \frac{1}{4\alpha}\sum_{i=1}^{m}((\alpha g_i(x) + y_i)_+^4 - y_i^4) + \sum_{i=m+1}^{k}(\frac{\alpha}{2} g_i(x)^2 + y_i g_i(x))$$

where we have used the notation

$$(z)_+^4 = \begin{cases} z^4 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

which is standard in spline function theory [6, 25]. (This notation besides being extremely helpful in simplifying the expressions for the Lagrangian L above in 2.10 and elsewhere in the paper, reveals the close connection between the λ and φ functions which go into the constitution of L . See for example 2.10, and 5.4 and 5.5 below.)

3. DUALITY

With the primal minimization problem 1.1 we shall associate, through the Lagrangian L(x, y) defined by 2.1, the following dual problem

3.1 $$\underset{x,y}{\text{maximize}}\ L(x,y)$$

$$\text{subject to } \nabla_1 L(x,y) = 0 \qquad \text{(L defined by 2.1)}$$

Under suitable conditions on the λ and φ functions such as conditions ii, ii' and conditions viii, viii' below, the primal problem 1.1 can be rewritten equivalently as

3.2 $$\underset{x,y}{\text{minimize}}\ L(x,y)$$

$$\text{subject to } \nabla_2 L(x,y) = 0 \qquad \text{(L defined by 2.1)}$$

Since we shall assume no convexity quite often in this section, the standard techniques of deriving duality results such as the use of minmax theorems [26,13,10,27] will not apply, nor will the elegant conjugate function theory of Rockafellar [20] apply directly, however see [24].

The results of this section consist of a weak duality theorem 3.3 (for which convexity is needed), a duality theorem 3.4 which relates a solution of 1.1 to a Kuhn-Tucker point of the dual problem 3.1 and to a second order maximum of 3.1 under no convexity assumptions and finally to a global solution of 3.1 under convexity. The converse duality theorem 3.6 similarly relates a local solution of the dual problem 3.1 to a Kuhn-Tucker point of the primal problem 1.1 and to a second order minimum under no convexity assumptions and finally to a global solution of 1.1 under convexity.

Probably the most important features of these duality theorems are the absence of inequality constraints from the dual problem 3.1 and the relations between second order optima of the dual problems obtained in theorems 3.4b and 3.6b without any convexity assumptions at all. Related local duality results for a specific L have also been given by Buys [3].

3.3 Weak duality Let $\hat{x}$ be a feasible point of the primal problem 1.1, that is $g_i(\hat{x}) \leqq 0$, $i=1,\ldots,m$, $g_i(\hat{x}) = 0$, $i=m+1,\ldots,k$ and let (x,y) be feasible point of the dual problem 3.1, that is $\nabla_1 L(x,y) = 0$. Let f, g_i, $i = 1,\ldots,m$, be differentiable and convex on R^n, g_i, $i=m+1,\ldots,k$, be affine functions, and let the λ and φ functions entering the definition 2.1 of L be differentiable functions on R^2 satisfying

(v) $\lambda_1(\xi,\eta) \geqq 0$ and $\lambda_1(\xi,\eta)\xi - \lambda(\xi,\eta) \geqq 0$ for all ξ,

(v') $\varphi_1(\xi,\eta)\xi - \varphi(\xi,\eta) \geqq 0$ for all ξ,

Then

$$f(\hat{x}) \geqq L(x,y). \quad \blacksquare$$

We observe that the second inequality of v and v' are both satisfied if λ and φ are convex in ξ for each fixed η and $\lambda(0,\eta) \leqq 0$ and $\varphi(0,\eta) \leqq 0$ for all η.

3.4 Duality theorem Let f and g_i, $i=1,\ldots,k$, be differentiable at $\bar{x}$.

(a) Let $\bar{x}$ be a local or global solution of the primal problem 1.1 such that a constraint qualification [12, pp. 171-173] is satisfied at $\bar{x}$, or let $\bar{x}$ and some $\bar{u} \in R^k$ satisfy the Kuhn-Tucker conditions

2.3. Let λ and φ be differentiable functions satisfying condition i and i'. Then $\bar{x}$ and $\bar{y} \in R^k$ determined by solving the system A.0 (given in the appendix) satisfy the Kuhn-Tucker conditions of the dual problem 3.1

3.5
$$\begin{cases} \nabla_2 L(\bar{x},\bar{y}) + \bar{v}\nabla_{12} L(\bar{x},\bar{y}) = 0 \\ \nabla_1 L(\bar{x},\bar{y}) + \bar{v}\nabla_{11} L(\bar{x},\bar{y}) = 0 \\ \nabla_1 L(\bar{x},\bar{y}) = 0 \end{cases}$$

with $\bar{v} = 0$.

(b) Let $\bar{x}$ be a local or global solution of the primal problem 1.1 such that a constraint qualification [12, pp. 171-173] is satisfied at $\bar{x}$, or let $\bar{x}$ and some $\bar{u} \in R^k$ satisfy the Kuhn-Tucker conditions 2.3 of problem 1.1. Let the assumptions of theorem 2.5 hold, let $\nabla g_i(\bar{x})$, $i \in I \cup \{m+1,\ldots,k\}$, where $I = \{i/g_i(\bar{x}) = 0,\ i=1,\ldots,m\}$, be linearly independent, and let the λ and φ functions of 2.1 be twice continuously differentiable and satisfy in addition

(vi) $\lambda_2(\xi,\eta) = 0 \implies \lambda_{22}(\xi,\eta) = 0$

(vi') $\varphi_2(\xi,\eta) = 0 \implies \varphi_{22}(\xi,\eta) = 0$

(vii) $\lambda_1(0,\eta) > 0 \implies \lambda_{12}(0,\eta) \not< 0$

(vii') $\varphi_1(0,\eta) \not< 0 \implies \varphi_{12}(0,\eta) \not< 0$

Then, for sufficiently large α (α enters L through the conditions iv and iv') $\bar{x}$ and $\bar{y} \in R^k$ (determined by solving the system A.0 in the appendix) form an isolated local maximum of the dual problem 3.1 subject to the additional constraints that $y_i = 0$, $i \in J = \{i \mid g_i(\bar{x}) < 0, i=1,\ldots,m\}$. That is there exists an open neighborhood of $(\bar{x},\bar{y})$ in $R^n \times R^k$ such that $L(\bar{x},\bar{y}) > L(x,y)$ for all $(x,y) \in R^n \times R^k$ such that $\nabla_1 L(x,y) = 0$ and $y_i = 0$, $i \in J$.

(c) If in addition to the assumptions of part a, $f, g_i, i=1,\ldots,m$ are differentiable and convex on R^n and $g_i, i=m+1,\ldots,k$, are affine and if λ and φ an differentiable on R^2 and satisfy conditions v and v' and

(viii) $\lambda_2(\xi,\eta) = 0$ implies $\lambda(\xi,\eta) = 0$

(viii') $\varphi_2(\xi,\eta) = 0$ implies $\varphi(\xi,\eta) = 0$

then $(\bar{x},\bar{y})$ solves the dual problem 3.1 and the extrema $f(\bar{x})$ and $L(\bar{x},\bar{y})$ are equal. ∎

3.6 Converse duality theorem Let $(\bar{x},\bar{y})$ be a local or global solution of the dual problem 3.1 or let $(\bar{x},\bar{y},\bar{v}_0,\bar{v}) \in R^n \times R^k \times R \times R^n$ satisfy the Fritz John conditions A.14 (given in the appendix) for the dual problem 3.1 and let $f, g_i, i=1,\ldots,k$ be twice continuously differentiable at $\bar{x}$.

a) If the $n \times n$ matrix $\nabla_{11} L(\bar{x},\bar{y})$ is nonsingular then $\bar{x}$ and $\bar{u} \in R^k$ where

3.7
$$\begin{aligned} \bar{u}_i &= \lambda_1(g_i(\bar{x}),\bar{y}_i) \quad i = 1,\ldots,m \\ &= \varphi_1(g_i(\bar{x}),\bar{y}_i) \quad i = m+1,\ldots,k \end{aligned}$$

satisfy the Kuhn-Tucker conditions 2.3 for the primal problem 1.1 provided that λ and φ are differentiable and satisfy conditions ii and ii'.

b) In addition to the assumptions of part a above, let $\nabla_{11}L(\bar{x},\bar{y})$ be positive definite[1], let $\lambda_1(g_i(\bar{x}),\bar{y}_i) > 0$, $i \in I = \{i/g_i(\bar{x}) = 0, i=1,\ldots,m\}$ and let condition iii hold. Then $\bar{x}$ and $\bar{u} \in R^k$ determined from 3.7 satisfy the second order sufficient optimality conditions for the primal problem 1.1.

c) If in addition to the assumptions of part a above, f is convex or pseudoconvex at $\bar{x}$, and $g_i, i=1,\ldots,m$, are convex or quasiconvex at $\bar{x}$, and $g_i, i=m+1,\ldots,k$ are affine or simultaneously quasiconvex and quasiconcave at $\bar{x}$, then $\bar{x}$ is a global solution of the primal problem 3.1. ▢

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[1] This implies that $L(\bar{x},\bar{y})$ is an isolated local <u>maximum</u> of $L(x,y)$ subject to $\nabla_1 L(x,y) = 0$ and $y_i - \bar{y}_i = 0$, $i \in J = \{i \mid g_i(\bar{x}) < 0, i=1,\ldots,m\}$ provided that conditions vii and vii' are also satisfied and $\nabla g_i(\bar{x})$, $i \in I \cup \{m+1,\ldots,k\}$ are linearly independent. (See proof of theorem 3.4b in appendix.)

4. COMPUTATIONAL ALGORITHMS

We shall present in this section two algorithms for the solution of problem 1.1 which are based on reducing the problem 1.1 to that of finding solutions of the n + k nonlinear equations $\nabla_1 L(x,y) = 0$ and $\nabla_2 L(x,y) = 0$. The first algorithm 4.1 is a Newton algorithm for which we establish, under suitable conditions, local superlinear or quadratic convergence rates. The second method is an extension of the method of multipliers investigated by Arrow-Solow [2]. Hestenes [8,9] Powell [19] Haarhoff and Buys [7] and Miele, Moseley and Cragg [14,15] for the case of equality constraints. Our extension is to inequality constraints and to a general Lagrangian. We establish linear convergence for the algorithm and indicate under what sort of conditions we may expect fast or slow convergence of the method. In [3] Buys gives, for a specific Lagrangian, a dual algorithm which is related to our stationary point problem 1.2. One specific implementation of his algorithm, for equality constraints only, turns out to be the method of multipliers [8,9] and for which he establishes local convergence. For inequalities however, a particular case of his algorithm gives a special case of our algorithm 4.4. He does not however establish convergence nor a rate of convergence for that algorithm.

4.1 <u>Newton algorithm for the solution of 1.1</u> Choose $\varepsilon > 0$ and $(x^0, y^0) \in R^n \times R^k$. Determine (x^{j+1}, y^{j+1}) from (x^j, y^j) as follows

a) Define

$$L^j(x,y) = f(x) + \sum_{i \in I(x^j)} \lambda(g_i(x), y_i) + \sum_{i=m+1}^{k} \varphi(g_i(x), y_i)$$

where

$$I(x^j) = \{i \mid g_i(x^j) \geq -\varepsilon, i=1,\ldots,m\}$$

and λ and φ are functions on R^2 satisfying conditions stated in theorem 4.3 below.

b) Set $\hat{y}_i^{j+1} = 0$ for $i \in J(x^j) = \{i \mid g_i(x^j) < -\varepsilon, i=1,\ldots,m\}$

c) Linearize the equations $\frac{\partial L^j(x,y)}{\partial x_i} = 0, i=1,\ldots,n,$ and $\frac{\partial L^j(x,y)}{\partial y_i} = 0,\ i \in I(x^j) \cup \{m+1,\ldots,k\}$ around the point (x^j, y^j) and solve for x^{j+1} and $\hat{y}_i^{j+1}$, $i \in I(x_j) \cup \{m+1,\ldots,k\}$ that is

$$4.2 \quad \begin{bmatrix} \nabla_{11} L^j(x^j,y^j) & \lambda_{12}(g_i(x^j),y_i^j)\nabla g_i(x^j)' & \varphi_{12}(g_i(x^j),y_i^j)\nabla g_i(x^j)' \\ & i \in I(x^j) & i \in \{m+1,\ldots,k\} \\ \lambda_{21}(g_i(x^j),y_i^j)\nabla g_i(x^j) & 0 & 0 \\ i \in I(x^j) & & \\ \varphi_{21}(g_i(x^j),y_i^j)\nabla g_i(x^j) & 0 & 0 \\ i \in \{m+1,\ldots,k\} & & \end{bmatrix} \cdot \begin{bmatrix} x^{j+1} - x^j \\ \hat{y}^{j+1} - y_i^j \\ i \in I(x^j) \\ \hat{y}^{j+1} - y_i^j \\ i \in \{m+1,\ldots,k\} \end{bmatrix} + \begin{bmatrix} \nabla_1 L(x^j,y^j) \\ \frac{\partial L(x^j,y^j)}{\partial y_i} \\ i \in I(x^j) \\ \frac{\partial L(x^j,y^j)}{\partial y_i} \\ i \in \{m+1,\ldots,k\} \end{bmatrix} = 0$$

d) Set

$$y_i^{j+1} = \begin{cases} 0 & \text{if } g_i(x^{j+1}) < -\varepsilon \text{ and } i=1,\dots,m \\ \hat{y}_i^{j+1} & \text{if } g_i(x^{j+1}) \geqq -\varepsilon \text{ and } i=1,\dots,m \\ \hat{y}_i^{j+1} & \text{if } i=m+1,\dots,k \end{cases}$$

The following practical additons to the above algorithm were made in an effort to globalize convergence. The purpose of these additions is to maintain nonsingularity of the matrix of 4.2 and they do not affect the convergence theorem 4.3:

If the cardinality $\bar{k}$ of $I(x^j) \cup \{m+1,\dots,k\}$ exceeds n and $k-m < n$ then replace $I(x^j)$ and $J(x^j)$ in steps a,b and c above by $I'(x^j)$ and $J'(x^j)$ where

$$I'(x^j) = \left\{ i_{k-n+1}, i_{k-n+2}, \dots, i_m \;\middle|\; \begin{array}{l} g_{i_1}(x^j) \leqq g_{i_2}(x^j) \leqq \dots \leqq g_{i_m}(x^j) \\ i_\ell = 1,\dots,m \end{array} \right\}$$

$$J'(x^j) = \left\{ i_1, i_2, \dots, i_{k-n} \;\middle|\; \begin{array}{l} g_{i_1}(x^j) \leqq g_{i_2}(x^j) \leqq \dots \leqq g_{i_m}(x^j) \\ i_\ell = 1,\dots,m \end{array} \right\}$$

If $\lambda_{12}(g_i(x^j), y_i^j) = 0$ for some $i \in I(x^j)$ or if $\varphi_{12}(g_i(x^j), y_i^j) = 0$ for some $i \in \{m+1,\dots,k\}$, then replace these zeros by ones in step c above.

If $\hat{y}_i^{j+1} = 0$ and $g_i(x^{j+1}) \geqq -\varepsilon$ for some $i \in \{1,\dots,m\}$, set $y_i^{j+1} = 1$ in step d above.

4.3 Local convergence and rate of convergence of the Newton algorithm 4.1. Let $\bar{x} \in R^n$ and some $\bar{u} \in R^k$ satisfy the Kuhn-Tucker conditions 2.3 for problem 1.1. Let f and $g_i, i=1,\ldots,k$, be twice differentiable at each point of an open neighborhood of $\bar{x}$ and let the λ and φ functions defining L in 2.1 be twice continuously differentiable on R^2. Let $\nabla^2 f$ and $\nabla^2 g_i, i=1,\ldots,k$, be continuous at $\bar{x}$, let the assumptions of theorem 2.5 above hold and let λ and φ satisfy in addition conditions vi, vi', vii, vii'. Let the vectors $\nabla g_i(\bar{x})$, $i \in I \cup \{m+1,\ldots,k\}$ be linearly independent, where $I = \{i \mid g_i(\bar{x}) = 0, i=1,\ldots,m\}$. Then for large enough but finite α (α entering L through conditions iv and iv') there exists an $\varepsilon > 0$ and an open neighborhood $N(\bar{x},\bar{y})$ of $(\bar{x},\bar{y})$ in $R^n \times R^k$ (where $\bar{y}$ is obtained from $(\bar{x},\bar{u})$ by solving the system A.0) such that for every (x^0,y^0) in $N(\bar{x},\bar{y})$, the Newton algorithms iterates of 4.1 are well defined and converge to $(\bar{x},\bar{y})$ and

$$\lim_{j \to \infty} \frac{\|z^{j+1} - \bar{z}\|}{\|z^j - \bar{z}\|} = 0 \qquad \text{(superlinear convergence)}$$

where $z = (x,y)$. Moreover, if $f, g_i, i=1,\ldots,k$, are three times differentiable on $N(\bar{x},\bar{y})$ and λ and φ are three times differentiable on R^2, then there is a constant c such that

$$\|z^{j+1} - \bar{z}\| \leqq c\|z^j - \bar{z}\|^2 \qquad \text{(quadratic convergence)}$$

for all $j \geqq j_0$ where j_0 depends on z^0. ∎

We remark that any positive number $\varepsilon > 0$ satisfying the condition A.19 of the appendix and starting with any $(x^0,y^0) \in N(\bar{x},\bar{y})$ as defined by A.24 will generate a sequence $\{x^j,y^j\}$ which will converge to $(\bar{x},\bar{y})$. However on small test problems both conditions were violated and convergence was still obtained.

We present now a second method which is an extension to inequality constraints and to more general Lagrangians of the method of multipliers. Originally this method was proposed for equality constraints by Arrow and Solow [2] by using differential equations to determine a small stepsize algorithm. Later and independently of Arrow and Solow and of each other Hestenes [8,9], Powell [19] and Haarhoff and Buys [7] used a similar Lagrangian approach for equality constraints and proposed a large stepsize method. Miele, Moseley and Cragg [14,15] made numerical tests of the algorithm and variants of it. More recently Buys [3] and Wierzbicki [28] considered extensions to inequality constraints. Buys suggested a dual problem approach for a specific Lagrangian function but did not give any convergence rates. Wierzbicki considers another specific but different Lagrangian.

4.4 Method of Lagrange multipliers Choose $\varepsilon > 0$, $\beta > 0$. Start with $(x^0, y^0) \in R^n \times R^k$ such that $y_i^0 = 0$ for $g_i(x^0) < -\varepsilon$, $i \in \{1,\ldots,m\}$. Determine (x^{j+1}, y^{j+1}) from (x^j, y^j) as follows:

(a) $$y_i^{j+1} = y_i^j + \beta \frac{\partial L(x^j, y^j)}{\partial y_i}, \quad i \in I(x^j) \cup \{m+1,\ldots,k\}$$

$$I(x^j) = \{i \mid g_i(x^j) \geqq -\varepsilon, i=1,\ldots,m\}$$

$$y_i^{j+1} = 0 \qquad i \in J(x^j) = \{i \mid g_i(x^j) < -\varepsilon, i=1,\ldots,m\}$$

(b) Determine x^{j+1} such that

$$L(x^{j+1}, y^{j+1}) = \underset{x \in R^n}{\text{minimum}}\ L(x, y^{j+1})$$

or

$$\nabla_1 L(x^{j+1}, y^{j+1}) = 0 . \quad \blacksquare$$

4.5 Local Convergence of the method of Lagrange multipliers

Let $\bar{x} \in R^n$ and some $\bar{u} \in R^k$ satisfy the Kuhn-Tucker conditions 2.3 of problem 1.1. Let $f, g_i, i=1,\ldots,k$ be twice continuously differentiable at each point of an open neighborhood of $\bar{x}$ and let the λ and φ functions defining L in 2.1 be twice continuously differentiable on R^2. Let the assumptions of theorem 2.5 above hold and let λ and φ satisfy the conditions vi, vi', vii and vii'. Let the vectors $\nabla g_i(\bar{x})$, $i \in I \cup \{m+1,\ldots,k\}$ be linearly independent where $I = \{i \mid g_i(\bar{x}) = 0, i=1,\ldots,m\}$. Then for α large enough but finite (α entering L through conditions iv and iv') there exists an $\varepsilon > 0$ and an open neighborhood $\bar{N}(\bar{x},\bar{y})$ of $(\bar{x},\bar{y})$ in $R^n \times R^k$ (where $\bar{y}$ is obtained from $(\bar{x},\bar{u})$ by solving the system A.0) such that for any (x^0, y^0) in $\bar{N}(\bar{x},\bar{y})$ the iterates $\{x^j, y^j\}$ of the method of Lagrange multipliers 4.4 are well defined and converge linearly to $(\bar{x},\bar{y})$ for $\beta \in (0,\bar{\beta})$ for some $\bar{\beta} > 0$. ∎

4.6 Remark about size of $\bar{\beta}$ If we let $\lambda_{\bar{k}} \geq \lambda_1 > 0$ denote respectively the largest and smallest eigenvalues of the $\bar{k} \times \bar{k}$ matrix

4.7 $\nabla_{21} L(\bar{x},\bar{y}_K,\bar{y}_J) \nabla_{11} L(\bar{x},\bar{y})^{-1} \nabla_{12} L(\bar{x},\bar{y}_K,\bar{y}_J)$

where

$\bar{y} = [\bar{y}_{i\in K}, \bar{y}_{i\in J}]$, $K = \{i \mid g_i(\bar{x}), i=1,\ldots,m\} \cup \{m+1,\ldots,k\}$,

$J = \{i \mid g_i(\bar{x}) < 0, i=1,\ldots,m\}$, $\bar{k}$ is the cardinality of K,

$\nabla_{21} L(x, y_K, y_J)$ is the $\bar{k} \times n$ Jacobian with respect to x of the gradient with respect to y_K of $L : \nabla_2 L(x, y_K, y_J)$, and

$\nabla_{12}L(x,y_K,y_J)$ is the $n \times \bar{k}$ Jacobian with respect to y_K of the gradient with respect to x of L: $\nabla_1 L(x,y_K,y_J \cdot)$, then

$$0 < \frac{\delta}{\lambda_1} \leqq \bar{\beta} \leqq \frac{2-\delta}{\lambda_{\bar{k}}}$$

where

$$\delta \in (0, 2/(1+\lambda_{\bar{k}}/\lambda_1))$$

Thus the condition number $\frac{\lambda_{\bar{k}}}{\lambda_1}$ of the matrix 4.7 plays a crucial role in determining the size of $\bar{\beta}$. The larger the condition number of the matrix 4.7 the smaller is the size of $\bar{\beta}$ and the slower is the convergence. We also note from expression A.11 that if $\bar{k} = n$ and $\nabla g_i(\bar{x})$, $i \in K$, are linearly independent, then the condition number of $\nabla_{11}L(\bar{x},\bar{y})$ remains finite as α (in iv and iv') tends to ∞, and hence the condition number $\frac{\lambda_{\bar{k}}}{\lambda_1}$ of the matrix 4.7 remains finite as $\alpha \to \infty$.[1)] However in the more general case of $\bar{k} < n$ and $\nabla g_i(\bar{x})$, $i \in K$, are linearly independent, then again by A.11 the condition number of $\nabla_{11}L(\bar{x},\bar{y})$ and hence $\frac{\lambda_{\bar{k}}}{\lambda_1}$ also approaches ∞ as $\alpha \to \infty$. Therefore for this latter case slow convergence may be encountered as α is increased. In both cases slow convergence

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[1)] For this special case where the solution lies on a "vertex", the intersection of n inequality and equality constraints whose gradients there are linearly independent, making α arbitrarily large should not, by the above analysis, slow convergence. It is assumed throughout remark 4.6 that $\lambda_{11}(0,\bar{y}_i)$, $i \in I$ and $\varphi_{11}(0,\bar{y}_i)$, $i = m+1,\ldots,k$, tend to ∞ at the same rate.

may be encountered with small values of α for which $\nabla_{11}L(\bar{x},\bar{y})$ may be almost singular and hence the condition number of 4.7 may be large. Numerical results of Miele, Moseley and Cragg [14, Table 2, Examples 6.2 and 6.3] where $\alpha = \beta$ (k in their notation), slow convergence occurred for both small and large values of α. Fastest convergence occurred for intermediate values of α. This agrees with our analysis above.

5. GENERATION OF UNCONSTRAINED LAGRANGIANS

The purpose of this section is to give simple sufficient conditions for the λ and φ functions entering into the definition 2.1 of L in order that conditions i to viii for the λ functions and conditions i' to viii' for the φ functions are satisfied. Although the form of Rockafeller's λ function [21,22]

5.1
$$\lambda(\xi,\eta) = \begin{cases} \frac{\alpha}{2}\xi^2 + \xi\eta & \text{if } \alpha\xi + \eta \leqq 0 \\ -\frac{\xi^2}{2\alpha} & \text{if } \alpha\xi + \eta \geqq 0 \end{cases}$$

is not obviously related to the φ function of Arrow-Solow[1)] [2]

5.2
$$\varphi(\xi,\eta) = \frac{\alpha}{2}\xi^2 + \xi\eta$$

we will show how they and other functions can be very simply generated from a single differentiable function $\psi: R \to R$. In particular we give the following sufficient conditions for the satisfaction of all the conditions i to viii and i' to viii'.

5.3 Theorem (Generation of λ and φ functions for the Lagrangian L of 2.1) Let $\psi: R \to R$ be a differentiable function on R. Let $\alpha \in R$ be a parameter and define the functions $\lambda: R^2 \to R$ and $\varphi: R^2 \to R$ as follows

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1) In [22] Rockafellar derives his specific λ function from a φ function by replacing inequalities with equalities containing slack variables. This is not the approach used here.

5.4 $$\lambda(\xi,\eta) = \psi(\alpha\xi + \eta)_+ - \psi(\eta)\ , \qquad (\xi,\eta) \in R \times R$$

5.5 $$\varphi(\xi,\eta) = \psi(\alpha\xi + \eta) - \psi(\eta)\ , \qquad (\xi,\eta) \in R \times R$$

where

$$\psi(\zeta)_+ = \begin{cases} \psi(\zeta) & \text{if } \zeta \geqq 0 \\ \psi(0) & \text{if } \zeta < 0 \end{cases}$$

Consider the following assumptions on $\psi'(\zeta) = \dfrac{d\psi(\zeta)}{d\zeta}$:

a) ψ' is a strictly increasing function on R, such that $\psi'(\zeta)_+$ maps $[0,\infty)$ onto itself and $\psi'(0) = 0$.

b) ψ is convex and nonnegative on R and $\psi(0) = 0$

c) ψ is twice differentiable on R, $\psi''(\zeta) > 0$ for $\zeta > 0$ and $\psi''(0) = 0$

a') ψ' is a strictly increasing function mapping R onto R

b') ψ is convex on R

c') ψ is twice differentiable on R and $\psi''(\zeta) > 0$ for $\zeta \neq 0$.

Then for $\alpha > 0$:

(a) $\Longrightarrow$ (i), (ii) & (viii)

(a) & (b) $\Longrightarrow$ (v)

(a) & (c) $\Longrightarrow$ (iii), (iv), (vi) and (vii)

and for $\alpha \neq 0$:

$$(a') \Longrightarrow (i'), (ii') \ \& \ (viii')$$
$$(a') \ \& \ (b') \Longrightarrow (v')$$
$$(a') \ \& \ (c') \Longrightarrow (iv'), (vi') \text{ and } (vii') \quad \blacksquare$$

Examples of ψ functions that satisfy <u>all</u> the conditions a, b, c, a', b' and c' of theorem 5.3 are the following:

5.6 $$\psi(\zeta) = \frac{1}{\alpha t}\zeta^t \qquad \alpha \in R, \ \alpha > 0, \ t = \text{even integer} \geqq 4$$

5.7 $$\psi(\zeta) = \cosh \zeta - \frac{\zeta^2}{2} - 1$$

5.8 $$\psi(\zeta) = \frac{1}{2}(\cosh \zeta - 1)^2$$

Hence the following λ and φ functions satisfy <u>all</u> the conditions i to viii and i' to viii' of this paper.

5.9 $$\lambda(\xi,\eta) = \frac{1}{\alpha t}((\alpha\xi + \eta)_+^t - \eta^t), \quad \alpha \in R, \ \alpha > 0, \ t = \text{even integer} \geqq 4$$

5.10 $$\lambda(\xi,\eta) = \cosh(\alpha\xi + \eta)_+ - \frac{(\alpha\xi + \eta)_+^2}{2} - \cosh \eta + \frac{\eta^2}{2}$$

5.11 $$\lambda(\xi,\eta) = \frac{1}{2}(\cosh(\alpha\xi + \eta)_+ - 1)^2 - \frac{1}{2}(\cosh \eta - 1)^2$$

5.12 $$\varphi(\xi,\eta) = \frac{1}{\alpha t}((\alpha\xi + \eta)^t - \eta^t), \quad \alpha \in R, \ \alpha > 0, \ t = \text{even integer} \geqq 4$$

5.13 $$\varphi(\xi,\eta) = \cosh(\alpha\xi + \eta) - \frac{(\alpha\xi + \eta)^2}{2} - \cosh \eta + \frac{\eta^2}{2}$$

5.14 $$\varphi(\xi,\eta) = \frac{1}{2}(\cosh(\alpha\xi + \eta) - 1)^2 - \frac{1}{2}(\cosh \eta - 1)^2$$

Examples of ψ functions that satisfy all the conditions a, b, c, a', b' and c' of theorem 5.3 <u>except</u> the condition $\psi''(0) = 0$, are

5.15 $\psi(\zeta) = \frac{1}{2\alpha}\zeta^2$ (set $t = 2$ in 5.6)

5.16 $\psi(\zeta) = \cosh\zeta - 1$

Hence the following λ functions satisfy conditions i, ii, v, viii, but not the remaining conditions iii, iv, vi and vii because $\psi(\zeta)_+$ is not twice differentiable at $\zeta = 0$

5.17 $\lambda(\xi,\eta) = \frac{1}{2\alpha}((\alpha\xi + \eta)_+^2 - \eta^2)$

5.18 $\lambda(\xi,\eta) = \cosh(\alpha\xi + \eta)_+ - \cosh\eta$

We note that if strict complementarity is assumed, then for a neighborhood of a stationary point $(\bar{x},\bar{y})$, $\alpha g_i(x) + y_i \neq 0$ and hence the λ functions of 5.17 and 5.18 can be differentiated twice in that neighborhood and conditions iii, iv, vi and vii hold because $\alpha\xi + \eta \neq 0$.

The following φ functions, derived from 5.5, 5.15 and 5.16 satisfy all the conditions i' to viii' of theorem 5.3

5.19 $\varphi(\xi,\eta) = \frac{1}{2\alpha}((\alpha\xi + \eta)^2 - \eta^2)$

5.20 $\varphi(\xi,\eta) = \cosh(\alpha\xi + \eta) - \cosh\eta$

We observe that the φ function 5.19 is that of Arrow-Solow [2] which was also studied by Arrow, Gould and Howe [1, function M5]. The λ function of 5.17 is that of Rockafellar [21,22] which was also studied by Arrow, Gould and Howe [1, function M4]. Rockafellar's results are for convex problems and those of Arrow, Gould and Howe are for sign restricted Lagrangians, that is $y_i \geqq 0$ for $g_i(x) \leqq 0$. Our principal results make neither of these assumptions. The functions 5.9 to 5.14, 5.18 and 5.20 define new unconstrained Lagrangians for not-necessarily convex problems.

We also note that the λ function of 5.9 is $(t - 1)$ times differentiable everywhere and the λ functions of 5.10 and 5.11 are twice differentiable everywhere. This is in contrast with the penalty Lagrangian of Rockafellar [21, 22] which is based on 5.17 which is only once differentiable everywhere but twice differentiable only in a neighborhood of the solution provided that strict complementarity is assumed. This $(t - 1)$-times differentiability property makes the case $t = 4$ for 5.9 particularly attractive since the quadratic convergence of the Newton method 4.3 requires that the Lagrangian $\dot{L}$ be three times differentiable. In any case, the most attractive unconstrained Lagrangian to work with is that generated from 5.9 with $t = 4$ and from 5.19. In particular we have for our original problem 1.1 the Lagrangian 2.10 which satisfies <u>all</u> the conditions i to viii and i' to viii' of this paper and hence can be used for <u>all</u> the results given. Another such Lagrangian is the following which is based on 5.11 and 5.20:

5.21 $$L(x,y) = f(x) + \frac{1}{2}\sum_{i=1}^{m}\left((\cosh(\alpha g_i(x) + y_i)_+ - 1)^2 - (\cosh y_i - 1)^2\right) + \sum_{i=m+1}^{k}\left(\cosh(\alpha g_i(x) + y_i) - \cosh y_i\right)$$

APPENDIX

Proof of Theorem 2.2

(a) Since a constraint qualification is satisfied at $\bar{x}$ when $\bar{x}$ is a local or global minimum of 1.1, the Kuhn-Tucker conditions 2.3 must be satisfied. For any two λ and φ functions defined on R^2 that satisfy conditions i and i' of the theorem define $\bar{y} \in R^k$ as a solution of

$$\text{A.0} \quad \left\langle \begin{array}{ll} \lambda_1(0,\bar{y}_i) = \bar{u}_i, \ \lambda_2(0,\bar{y}_i) = 0 & \text{for } i \in \{i \mid g_i(\bar{x}) = 0, \ i=1,\ldots,m\} \\ \lambda_1(g_i(\bar{x}),\bar{y}_i) = 0, \ \lambda_2(g_i(\bar{x}),\bar{y}_i) = 0 & \text{for } i \in \{i \mid g_i(\bar{x}) < 0, \ i=1,\ldots,m\} \\ \varphi_1(0,\bar{y}_i) = \bar{u}_i, \ \varphi_2(0,\bar{y}_i) = 0 & \text{for } i \in \{m+1,\ldots,k\} \end{array} \right.$$

Conditions i and i' insure the existence of such $\bar{y}_i$, $i = 1,\ldots,k$. The Kuhn-Tucker conditions 2.3 and the above relations give

$$\text{A.1} \qquad \nabla_1 L(\bar{x},\bar{y}) = \nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_1(g_i(\bar{x}),\bar{y}_i) \nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} \varphi_1(g_i(\bar{x}),\bar{y}_i) \nabla g_i(\bar{x}) = 0$$

$$\text{A.2} \qquad \frac{\partial L}{\partial y_i}(\bar{x},\bar{y}) = \lambda_2(g_i(\bar{x}),\bar{y}_i) = 0, \qquad i = 1,\ldots,m$$

$$\text{A.3} \qquad \frac{\partial L}{\partial y_i}(\bar{x},\bar{y}) = \varphi_2(g_i(\bar{x}),\bar{y}_i) = 0, \qquad i = m+1,\ldots,k$$

(b) Conversely now, let $(\bar{x},\bar{y})$ satisfy conditions A.1 to A.3 above, and let conditions ii and ii' hold. Define

A.4 $$\bar{u}_i = \lambda_1(g_i(\bar{x}), \bar{y}_i) \qquad i = 1,\ldots,m$$
$$= \varphi_1(g_i(\bar{x}), \bar{y}_i) \qquad i = m+1,\ldots,k$$

Condition A.1 becomes then

A.5 $$\nabla f(\bar{x}) + \sum_{i=1}^{k} \bar{u}_i \nabla g_i(\bar{x}) = 0$$

The first condition of ii and A.4 give

A.6 $$\bar{u}_i \geqq 0, \quad i \in \{i \mid g_i(\bar{x}) = 0, \quad i = 1,\ldots,m\}$$

and the third condition of ii and A.2 give

A.7 $$g_i(\bar{x}) \leqq 0 \qquad i = 1,\ldots,m$$

A.8 $$\bar{y}_i \, g_i(\bar{x}) = 0 \qquad i = 1,\ldots,m$$

Condition A.4, condition A.8 and the second condition of ii imply that

A.9 $$\bar{u}_i = \lambda_1(g_i(\bar{x}), \bar{y}_i) = \lambda_1(g_i(\bar{x}), 0) = 0 \quad \text{for} \quad i \in \{i \mid g_i(\bar{x}) < 0, \ i = 1,\ldots,m\}$$

Finally from A.3 and ii' we have that

A.10 $$g_i(\bar{x}) = 0 \qquad i = m+1,\ldots,k$$

Conditions A.5, A.6, A.9, A.7 and A.10 are equivalent to the Kuhn-Tucker conditions 2.3. The last part of the theorem follows from the sufficiency theorem of the Kuhn-Tucker conditions [12, theorem 11.1.2, p. 162]. ∎

Proof of Theorem 2.5

The proof follows closely that of [1] even though our conditions are different from those of [1]. Let $\bar{x}$ be a local or global solution of 1.1 satisfying a constraint qualification or let $(\bar{x},\bar{u})$ be a Kuhn-Tucker point of 1.1. Let $\nabla^2 f$ denote the $n\times n$ Hessian of f with respect to x, let $L^0(x,u)$ denote the standard Lagrangian as defined in 2.8, let $I = \{i \mid g_i(\bar{x}) = 0,\ i = 1,\ldots,m\}$ and let a prime denote the transpose. Then by the proof of theorem 2.2, $\bar{x}$, and some $\bar{y} \in R^k$ determined by solving the system A.0 form a stationary point of L, that is $\nabla_1 L(\bar{x},\bar{y}) = 0$ and $\nabla_2 L(\bar{x},\bar{y}) = 0$, where L is defined by 2.1. We also have that[1]

$$\nabla_{11}L(\bar{x},\bar{y}) = \nabla^2 f(\bar{x}) + \sum_{i=1}^{m} (\lambda_1(g_i(\bar{x}),\bar{y}_i)\nabla^2 g_i(\bar{x}) + \lambda_{11}(g_i(\bar{x}),\bar{y}_i)\nabla g_i(\bar{x})'\nabla g_i(\bar{x}))$$
$$+ \sum_{i=m+1}^{k} (\varphi_1(g_i(\bar{x}),\bar{y}_i)\nabla^2 g_i(\bar{x}) + \varphi_{11}(g_i(\bar{x}),\bar{y}_i)\nabla g_i(\bar{x})'\nabla g_i(\bar{x}))$$
$$= \nabla_{11}L^0(\bar{x},\bar{u}) + \sum_{i\in I} \lambda_{11}(0,\bar{y}_i)\nabla g_i(\bar{x})'\nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} \varphi_{11}(0,\bar{y}_i)\nabla g_i(\bar{x})'\nabla g_i(\bar{x})$$

(by iii)

Hence for any real number γ

A.11
$$\nabla_{11}L(\bar{x},\bar{y}) = [\nabla_{11}L^0(\bar{x},\bar{u}) + \gamma(\sum_{i\in I} \nabla g_i(\bar{x})'\nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} \nabla g_i(\bar{x})'\nabla g_i(\bar{x}))]$$
$$+ [\sum_{i\in I} (\lambda_{11}(0,\bar{y}_i) - \gamma)\nabla g_i(\bar{x})'\nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} (\varphi_{11}(0,\bar{y}_i) - \gamma)\nabla g_i(\bar{x})'\nabla g_i(\bar{x})]$$

By Debreu's theorem [4, theorem 3] which states that

A.12
$$\left\langle x \neq 0,\ Mx = 0 \Longrightarrow xLx > 0 \right\rangle \Longleftrightarrow \left\langle \begin{array}{l} L + \gamma M'M \text{ is positive} \\ \text{definite for } \gamma \text{ sufficiently} \\ \text{large} \end{array}\right.$$

---

[1] The expression $\nabla g_i(\bar{x})'\nabla g_i(\bar{x})$ denotes the $n\times n$ tensor product of the $n\times 1$ vector $\nabla g_i(\bar{x})'$ by the $1\times n$ vector $\nabla g_i(\bar{x})$.

and by 2.7 it follows that the term in the first square bracket in A.11 is positive definite for γ large enough, and since $\nabla g_i(x)'\nabla g_i(x)$ is a positive semidefinite matrix it follows by iv, iv' and 2.6 that the terms in the second square bracket in A.11 are positive semidefinite for α large enough. Hence $\nabla_{11}L(\bar{x},\bar{y})$ is positive definite for α large enough. Hence [5] $L(x,\bar{y})$ has an isolated unconstrained local minimum at $\bar{x}$. ∎

Proof of Remark 2.9

The validity of remark 2.9 follows from the following observations. When 2.6 holds then

$$\bar{u}_i = \lambda_1(0,\bar{y}_i) > 0 \qquad i \in I$$

$$= \varphi_1(0,\bar{y}_i) \neq 0 \qquad i = m+1,\ldots,k$$

then by iv and iv', $\lambda_{11}(0,\bar{y}_i) \to \infty$, $i \in I$, $\varphi_{11}(0,\bar{y}_i) \to \infty$, $i = m+1,\ldots,k$, and hence the expression in the second square bracket of A.11 is positive semidefinite. When 2.6 does not hold, then

$$\bar{u}_i = \lambda_1(0,\bar{y}_i) \geqq 0 \qquad i \in I$$

$$= \varphi_1(0,\bar{y}_i) \qquad i = m+1,\ldots,k$$

but again since both iv and iv' have been strengthened now by deletion of the conditions $\lambda_1(0,\eta) > 0$ and $\varphi_1(0,\eta) \neq 0$ respectively, it again follows that $\lambda_{11}(0,\bar{y}_i) \to \infty$, $i \in I$ and $\varphi_{11}(0,\bar{y}_i) \to \infty$, $i = m+1, \ldots,k$, and hence the expression in the second square bracket of A.11 is positive semidefinite. The case 2.9b corresponds to

$$\bar{u}_i = \lambda_1(0,\bar{y}_i) > 0, \qquad i \in I$$
$$= \varphi_1(0,\bar{y}_i) \quad , \qquad i = m+1,\dots,k$$

and the case 2.9c corresponds to

$$\bar{u}_i = \lambda_1(0,\bar{y}_i) \geqq 0, \qquad i \in I$$
$$= \varphi_1(0,\bar{y}_i) \neq 0 \ , \qquad i = m+1,\dots,k$$

and again because in the first case iv' is strengthened and in the second case iv is strengthened, it again follows that $\lambda_{11}(0,\bar{y}_i) \to \infty$, $i \in I$, and $\varphi_{11}(0,\bar{y}_i) \to \infty$, $i = m+1,\dots,k$, and hence the expression in the second square bracket of A.11 is positive semidefinite. ∎

Proof of Theorem 3.3

$$f(\hat{x}) \geqq f(x) + \nabla f(x)(\hat{x} - x) \qquad \text{(by convexity of } f)$$

$$= f(x) - \sum_{i=1}^{m} \lambda_1(g_i(x),y_i)\nabla g_i(x)(\hat{x}-x) - \sum_{i=m+1}^{k} \varphi_1(g_i(x),y_i)\nabla g_i(x)(\hat{x}-x)$$

(by dual feasibility of (x,y))

$$\geqq f(x) + \sum_{i=1}^{m} \lambda_1(g_i(x),y_i)(g_i(x)-g_i(\hat{x})) + \sum_{i=m+1}^{k} \varphi_1(g_i(x),y_i)(g_i(x)-g_i(\hat{x}))$$

(by convexity of g_i, $i = 1,\dots,m$, first inequality of v and affineness of g_i, $i = m+1,\dots,k$)

$$\geqq f(x) + \sum_{i=1}^{m} \lambda_1(g_i(x),y_i)\, g_i(x) + \sum_{i=m+1}^{k} \varphi_1(g_i(x),y_i)\, g_i(x)$$

(by primal feasibility of $\hat{x}$ and first inequality of v)

$$\geqq f(x) + \sum_{i=1}^{m} \lambda(g_i(x),y_i) + \sum_{i=m+1}^{k} \varphi(g_i(x),y_i)$$

(by second inequality of v and v')

$$= L(x,y)$$ ∎

Proof of Theorem 3.4

(a) By theorem 2.2a $\bar{x}$ and some $\bar{y} \in R^k$ satisfy $\nabla_1 L(\bar{x},\bar{y}) = 0$ and $\nabla_2 L(\bar{x},\bar{y}) = 0$. These are the Kuhn-Tucker conditions 3.5 for the dual problem 3.1 with $\bar{v} = 0$.

(b) By part (a) of this theorem, $(\bar{x},\bar{y})$ satisfies the Kuhn-Tucker conditions 3.5 for the dual problem 3.1 with $\bar{v} = 0$. To show that $(\bar{x},\bar{y})$ is an isolated local maximum of 3.1 with the added constraints $y_i = 0$, $i \in J = \{i \mid g_i(\bar{x}) < 0,\ i = 1,\dots,m\}$ we need to show that the second order sufficiency conditions [5] are satisfied at $(\bar{x},\bar{y})$. That is for $(x,y) \neq 0$, $x \in R^n$, $y \in R^k$, $y_i = 0$, $i \in J$.

A.13 $$\left\langle (\nabla_{11}L(\bar{x},\bar{y}) \quad \nabla_{12}L(\bar{x},\bar{y}))\begin{pmatrix} x \\ y \end{pmatrix} = 0 \right\rangle \Longrightarrow \left\langle (x \quad y)\begin{pmatrix} \nabla_{11}L(\bar{x},\bar{y}) & \nabla_{12}L(\bar{x},\bar{y}) \\ \nabla_{21}L(\bar{x},\bar{y}) & \nabla_{22}L(\bar{x},\bar{y}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} < 0 \right.$$

From the left side above we have that

$$x\nabla_{11}L(\bar{x},\bar{y})x + x\nabla_{12}L(\bar{x},\bar{y})y = 0$$

and hence the right side of the above implication becomes since $\nabla_{12}L(\bar{x},\bar{y}) = \nabla_{21}L(\bar{x},\bar{y})'$, and since by assumptions vi and vi', $\nabla_{22}L(\bar{x},\bar{y}) = 0$:

$$(x \quad y)\begin{pmatrix} \nabla_{11}L(\bar{x},\bar{y}) & \nabla_{12}L(\bar{x},\bar{y}) \\ \nabla_{21}L(\bar{x},\bar{y}) & \nabla_{22}L(\bar{x},\bar{y}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\nabla_{11}L(\bar{x},\bar{y})x + 2y\nabla_{21}L(\bar{x},\bar{y})x$$

$$= -x\nabla_{11}L(\bar{x},\bar{y})x < 0 \qquad \text{for } x \neq 0$$

where the last inequality follows from the positive definiteness of $\nabla_{11}L(\bar{x},\bar{y})$ which is established in Theorem 2.5 for large enough α. We have thus established implication A.13 for the case $x \neq 0$. Suppose now $x = 0$, then the left hand side of implication A.13 gives that

$$0 = \sum_{i \in I} y_i \lambda_{12}(g_i(\bar{x}), \bar{y}_i) \nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} y_i \varphi_{12}(g_i(\bar{x}), \bar{y}_i) \nabla g_i(\bar{x})$$

$$= \sum_{i \in I} y_i \lambda_{12}(0, \bar{y}_i) \nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} y_i \varphi_{12}(0, \bar{y}_i) \nabla g_i(\bar{x})$$

But by assumptions vii and vii', $\lambda_{12}(0, \bar{y}_i) \neq 0$, $i \in I$ and $\varphi_{12}(0, \bar{y}_i) \neq 0$, $i = m+1, \ldots, k$. Hence by the linear independence assumption of $\nabla g_i(\bar{x})$, $i \in I \cup \{m+1, \ldots, k\}$, it follows that $y_i = 0$, $i \in I \cup \{m+1, \ldots, k\}$, and since $y_i = 0$, $i \in J$, the implication A.13 is vacuously satisfied.

(c) By part a of this theorem $\bar{x}$ and $\bar{y} \in R^k$ determined by solving A.0 satisfy 3.5 with $\bar{v} = 0$. Hence $\nabla_1 L(\bar{x}, \bar{y}) = 0$ and $(\bar{x}, \bar{y})$ is a feasible point of the dual problem 3.1. For any dual feasible point (x, y), we have by the weak duality theorem that $f(\bar{x}) \geq L(x, y)$. But by assumptions viii and viii', $L(\bar{x}, \bar{y}) = f(\bar{x})$. Hence $L(\bar{x}, \bar{y}) \geq L(x, y)$. ∎

Proof of Theorem 3.6

(a) Since $(\bar{x}, \bar{y})$ is a solution of the dual problem 3.1, $(\bar{x}, \bar{y})$ and some $(\bar{v}_0, \bar{v}) \in R \times R^n$ satisfy the following Fritz John conditions [12, p. 170]

A.14
$$\begin{cases} \bar{v}_0 \nabla_1 L(\bar{x}, \bar{y}) + \bar{v} \nabla_{11} L(\bar{x}, \bar{y}) = 0 \\ \bar{v}_0 \nabla_2 L(\bar{x}, \bar{y}) + \bar{v} \nabla_{12} L(\bar{x}, \bar{y}) = 0 \\ \nabla_1 L(\bar{x}, \bar{y}) = 0 \\ (\bar{v}_0, \bar{v}) \neq 0 \end{cases}$$

From the first and third equations and the nonsingularity of $\nabla_{11}L(\bar{x},\bar{y})$ it follows that $\bar{v} = 0$ and hence $\bar{v}_0 \neq 0$. So $\nabla_1 L(\bar{x},\bar{y}) = 0$ and $\nabla_2 L(\bar{x},\bar{y}) = 0$. Hence by theorem 2.2b $\bar{x}$ and $\bar{u}$ defined by 3.7 satisfy the Kuhn-Tucker conditions 2.3 for the primal problem 1.1.

(b) By part a above, $\bar{x}$ and $\bar{u}$ determined by 3.7 satisfy the Kuhn-Tucker conditions 2.3 of the primal problem 1.1. Now if we let $L^0(x,u)$ denote the standard Lagrangian defined by 2.8 and $I = \{i \mid g_i(\bar{x}) = 0,\ i = 1,\ldots,m\}$, then by the equation preceding A.11

A.15 $$\nabla_{11}L(\bar{x},\bar{y}) = \nabla_{11}L^0(\bar{x},\bar{u}) + \sum_{i\in I} \lambda_{11}(0,\bar{y}_i)\nabla g_i(\bar{x})'\nabla g_i(\bar{x}) + \sum_{i=m+1}^{k} \varphi_{11}(0,\bar{y}_i)\nabla g_i(\bar{x})'\nabla g_i(\bar{x}) .$$

Since $\nabla_{11}L(\bar{x},\bar{y})$ is positive definite by assumption it follows that

A.16 $$\left\langle x \neq 0,\ \nabla g_i(\bar{x})x = 0,\ i \in I \cup \{m+1,\ldots,k\}\right\rangle \Longrightarrow x\nabla_{11}L^0(\bar{x},\bar{u})x > 0$$

for if not, then for some $\hat{x} \neq 0$, $\nabla g_i(x)\hat{x} = 0$, $i \in I \cup \{m+1,\ldots,k\}$, $\hat{x}\nabla_{11}L^0(\bar{x},\bar{u})\hat{x} \leqq 0$ which by A.15 gives $\hat{x}\nabla_{11}L(\bar{x},\bar{y})\hat{x} \leqq 0$ which contradicts the positive definiteness of $\nabla_{11}L(\bar{x},\bar{y})$. Condition A.16 is the second order sufficient optimality condition for the primal problem 1.1 because we have assumed that for $i \in I$, $\bar{u}_i = \lambda_1(0,\bar{y}_i) > 0$ which implies strict complementarity with respect to the inequality constraints $g_i(x) \leqq 0$, $i = 1,\ldots,m$ [5].

(c) This part follows from the sufficiency theorem of the Kuhn-Tucker conditions [12, theorem 11.1.2, p. 162]. ∎

We shall need the following lemma in the convergence proof of the Newton method 4.1 and the method of multipliers 4.4.

A.17 <u>Lemma</u> Let $I = \{i \mid g_i(\bar{x}) = 0,\ i = 1, \ldots, m\}$, $J = \{i \mid g_i(\bar{x}) < 1, \ldots, m\}$, let $I \cup J \neq \phi$, let $N_0(\bar{x})$ be any open neighborhood of $\bar{x}$ such that g_i, $i = 1, \ldots, m$, are continuous on $N_0(\bar{x})$. Then there exists an open neighborhood $N_1(\bar{x}) \subset N_0(\bar{x})$ defined as follows

A.18 $$N_1(\bar{x}) \subset \{x \mid x \in N_0(\bar{x}),\ \ g_i(x) > -\varepsilon,\ \ i \in I,\ \ g_i(x) < -\varepsilon,\ i \in J\}$$

where

A.19 $$\varepsilon = \begin{cases} -\frac{1}{2} \max\limits_{i \in J} g_i(\bar{x}) & \text{if } J \neq \phi \\ \text{any positive number if } J = \phi \end{cases}$$

such that

$$I(x) = I \ \text{ and } \ J(x) = J \quad \text{for all } \ x \in N_1(\bar{x})$$

where

A.20 $$\begin{cases} I(x) = \{i \mid g_i(x) \geqq -\varepsilon,\ \ i = 1, \ldots, m\} \\ J(x) = \{i \mid g_i(x) < -\varepsilon,\ \ i = 1, \ldots, m\} \end{cases}$$

<u>Proof</u> We observe first that $N_1(\bar{x})$ is an open set in R^n because of the continuity of g_i, $i = 1, \ldots, m$, and is nonempty because $g_i(\bar{x}) = 0 > -\varepsilon$ for $i \in I$, and $g_i(\bar{x}) \leqq \max\limits_{i \in J} g_i(\bar{x}) = -2\varepsilon < -\varepsilon$ for $i \in J$, and hence $\bar{x} \in N_1(\bar{x})$. We have three cases to consider now for $x \in N_1(\bar{x})$:

<u>First case $I \neq \phi$, $J \neq \phi$</u>

$$\langle i \in I \implies g_i(x) > -\varepsilon \implies i \in I(x) \rangle \implies I \subset I(x)$$

$$\langle i \in J \implies g_i(x) < -\varepsilon \implies i \in J(x) \rangle \implies J \subset J(x)$$

$$\left\langle i \in I(x) \implies i \notin J(x) \implies i \notin J \implies i \in I \right\rangle \implies I(x) \subset I$$

$$\left\langle i \in J(x) \implies i \notin I(x) \implies i \notin I \implies i \in J \right\rangle \implies J(x) \subset J$$

Hence $I(x) = I$ and $J(x) = J$ for $x \in N_1(\bar{x})$.

<u>Second case $I \neq \phi$, $J = \phi$</u>

$$\left\langle i \in I \implies g_i(x) > -\varepsilon \implies i \in I(x) \right\rangle \implies I \subset I(x)$$

$$J = \varphi \implies g_i(x) > -\varepsilon,\ i = 1,\dots,m \implies J(x) = \phi = J$$

$$\left\langle i \in I(x) \implies i \notin J \implies i \in I \right\rangle \implies I(x) \subset I$$

Hence $I(x) = I$ and $J(x) = J$ for $x \in N_1(\bar{x})$.

<u>Third case $I = \phi$, $J \neq \phi$</u>

$$I = \phi \implies g_i(x) < -\varepsilon,\ i = 1,\dots,m \implies I(x) = \phi = I$$

$$\left\langle i \in J \implies g_i(x) < -\varepsilon \implies i \in J(x) \right\rangle \implies J \subset J(x)$$

$$\left\langle i \in J(x) \implies i \notin I \implies i \in J \right\rangle \implies J(x) \subset J$$

Hence $I(x) = I$ and $J(x) = J$ for $x \in N_1(\bar{x})$. ▪

<u>Proof of Theorem 4.3</u>

Since $(\bar{x},\bar{y}) \in R^n \times R^k$ is obtained from $(\bar{x},\bar{u})$ by solving the system A.0 it follows from A.8 that $\bar{y}_J = [\bar{y}_{i\in J}] = 0$ where $J = \{i \mid g_i(\bar{x}) < 0,\ i = 1,\dots,m\}$. Hence we have that

A.21 $$\nabla_1 L(\bar{x},\bar{y}_K,0) = 0, \quad \nabla_2 L(\bar{x},\bar{y}_K,0) = 0$$

where $\nabla_1 L(x, y_K, y_J)$ denotes the gradient with respect to the first argument x, $\nabla_2 L(x, y_K, y_J)$ denotes the gradient with respect to the second argument y_K and $K = \{i | g_i(\bar{x}) = 0,\ i = 1, \ldots, k\}$. Because of conditions vi and vi', the Hessian of $L(x, y_K, y_J)$ with respect to its first two arguments x and y_K evaluated at $(\bar{x}, \bar{y})$ is given by

$$\text{A.22} \qquad \begin{bmatrix} \nabla_{11} L(\bar{x}, \bar{y}) & \underset{i \in I}{\lambda_{12}(0, \bar{y}_i) \nabla g_i(\bar{x})'} & \underset{i \in \{m+1, \ldots, k\}}{\varphi_{12}(0, \bar{y}_i) \nabla g_i(\bar{x})'} \\ \underset{i \in I}{\lambda_{21}(0, \bar{y}_i) \nabla g_i(\bar{x})} & 0 & 0 \\ \underset{i \in \{m+1, \ldots, k\}}{\varphi_{21}(0, \bar{y}_i) \nabla g_i(\bar{x})} & 0 & 0 \end{bmatrix}$$

We will now show that the matrix A.22 is nonsingular for large enough α. By Theorem 2.5 $\nabla_{11} L(\bar{x}, \bar{y})$ is nonsingular for α large enough. Hence the matrix A.22 is nonsingular if the following product of matrices is also nonsingular

$$\begin{bmatrix} \underset{i \in I}{\lambda_{21}(0, \bar{y}_i) \nabla g_i(\bar{x})} \\ \underset{i \in \{m+1, \ldots, k\}}{\varphi_{21}(0, \bar{y}_i) \nabla g_i(\bar{x})} \end{bmatrix} \nabla_{11} L(\bar{x}, \bar{y})^{-1} \begin{bmatrix} \underset{i \in I}{\lambda_{12}(0, \bar{y}_i) \nabla g_i(\bar{x})'} & \underset{i \in \{m+1, \ldots, k\}}{\varphi_{12}(0, \bar{y}_i) \nabla g_i(\bar{x})'} \end{bmatrix}$$

Since by assumption, $\nabla g_i(\bar{x})$, $i \in K = I \cup \{m+1, \ldots, k\}$ are linearly independent, and since by 2.6, A.0, vii and vii', $\lambda_{21}(0, \bar{y}_i)$, $i \in I$, and

$\varphi_{21}(0,\bar{y}_i)$, $i \in \{m+1,\ldots,k\}$ are nonzero, the rank of the matrices appearing in square brackets in the above product is $\bar{k} \leq n$ where $\bar{k}$ is the cardinality of K. Hence the rank of the product is also $\bar{k}$ and thus is nonsingular and so is the matrix A.22.

It follows by the local convergence of Newton's method [16, p. 148] that there exists an open neighborhood $N_2(\bar{x},\bar{y}_K)$ of $(\bar{x},\bar{y}_K)$ in $R^n \times R^{\bar{k}}$ such that a Newton method applied to

A.23 $$\nabla_1 L(x,y_K,0) = 0, \quad \nabla_2 L(x,y_K,0) = 0$$

will have all the convergence properties stated in theorem 4.3 if we set $z = (x,y_K)$.

It only remains to show that for some $\varepsilon > 0$ algorithm 4.1 is indeed equivalent to Newton's method applied to A.23 above. Choose $\varepsilon > 0$ as defined by A.19 in Lemma A.17 above and a neighborhood $N_1(\bar{x})$ of $\bar{x}$ as defined by A.18. Define now $N(\bar{x},\bar{y})$ such that

A.24 $$N(\bar{x},\bar{y}) \subset \{(x,y) \mid (x,y) \in R^n \times R^k,\ x \in N_1(\bar{x}),\ (x,y_K) \in N_2(\bar{x},\bar{y}_K)\}$$

For any $(x^0,y^0) \in N(\bar{x},\bar{y})$, we have by lemma A.17 that $I(x^0) = I$ and $J(x^0) = J$. Hence starting with $(x^0,y^0) \in N(\bar{x},\bar{y})$ algorithm 4.1 is equivalent to setting $y_J^j = 0$ and determining (x^j,y_K^j) from a Newton method applied to A.23 starting with (x^0,y_K^0). Hence we have all the convergence properties stated in Theorem 4.3 for $z = (x,y_K,y_J) = (x,y_K,0)$. ∎

Proof of Theorem 4.5

As in the proof of Theorem 4.3 we have that equations A.21 are satisfied at the solution $(\bar{x},\bar{y})$. Since for sufficiently large α, $\nabla_{11} L(\bar{x},\bar{y}_K,0)$ is nonsingular, it follows from the implicit function theorem that for an open neighborhood $N_3(\bar{y}_K)$ in $R^{\bar{k}}$ there exists a function $e: R^{\bar{k}} \to R^n$ which is continuously differentiable on $N_3(\bar{y}_K)$ and such that

A.25 $$\bar{x} = e(\bar{y}_K) \quad \text{and} \quad \nabla_1 L(e(y_K), y_K, 0) = 0, \quad y_K \in N_3(\bar{y}_K)$$

Choose now ε and $N_1(\bar{x})$ as defined by A.19 and A.18 respectively in lemma A.17. Define now

A.26 $$N(\bar{x},\bar{y}) \subset \{(x,y) \mid (x,y) \in R^n \times R^k, \ \ x \in N_1(\bar{x}), \ \ y_K \in N_3(\bar{y}_K)\}$$

By lemma A.17, for any $(x,y) \in N(\bar{x},\bar{y})$ we have that

$$I(x) \cup \{m+1,\ldots,k\} = I \cup \{m+1,\ldots,k\} = K$$

Hence starting with $(x^0, y^0) \in N(\bar{x},\bar{y})$, algorithm 4.4 is equivalent to

A.27 $$y_K^{j+1} = y_K^j + \beta \nabla_2 L(e(y_K^j), y_K^j, y_J^j) = y_K^j + \beta \nabla_2 L(e(y_K^j), y_K^j, 0)$$

A.28 $$y_J^{i+1} = 0$$

A.29 $$x^{j+1} = e(y_K^{j+1})$$

where the last equality in A.27 above follows from the fact that $\frac{\partial L}{\partial y_i}(x,y) = \lambda_2(g(x_i), y_i)$, $i = 1,\ldots,m$, and hence $\nabla_2 L(e(y_K), y_K, y_J)$ is independent of y_J. Consider the mapping $G(y): R^{\bar{k}} \to R^{\bar{k}}$ derived from A.27 and defined by

A.30 $$G(y_K) = y_K + \beta \nabla_2 L(e(y_K), y_K, 0)$$

and its gradient

A.31 $$\nabla G(\bar{y}_K) = I + \beta \nabla_{21} L(e(\bar{y}_K), \bar{y}_K, 0) \nabla e(\bar{y}_K) + \beta \nabla_{22} L(e(\bar{y}_K), \bar{y}_K, 0)$$

$$= I + \beta \nabla_{21} L(\bar{x}, \bar{y}_K, 0) \nabla e(\bar{y}_K) \qquad \text{(by vi and vi')}$$

Differentiating $\nabla_1 L(e(y_K), y_K, 0) = 0$ with respect to y_K and evaluating at $\bar{y}_K$ gives

$$\nabla_{11} L(\bar{x}, \bar{y}_K, 0) \nabla e(\bar{y}_K) + \nabla_{12} L(\bar{x}, \bar{y}_K, 0) = 0$$

and since by theorem 2.5, $\nabla_{11} L(\bar{x}, \bar{y})$ is positive definite for large enough α, we get that

$$\nabla e(\bar{y}_K) = -\nabla_{11} L(\bar{x}, \bar{y}_K, 0)^{-1} \nabla_{12} L(\bar{x}, \bar{y}_K, 0)$$

Substitution in A.31 gives

$$\nabla G(\bar{y}_K) = I - \beta \nabla_{21} L(\bar{x}, \bar{y}_K, 0) \nabla_{11} L(\bar{x}, \bar{y})^{-1} \nabla_{12} L(\bar{x}, \bar{y}_K, 0)$$

The linear independence assumption of $\nabla g_i(\bar{x})$, $i \in K$ gives that $\nabla_{21} L(\bar{x}, \bar{y}_K, 0)$ and $\nabla_{12} L(\bar{x}, \bar{y}_K, 0)$ are of rank $\bar{k} \leq n$ and since $\nabla_{11} L(\bar{x}, \bar{y})^{-1}$ is positive definite it follows that for β small enough, $\beta \in (0, \bar{\beta})$, the eigenvalues of $\nabla G(\bar{y}_K)$ are all strictly less than one and hence its spectral radius $\rho(\nabla G(\bar{y}_K)) < 1$. Hence by Ostrowski's point of attraction theorem [16, p. 145] there exists an open neighborhood $N_4(\bar{y}_K)$ of $\bar{y}_K$ such that when $y_K^0 \in N_4(\bar{y}_K)$, the iterates A.27 are well defined and converge linearly to $\bar{y}_K$. Since $e(y_K)$ is differentiable the iterates of A.29 converge also linearly to $\bar{x} = e(\bar{y}_K)$. Hence the iterates A.27, A.28 and A.29 converge linearly to $\bar{y}_K, \bar{y}_J = 0$ and $\bar{x}$ respectively. The neighborhood $\bar{N}(\bar{x}, \bar{y})$ of convergence mentioned in the statement of the theorem can be taken as follows

$$\bar{N}(\bar{x}, \bar{y}) \subset \{(x, y) \mid (x, y) \in N(\bar{x}, \bar{y}),\ y_K \in N_4(\bar{y}_K)\}$$

where $N(\bar{x}, \bar{y})$ is defined by A.26. ∎

Proof of Theorem 5.3

(a) $\Longrightarrow$ (i): Because $\psi'(0) = 0$ it follows that

$$\lambda_1(\xi,\eta) = \frac{\partial\lambda(\xi,\eta)}{\partial\xi} = \alpha\,\psi'(\alpha\xi+\eta)_+$$

$$\lambda_2(\xi,\eta) = \frac{\partial\lambda(\xi,\eta)}{\partial\eta} = \psi'(\alpha\xi+\eta)_+ - \psi'(\eta)$$

Hence the equations

$$\lambda_1(0,\eta) = \alpha\psi'(\eta)_+ = \mu, \quad \lambda_2(0,\eta) = \psi'(\eta)_+ - \psi'(\eta) = 0$$

have a solution $\eta \geqq 0$ for each $\mu \geqq 0$ because $\psi'(\eta)_+$ maps $[0,\infty)$ onto itself.

Also the equations

$$\lambda_1(\xi,\eta) = \alpha\,\psi'(\alpha\xi+\eta)_+ = 0, \quad \lambda_2(\xi,\eta) = \psi'(\alpha\xi+\eta)_+ - \psi'(\eta) = 0$$

have a solution $\eta = 0$ for each $\xi < 0$ because $\psi'(0) = 0$.

(a) $\Longrightarrow$ (ii): $\lambda_1(0,\eta) = \alpha\,\psi'(\eta)_+ \geqq 0$ (since ψ' is strictly increasing and $\psi'(0) = 0$)

$$\lambda_1(\xi,0) = \alpha\,\psi'(\alpha\xi)_+ = 0 \quad \text{for } \xi < 0 \quad (\text{since } \psi'(0) = 0)$$

Now

A.32 $$\left\langle \lambda_2(\xi,\eta) = \psi'(\alpha\xi+\eta)_+ - \psi'(\eta) = 0 \right\rangle$$

$$\Longrightarrow \left\langle \begin{matrix} \alpha\xi+\eta \geqq 0, & \alpha\xi+\eta = \eta \\ & \text{or} \\ \alpha\xi+\eta < 0, & \eta = 0 \end{matrix} \right\rangle \Longrightarrow \left\langle \begin{matrix} \xi = 0, & \eta \geqq 0 \\ & \text{or} \\ \xi < 0, & \eta = 0 \end{matrix} \right\rangle \Longrightarrow \begin{matrix} \xi \leqq 0 \\ \text{and} \\ \xi\eta = 0 \end{matrix}$$

Hence $\lambda_2(\xi,\eta) = 0$ implies that $\xi \leqq 0$ and $\xi\eta = 0$.

<u>(a) ==> (viii):</u>

$$\lambda_2(\xi,\eta) = 0 \implies \begin{cases} \xi = 0, & \eta \geqq 0 \\ & \text{or} \\ \xi < 0, & \eta = 0 \end{cases} \implies \lambda(\xi,\eta) = \begin{cases} \psi(\eta)_+ - \psi(\eta) = 0 \\ \text{or} \\ \psi(\alpha\xi)_+ - \psi(0) = 0 \end{cases}$$

(by A.32)

<u>(a) & (b) ==> (v):</u>

$$\lambda_1(\xi,\eta) = \alpha\,\psi'(\alpha\xi + \eta)_+ \geqq 0 \quad \text{(since } \psi'(0) = 0 \text{ and } \psi' \text{ is strictly increasing)}$$

$$\lambda_1(\xi,\eta)\xi - \lambda(\xi,\eta) = \alpha\xi\,\psi'(\alpha\xi + \eta)_+ - \psi(\alpha\xi + \eta)_+ + \psi(\eta)$$

$$= \begin{cases} \psi(\eta) & \text{if } \alpha\xi + \eta < 0 \quad \text{(since } \psi(0) = 0 \text{ \& } \psi'(0) = 0) \\ \alpha\xi\psi'(\alpha\xi + \eta) - \psi(\alpha\xi+\eta) + \psi(\eta) & \text{if } \alpha\xi + \eta \geqq 0 \end{cases}$$

$$\geqq \begin{cases} 0 & \text{(since } \psi(\eta) \geqq 0) \\ 0 & \text{(by convexity of } \psi) \end{cases}$$

(a) & (c) $\Longrightarrow$ (iii):

Because $\psi''(0) = 0$ it follows that

$$\lambda_{11}(\xi,\eta) = \alpha^2 \psi''(\alpha\xi + \eta)_+$$

$$\lambda_1(\xi,\eta) = \alpha\psi'(\alpha\xi + \eta)_+ = 0 \implies \alpha\xi + \eta \leqq 0 \quad \text{(since } \psi'(0) = 0 \text{ and } \psi' \text{ is strictly increasing).}$$

$$\implies \lambda_{11}(\xi,\eta) = \alpha^2\psi''(\alpha\xi + \eta)_+ = 0$$

$$\text{(since } \psi''(0) = 0\text{)}$$

(a) & (c) $\Longrightarrow$ (iv):

Since $\lambda_1(0,\eta) = \alpha\, \psi'(\eta)_+ > 0$ it follows that $\eta > 0$. Hence $\lambda_{11}(0,\eta) = \alpha^2 \psi''(\eta) > 0$, because $\psi''(\eta) > 0$ for $\eta > 0$. Hence $\lambda_{11}(0,\eta) \to \infty$ as $\alpha \to \infty$ for $\lambda_1(0,\eta) > 0$.

(a) & (c) $\Longrightarrow$ (vi):

$$\lambda_{22}(\xi,\eta) = \psi''(\alpha\xi + \eta)_+ - \psi''(\eta)$$

$$\lambda_2(\xi,\eta) = 0 \implies \begin{cases} \xi = 0, \quad \eta \geqq 0 \\ \quad\text{or} \\ \xi < 0, \quad \eta = 0 \end{cases} \qquad \text{(by A.32)}$$

$$\implies \begin{cases} \lambda_{22}(\xi,\eta) = \psi''(\eta)_+ - \psi''(\eta) = 0 \\ \quad\text{or} \\ \lambda_{22}(\xi,\eta) = \psi''(\alpha\xi)_+ - \psi''(0) = 0 \end{cases}$$

(a) & (c) $\Longrightarrow$ (vii):

$$\lambda_1(0,\eta) = \alpha\, \psi'(\eta)_+ > 0 \Big\rangle \implies \eta > 0$$

$$\implies \lambda_{12}(0,\eta) = \alpha\, \psi''(\eta)_+ > 0 \quad \text{(since } \psi''(\eta) > 0 \text{ for } \eta > 0\text{)}$$

(a') $\Longrightarrow$ (i'):

The conditions

$$\varphi_1(0,\eta) = \alpha\,\psi'(\eta) = \mu, \qquad \varphi_2(0,\eta) = \psi'(\eta) - \psi'(\eta) = 0$$

are satisfied by some $\eta \in R$ because $\alpha \neq 0$ and ψ' maps R onto R.

(a') $\Longrightarrow$ (ii'):

$\varphi_2(\xi,\eta) = \psi'(\alpha\xi + \eta) - \psi'(\eta) = 0$ implies, since ψ' is strictly increasing, $\alpha\xi + \eta = \eta$ and hence $\xi = 0$, since $\alpha \neq 0$.

(a') $\Longrightarrow$ (viii'):

$$\varphi_2(\xi,\eta) = \psi'(\alpha\xi + \eta) - \psi'(\eta) = 0 \Longrightarrow \xi = 0 \Longrightarrow \varphi(\xi,\eta) = \psi(\eta) - \psi(\eta) = 0.$$

(a') & (b') $\Longrightarrow$ (v'):

$$\varphi_1(\xi,\eta)\xi - \varphi(\xi,\eta) = \alpha\psi'(\alpha\xi+\eta)\xi - \psi(\alpha\xi+\eta) + \psi(\eta) \geqq 0$$

(by convexity of ψ)

(a') & (c') $\Longrightarrow$ (iv'):

$\varphi_{11}(0,\eta) = \alpha^2\,\psi''(\eta) \to \infty$ as $\alpha \to \infty$ because $\varphi_1(0,\eta) = \alpha\psi'(\eta) \neq 0$ implies that $\eta \neq 0$ and hence $\psi''(\eta) > 0$.

(a') & (c') $\Longrightarrow$ (vi'):

$$\left.\varphi_2(\xi,\eta) = \psi'(\alpha\xi + \eta) - \psi'(\eta) = 0 \right\rangle \Longrightarrow \xi = 0$$

$$\Longrightarrow \varphi_{22}(\xi,\eta) = \psi''(\eta) - \psi''(\eta) = 0$$

<u>(a') & (c') $\Longrightarrow$ (vii'):</u>

$$\varphi_1(0,\eta) = \alpha\, \psi'(\eta) \neq 0 \quad \Longrightarrow \quad \eta \neq 0$$

$$\Longrightarrow \quad \varphi_{12}(0,\eta) = \alpha\, \psi''(\eta) \neq 0 \ .$$