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SOLUTIONS OF A DIFFERENTIAL EQUATION ARISING IN CHEMICAL REACTOR PROCESSES*

by

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1. Introduction.

Consider the nonlinear boundary-value problem

$$(1.1) \qquad \begin{cases} u'' + \frac{1}{x} u' + \beta f(u(x) + \tau) = 0 , & 0 < x < 1 \\ u'(0) = u(1) = 0 \end{cases}$$

where $\tau \geq 0$, $\beta \geq 0$ are nonnegative constants and the unknown function $u(x)$ is also required to be nonnegative. The function $f(\alpha)$ is assumed to satisfy

$$(1.2a) \qquad f'(\alpha) \geq 0 , \quad \alpha > 0 .$$

There is an $\alpha_0 > 0$ such that

$$(1.2b) \qquad \begin{aligned} f''(\alpha) > 0 , &\quad 0 < \alpha < \alpha_0 \\ f''(\alpha) < 0 , &\quad \alpha_0 < \alpha \end{aligned}$$

$$(1.2c) \qquad f(0) = 0 , \quad \lim_{\alpha \to +\infty} f(\alpha) = 1 .$$

It is natural to refer to the interval $[0, \alpha_0]$ as the convex region and the interval $[\alpha_0, \infty)$ as the concave region.

The special case

$$(1.3) \qquad f(\alpha) = f_0(\alpha) = \exp\left\{- \frac{1}{|\alpha|}\right\}$$

arises in the study of chemically reacting systems as described by T. Cotter [5]. Working with Cotter, Paul Stein and Myron Stein have done extensive computation on this special problem. This report was motivated by the desire for a more precise understanding of these problems and their computational results. Our results are far from complete and much more mathematical analysis of these problems arising in the study of chemical reactions is certainly necessary. Nevertheless, in this special case, our results together with the computational results of P. Stein and M. Stein seem to give a reasonably clear understanding of the phenomena of multiple solutions. The interested reader will find more basic material in the books by Gravalas [10] and Frank-Kamenetskii [8].

In Section 2 we describe the computations of P. Stein and M. Stein and comment briefly on their conclusions. In Section 3 we collect some basic facts about monotone iterations for problem (1.1). These results depend on the work of D. Sattinger [22], H. B. Keller [11], D. S. Cohen and H. B. Keller [12], and the author [17], [18], and [19]. This fundamental approach is also described in Courant-Hilbert [7]. In Section 4 we employ a basic result of M. A. Krasnosel'skii and V. Ja. Stecenko [14] together with a recent result of H. Amann [2] to establish regions (in the τ-β plane) in which there are at least three solutions of problem (1.1). In Section 5 we turn to uniqueness questions. We derive regions (in the τ-β plane) for which problem (1.1) has exactly one solution. In addition we derive other useful estimates.

Of course, problem (1.1) can be rewritten as a Hammerstein equation

$$(1.4) \qquad u(x) = \beta \int_0^1 k(x, t)\ f(u(t) + \tau)\, dt$$

where

$$(1.4a) \qquad k(s, t) = \begin{cases} -t \log t \ , & 0 \le s \le t \le 1 \\ -t \log s \ , & 0 \le t \le s \le 1 \ . \end{cases}$$

We write for fixed t, β, -

$$(1.4b) \qquad u = A u \ .$$

Such problems have a long history and the interested reader is referred to the lecture notes [20] of G. Pimbley.

In addition, it is worth observing that

(i) if $f(u+\tau) \geq 0$ then $u(x) \geq 0$, and

(ii) in that case $u'(x) \leq 0$.

The first remark follows from the maximum principle while the second follows from the formula

$$u'(x) = -\frac{\beta}{x}\int_0^x t\ f(u(t)+\tau)\,dt \quad . \tag{1.5}$$

Finally, we remark that there are many related problems which arise in chemical reactor theory. In recent works D. S. Cohen [4] and D. H. Sattinger [22, Sec. 5] use the theory of monotone iterations <u>together</u> with the theory of singular perturbations to obtain results which appear very similar to the results of this report.

We are indebted to our many friends, both in Los Alamos and Madison who discussed this problem with us. We want to give particular thanks to Paul Stein, George Pimbley, Carl de Boor, and Charles Conley.

2. Computational Results.

In their study of the important special case where $f(\alpha) = f_0(\alpha)$ is given by (1.3), Paul Stein and Myron Stein proceeded as follows. Choosing a value of $\beta > 0$, they used a Runge-Kutta method to "solve" the initial-value problem

$$y'' + \frac{1}{x} y' + \beta e^{-\frac{1}{|y|}} = 0 \ , \ 0 < x < 1 \tag{2.1}$$

$$y'(0) = 0 \ , \ y(0) = y_0 \geq 0 \ . \tag{2.2}$$

Thus, they constructed an approximation to the function

$$\tau = y(1 ; \beta, y_0) \ , \tag{2.3}$$

the final value. Since this function is a very smooth function of the parameters β, y_0, and since their curves are very smooth; it is reasonable to assign great validity to their results. The value of β was sampled throughout the range

$$0 < \beta \leq 10^{13} \ ,$$

a rather large range. They found a region in the first quadrant of the τ, β plane, say R, and for $(\tau, \beta) \in R$ they found exactly three solutions. On the boundary of R they found exactly two solutions. Outside of the closure of R, there was a unique solution. The region R (as computed) is bounded by two curves. An upper branch, the explosion limit curve and a lower branch, the quenching curve. These curves meet in the point

$$\tau = 0.2421 \ , \ \beta = 10.961 \ .$$

The explosion limit curve (the upper curve) is monotone decreasing in the interval $0 \leq \tau \leq 0.2421$ and is easily described by the equation

$$\tilde{\beta}(\tau) \equiv \frac{1}{2}\, s(\tau) \left\{ (1 - 2\tau) - \sqrt{1 - 4\tau} \right\} e^{\frac{2}{1 - \sqrt{1 - 4\tau}}} \tag{2.4}$$

where $s(\tau)$ is a slowly varying monotone increasing function of τ. For example

$$\begin{cases} s(2.7473 \times 10^{-2}) = 5.4505772 \\ s(1.355667 \times 10^{-1}) = 5.5220439 \\ s(2.421 \times 10^{-1}) = 5.695645 \end{cases} \tag{2.5}$$

while

$$\begin{cases} \tilde{\beta}(2.7473 \times 10^{-2}) = 10^{13} \\ \tilde{\beta}(1.355667 \times 10^{-1}) = 70 \\ \tilde{\beta}(2.421 \times 10^{-1}) = 10.961 \end{cases} \tag{2.6}$$

The quenching curve $\underset{\sim}{\beta}(\tau)$ is also monotone decreasing in the interval $0 \leq \tau \leq 0.2421$ and is described by the following table.

τ	$\underset{\sim}{\beta}(\tau)$
0.00000	16.84
0.04200	16.00
0.08969	15.00
0.13438	14.00
0.17546	13.00
0.21191	12.00
0.24125	11.00
0.24210	10.961

Remark: Recall that the function $\tau(\beta)$ was actually computed rather than $\beta(\tau)$.

Additional computations were carried out by Carl de Boor using a program based on a collocation method of de Boor and Swartz [3]. In figures 1, 2, and 3 we show his results for

$$y(0\,;\,\beta,\,\tau) = u(0\,;\,\beta,\,\tau) + \tau$$

for fixed values of τ.

Remark: The half ray, $\{16.84 < \beta < \infty,\ \tau = 0\}$ is part of the region R in which there were found exactly three solutions.

3. Preliminaries.

In this section we collect some of the basic facts concerning problems of the form (1.1). Our first concern is with monotone iterative methods. While such methods have been discussed by many authors (see [2], [4], [17], and [18]) the recent work [22] of Sattinger is an excellent reference for the interested reader.

Let the operator L be defined by

$$Lu \equiv u'' + \frac{1}{x} u' \quad .$$

Definition 3.1. A function $\tilde{u}(x) \geq 0$ is called an "upper solution" of problem (1.1) if

$$(3.1) \qquad \begin{cases} L\tilde{u} + \beta\, f(\tilde{u}+\tau) \leq 0 \quad , \quad 0 < x < 1 \\ \tilde{u}'(0) = 0 \ , \quad \tilde{u}(1) \geq 0 \quad . \end{cases}$$

A function $\underset{\sim}{u} \geq -\tau$ is called a "lower solution" of problem (1.1) if

$$(3.2) \qquad \begin{cases} L\underset{\sim}{u} + \beta\, f(\underset{\sim}{u}+\tau) \geq 0 \quad , \quad 0 < x < 1 \\ \underset{\sim}{u}'(0) = 0 \ , \quad -\tau \leq \underset{\sim}{u}(1) \leq 0 \quad . \end{cases}$$

Definition 3.2. A function $\varphi(x) \in C^1[0,1]$ is said to "dominate" a function $\psi(x) \in C^1[0,1]$ if:

(i) $\psi(x) < \varphi(x)$, $0 < x < 1$,

(ii) if $\psi(0) = \varphi(0)$, then $\psi'(0) < \varphi'(0)$,

(iii) if $\psi(1) = \varphi(1)$, then $\psi'(1) > \varphi'(1)$.

And, if φ dominates ψ , we write

$$(3.3) \qquad \psi \prec \varphi \quad .$$

Since we are dealing with continuous functions we will use the supremum norm,

$$(3.4) \qquad \|u\| \equiv \max\{|u(x)| \ ; \ 0 \leq x \leq 1\} \quad .$$

Let $A: C[0,1] \to C^2[0,1]$ denote the basic solution operator described by the Hammerstein operator. That is: if $v \in C[0,1]$, then $u = Av$ is the solution of the linear boundary value problem

$$(3.5) \qquad \begin{cases} Lu + \beta f(v+\tau) = 0 \\ u'(0) = u(1) = 0 \ . \end{cases}$$

Observe that A can also be considered as a mapping from the bounded piecewise continuous functions into $C^1[0,1]$.

Lemma 3.1. Suppose $u_1(x) \le u_2(x)$. Then

$$(3.6) \qquad Au_1 < Au_2$$

unless $u_1(x) \equiv u_2(x)$.

Proof: Let $w = Au_1 - Au_2$. Then, using (1.2a) we have

$$Lw = -\beta \left[f(u_1+\tau) - f(u_2+\tau) \right] \ge 0 \ .$$

Applying the Hopf form of the "maximum principle" (see [21]) we obtain (3.6).

Our basic iterative process is described by

$$(3.7) \qquad Z_{n+1} = AZ_n \ .$$

Lemma 3.2. Suppose $\tilde{u}(x)$ is an upper solution of problem (1.1) and is not a solution. Let $\tilde{u}_n$, $n = 1, 2, \ldots$ denote the successive iterates in (3.7) with $\tilde{u}_0 = \tilde{u} = Z_0$. Then

$$(3.8a) \qquad \tilde{u}_{n+1} < \tilde{u}_n \ , \quad n = 0, 1, \ldots \ .$$

Moreover, each $\tilde{u}_n$ is also an upper solution. Similarly, if $\underset{\sim}{u}$ is any lower solution and $\underset{\sim}{u}_n$ denotes the successive iterates in (3.7) with $\underset{\sim}{u}_0 = \underset{\sim}{u} = Z_0$, then

$$(3.8b) \qquad \underset{\sim}{u}_n < \underset{\sim}{u}_{n+1} \ , \quad n = 0, 1, \ldots \ .$$

Moreover, each $\underset{\sim}{u}_n$ is also a lower solution. In fact, if $\underset{\sim}{u} \le \tilde{u}$ are respectively a lower solution and an upper solution, then

(3.8c) $$\underset{\sim}{u}_j \prec \tilde{u}_k \quad \forall j,k \ge 0 \quad .$$

Finally, if $\bar{u}$ is a solution of problem (1.1) and

(3.8d) $$\underset{\sim}{u} \le \bar{u} \quad (\text{or } \bar{u} \le \tilde{u})$$

then

(3.8e) $$\underset{\sim}{u}_n \prec \bar{u} \quad (\text{or } \bar{u} \prec \tilde{u}_n) \quad .$$

Proof: This is the basic lemma of the theory of monotone iteration (see [7], [17], [18], [22]) and follows directly by induction and repeated use of the Hopf form of the maximum principle.

Corollary 3.2.1. Problem (1.1) has at least one solution. In fact, there exists a unique minimal solution $v(x)$ and a unique maximal solution $u(x)$ which satisfy: If $z(x)$ is any other solution of problem (1.1), then

(3.9) $$v \prec z \prec u \quad .$$

Of course, there may be only one solution and $u(x) \equiv v(x)$. Furthermore, the minimal solution and the maximal solution are monotone increasing in τ and β. Finally, the maximal solution is continuous from the right in τ and β while the minimal solution is continuous from the left in τ and β.

Proof: If $\tau = 0$, then $\underset{\sim}{v}(x) \equiv 0$ is a solution. In any case, $\underset{\sim}{v}(x) \equiv 0$ is a lower solution. And, if $z(x)$ is any solution of problem (1.1), then

(3.10a) $$0 \equiv \underset{\sim}{v}(x) \le z(x) \quad .$$

The function

$$\widetilde{u}(x) = \beta/4\,(1 - x^2) \tag{3.10b}$$

satisfies

$$L\widetilde{u} + \beta = 0\ ,\quad \widetilde{u}'(0) = \widetilde{u}(1) = 0\ ,$$

and is an upper solution. Moreover, it is an easy matter to see that $\widetilde{u}(x)$ is greater than all solutions of problem (1.1). Thus

$$\underset{\sim}{v}_n \prec \widetilde{u}_n$$

and the sequences $\{\underset{\sim}{v}_n\}$, $\{\widetilde{u}_n\}$ converge. It is an easy matter to see that the limit functions are "weak" solutions of problem (1.1). And, hence are solutions of problem (1.1). Moreover, Dini's theorem implies that the convergence is uniform.

If $z(x)$ is a solution of problem (1.1) an easy inductive argument shows that

$$\underset{\sim}{v}_n \prec z \prec \widetilde{u}_n\ .$$

Finally, the monotone behavior and one sided continuity of the maximal and minimal solution follow as in [18].

These results concerning the iterative method (3.7) and the corresponding existence theorems are well known. We now present a recent result of H. Amann [2] in the special form that we require.

<u>Lemma 3.3</u>. Let $\underset{\sim}{v}(x)$ and $\widetilde{u}(x)$ be the lower and upper solutions of Corollary 3.2.1. Suppose $\underset{\sim}{w}(x)$ and $\widetilde{W}(x)$ are lower and upper solutions respectively which are not solutions of problem (1.1).

Suppose

(3.11a) $$\underset{\sim}{v} \prec \tilde{W}\ ,\ \underset{\sim}{w} \prec \tilde{u}\ ,$$

and

(3.11b) $$\tilde{W}(0) < \underset{\sim}{w}(0)\ .$$

Then there exists a solution $\overline{u}(x)$ of problem (1.1) which satisfies

(3.12) $$\tilde{W}(0) < \overline{u}(0) < \underset{\sim}{w}(0).$$

Proof: Here we merely sketch the proof. See [2] for a more detailed discussion. We consider the "degree" (see [13], [23]) of the map

$$\Phi u = u - A u\ .$$

Let $d(\Phi, \Omega, 0)$ denote the degree of Φ relative to the open set Ω at 0. Let

(3.13a) $$\Omega \equiv \{ V \in C[0,1];\ \underset{\sim}{v} < V < \tilde{u} \}$$

(3.13b) $$\Omega_1 \equiv \{ V \in C[0,1];\ \underset{\sim}{v} < V < \tilde{W} \}$$

(3.13c) $$\Omega_2 \equiv \{ V \in C[0,1];\ \underset{\sim}{w} < V < \tilde{u} \}$$

(3.13d) $$\Omega_3 \equiv \Omega - (\overline{\Omega_1 \cup \Omega_2})$$

The inequalities (3.11a) together with the basic facts about $\underset{\sim}{v}$ and $\tilde{u}$ imply

$$d(\Phi\,;\Omega, 0) = 1\ ,$$

$$d(\Phi\,;\Omega_1, 0) = d(\Phi\,;\Omega_2, 0) = 1\ .$$

Thus, by the additivity of the degree,

$$d(\Phi\,;\Omega_3, 0) = -1$$

and there is a solution $\bar{u}(x) \in \Omega_3$. Consider $\bar{u}(0)$. Returning to the inequalities (3.11a), (3.11b) we see that (3.12) must hold.

Definition 3.3. Let $\bar{u}(x)$ be a solution of problem (1.1). We say that $\bar{u}(x)$ is a "stable" solution if there exists $\underset{\sim}{w}(x)$, $\tilde{W}(x)$ lower and upper solutions respectively such that

(i) $$\underset{\sim}{w} \prec \bar{u} \prec \tilde{W}$$

and

(ii) $$\underset{\sim}{w}_n \uparrow \bar{u}\ ,\ \tilde{W}_n \downarrow \bar{u} \quad \text{as } n \to \infty\ .$$

Stable solutions are important for many reasons. Two of the most important follow (see Section 4 of [22]).

Lemma 3.4. Let $\bar{u}$ be a stable solution (in the sense of Definition 3.3). Then there is a neighborhood of $\bar{u}$, say

(3.14)
$$\Omega \equiv \{ V \in C[0,1]\ ;\ \bar{u}(x) - \delta_1 < V(x) < \bar{u}(x) + \delta_2,\ 0 < \delta_1 < \tau,\ \delta_2 > 0 \}$$

such that: If $V(x,t)$ is the solution of

(3.15)
$$\begin{cases} \dfrac{\partial}{\partial t} V = LV + \beta f(V+\tau),\ 0 < x < 1, \quad t > 0 \\ V'(0,t) = V(1,t) = 0\ , \\ V(x,0) = V(x) \in \Omega\ , \end{cases}$$

then

(3.16) $$\| V(\cdot,t) - \bar{u} \| \to 0 \quad \text{as } t \to \infty\ .$$

Remark: If $\tau = 0$, we obtain a neighborhood in the positive solutions, i.e., Ω takes the form

$$\Omega \equiv \{V \in C[0,1],\ V \geq 0\ ;\ \bar{u}(x) - \delta_1 < V(x) < \bar{u}(x) + \delta_2,\ \delta_1, \delta_2 > 0 \}\ .$$

Proof: This result is essentially theorem 4.1 of [22]. Our formulation is technically stronger and requires a slightly different proof which we omit.

Remark: The above result is not used in our discussion. However, it is important in the further study of the problems of chemical reactions.

Lemma 3.5. Let $\bar{u}(x)$ be a solution of problem (1.1). Consider the linear eigenvalue problem

$$(3.17) \qquad \begin{cases} L\varphi + \lambda\beta f'(\bar{u}+\tau)\varphi = 0 \\ \varphi'(0) = \varphi(1) = 0 \end{cases} .$$

Let $\lambda_0(\beta f'(\bar{u}+\tau)) = \lambda_0$ denote the lowest eigenvalue of this problem. Then

(i) if $\bar{u}$ is stable, then $1 \le \lambda_0$

(ii) if $1 < \lambda_0$, then $\bar{u}$ is stable

(iii) if $\lambda_0 < 1$, then $\bar{u}$ is unstable in the sense that: for every $\epsilon > 0$ there exists $\underset{\sim}{w}(x)$ and $\tilde{W}(x)$, lower and upper solutions respectively, such that

(iiia) $\tilde{W} \prec \bar{u} \prec \underset{\sim}{w}$

(iiib) $\| \underset{\sim}{w} - \bar{u} \| < \epsilon$, $\| \tilde{W} - \bar{u} \| < \epsilon$.

Proof: See theorem 4.2 of [22].

Lemma 3.6: Suppose $V_0(x) \ge 0$ is a piecewise continuous function and

$$(3.18) \qquad V_1(x) = AV_0 \ge V_0(x) \quad .$$

Then V_1 is a lower solution in the sense that

$$(3.19) \qquad \begin{cases} LV_1 + \beta\, f(V_1 + \tau) \geq 0, & 0 < x < 1 \\ V_1'(0) = V_1(1) = 0 \end{cases}$$

wherever LV_1 is defined and at a jump in V_0, the limits from both sides satisfy (3.19). Indeed,

$$V_2 = AV_1$$

is a genuine lower solution.

Proof: Equation (3.18) means that $V_1(x)$ satisfies the boundary conditions of (3.19) and

$$LV_1 + \beta\, f(V_0 + \tau) = 0$$

(properly interpreted at the points of discontinuity of V_0). Thus, the monotonicity of $f(\alpha)$ gives

$$LV_1 + \beta\, f(V_1 + \tau) = \beta[f(V_1 + \tau) - f(V_0 + \tau)] \geq 0 \quad .$$

Remark: A function $V_0(x)$ which satisfies (3.18) is called a generalized lower solution.

Finally we turn to results on a closely related problem. This result is due to H. Fujita [9] but this proof is due to S. Karlin.

Lemma 3.7: Suppose $f(\alpha)$ is strictly convex for all α. Of course f cannot be bounded. Hence (1.2c) does not hold. Suppose instead we have

$$(1.2c') \qquad f(\alpha + \tau) \geq 0 \; .$$

Suppose problem (1.1) has at least one nonnegative solution. Then it has only one stable solution, say $u_0(x)$. (Note: all solutions are nonnegative.)

And, if $U(x)$ is any other solution, then

(3.20) $$u_0 \prec U .$$

Moreover, it is not possible to find an upper solution $\tilde{u}$ such that

* $$\tilde{u}_n \downarrow U .$$

And, it is not possible to find a lower solution $\underset{\sim}{u}$ such that

** $$\underset{\sim}{u}_n \uparrow U .$$

Proof: If $f(\tau) = 0$ then $u_0(x) \equiv 0$ is a solution. If $f(\tau) > 0$ then $u_0(x) \equiv 0$ is a lower solution and the sequence $\tilde{u}_n$ (with $\tilde{u}_0 = u_0$) converges monotonically upward to a unique minimal solution $u_0(x)$. If there is another solution U, a straight forward induction gives (3.20). Suppose U is a solution satisfying (3.20). Consider the function

(3.21) $$w(x;\lambda) = \lambda U(x) + (1-\lambda)\ u_0(x) .$$

For all $\lambda > 1$, we have

$$U \prec w(\cdot,\lambda) .$$

Using the strict convexity of $f(\alpha)$, we obtain

$$f(w+\tau) > \lambda f(U+\tau) + (1-\lambda)\ f(u_0 + \tau)$$

Thus, applying lemma 3.1, we have

$$w_1 = Aw \succ \lambda\ AU + (1-\lambda)\ A\ u_0 = w .$$

On the other hand, if $0 < \lambda < 1$, we have

$$u_0 \prec w(\cdot,\lambda) \prec U$$

and

$$f(w+\tau) \le \lambda\, f(U+\tau) + (1-\lambda)\, f(u_0+\tau).$$

Thus

$$w_1 = Aw \prec w \,.$$

The lemma now follows immediately from lemma 3.1.

Corollary 3.7.1. Let $\overline{u}$ be a solution of the original problem (1.1), i.e., $f(\alpha)$ satisfies (1.2a), (1.2b) and (1.2c). Suppose

$$\overline{u}(0) \le \alpha_0 \,.$$

Suppose there exists an upper solution $\tilde{u}(x)$, such that

$$\tilde{u}_n \downarrow \overline{u} \,.$$

Then $\overline{u}(x)$ is the minimal solution and, $\overline{u}(x)$ is stable.

Suppose there exists a lower solution $\underset{\sim}{u}$ such that

$$\underset{\sim}{u}_n \uparrow \overline{u} \,.$$

Then $\overline{u}(x)$ is the minimal solution. And, if there is another solution $\hat{u}(x)$ with

$$\hat{u}(0) \le \alpha_0 \,.$$

Then, $\overline{u}(x)$ is stable.

Remark: It is important to observe that this argument really depends only on the positivity of $k(s,t)$. Thus, in particular, the results of lemma 3.7 and corollary 3.7.1 apply to the problem

$$(3.22)\qquad \begin{cases} L\varphi + \beta\, f(\varphi + \tau) = 0 & 0 < x_0 \le x \le x_1 \le 1 \\ \varphi(x_0) = \alpha,\ \varphi(x_1) = \gamma\ . \end{cases}$$

4. Multiple Solutions.

The results of this section are based on the work of M. A. Krasnosel'skii and V. Ja. Stecenko [14] and the work of H. Amann [2], i.e., lemma 3.3.

Lemma 4.1. Let $k(s,t)$ be the kernel of the Hammerstein equation (1.4a). Then for $0 \le s \le e^{-1/2}$

$$\int_0^{e^{-1/2}} k(s,t)\,dt \ge \frac{1}{4e} \quad . \tag{4.1}$$

Proof: The left-hand-side of equation (4.1) is the solution of the linear boundary-value problem

$$\begin{cases} LQ + X\left(x;e^{-1/2}\right) = 0 \quad , \\ Q'(0) = Q(1) = 0 \quad , \end{cases}$$

where, for all $\xi \in (0,1)$ we have

$$X(x;\xi) = \begin{cases} 1 \ , \ 0 \le x < \xi \quad , \\ 0 \ , \ \xi < x \le 1 \quad . \end{cases} \tag{4.2}$$

A direct calculation shows that

$$Q(x) = \frac{1}{2e} - \begin{cases} \dfrac{x^2}{4} \quad , \quad 0 \le x \le e^{-1/2} \\ \dfrac{1}{2e} + \dfrac{1}{2e}\log x \quad , \quad e^{-1/2} \le x \le 1 \quad . \end{cases}$$

Since $Q'(x) \le 0$ we have

$$Q(s) \ge Q\left(e^{-1/2}\right) = \frac{1}{4e} \ , \quad 0 \le s \le e^{-1/2} \ .$$

Remark: A direct calculation (based on the above work) shows that for all $x_0 \epsilon (0,1)$

$$\sup_{x_0} \text{Min} \left\{ \int_0^{x_0} k(s,t)dt \, ; \; 0 \le s \le x_0 \right\} = \frac{1}{4e}$$

and this supremum is taken on when $x_0 = e^{-1/2}$.

Lemma 4.2: Let $\alpha \ge 0$ satisfy

(4.3) $$4e\alpha \le \beta f(\alpha + \tau) .$$

Let

(4.4) $$V_0(x;\alpha) = \alpha X\left(x, e^{-1/2}\right) .$$

Let

(4.5) $$V_1(x;\alpha) = A V_0 ,$$

Then

(4.6a) $$V_0(x,\alpha) \le V_1(x,\alpha)$$

(4.6b) $$V_0(x,\alpha) \not\equiv V_1(x,\alpha) \text{ if } \alpha\tau > 0 .$$

And from lemma 3.6 we see that $V_1(x;\alpha)$ is a lower solution. Moreover, unless $\alpha = \tau = 0$, $V_1(x;\alpha)$ is not a solution of problem (1.1).

Proof: For $x \epsilon [e^{-1/2}, 1]$ we know that $V_1(x;A) \ge 0$. Hence (4.6a) is certainly true in that range of x. For $x \epsilon [0, e^{-1/2}]$ lemma 4.1 gives

$$V_1(x,\alpha) \ge \beta \; f(\alpha + \tau) \int_0^{e^{-1/2}} k(x,t)dt \ge \beta \, \frac{f(\alpha + \tau)}{4e} \ge \alpha .$$

Lemma 4.3. Let $\gamma \geq 0$ satisfy

(4.7) $$4\gamma \geq \beta f(\gamma + \tau) \quad .$$

Let

(4.8) $$U_0(x,\gamma) = \gamma(1 - x^2) \quad .$$

Then $U_0(x;\gamma)$ is an upper solution. Moreover, unless $\gamma = \tau = 0$, $U_0(x,\gamma)$ is not a solution of problem (1.1).

Proof: Using (1.2a) we have

$$L\, U_0 + \beta f(U_0 + \tau) = -4\gamma + \beta f(U_0 + \tau) \leq 0 \quad .$$

These two lemmas lead us to consider our next basic result. Despite its simplicity, a complete proof is rather lengthy and is found in the appendix. However, it is easily understood after a glance at a sketch.

Lemma 4.4. Let $\tau \geq 0$ be fixed. Let $\rho > 0$ be a fixed constant. Consider the equation

(4.9.ρ) $$\rho\alpha = \beta f(\alpha + \tau) \quad .$$

This equation may have, one, two or three solutions depending on the value of $\beta \geq 0$. In fact, if there is a value $\beta_0 > 0$ for which equation (4.9.ρ) has three distinct solutions, then there are two values $\beta_*(\tau,\rho)$, $\beta^*(\tau,\rho)$ with

(4.10) $$\beta_*(\tau,\rho) < \beta_0 < B^*(\tau,\rho)$$

such that:

(i) for all values β_0 satisfying (4.10) equation (4.9.ρ) has exactly three solutions; $\alpha_1(\tau,\rho,\beta_0) < \alpha_2(\tau,\rho,\beta_0) < \alpha_3(\tau,\rho,\beta_0)$

(ii) for $\beta = \beta_*(\tau,\rho)$ there are exactly two solutions $\alpha_1(\tau,\rho,\beta_*(\tau,\rho)) < \alpha_3(\tau,\rho,\beta_*(\tau,\rho))$ with $\alpha_3(\tau,\rho,\beta_*(\tau,\rho))$ a "double" solution in the sense that

$$(4.11a)\qquad \begin{cases} \rho\,\alpha_3 = \beta_*\, f(\alpha_3 + \tau) \quad , \\ \rho = \beta_*\, f'(\alpha_3 + \tau) \ . \end{cases}$$

Moreover,

$$(4.11b)\qquad \alpha_3 + \tau > \alpha_0 > \alpha_1 + \tau \ .$$

Further, if $\beta < \beta_*(\tau,\rho)$, then equation (4.9.ρ) has exactly one solution $\alpha_1(\tau,\rho,\beta)$ and this solution satisfies

$$\alpha_1 + \tau < \alpha_0 \quad .$$

(iii) For $\beta = \beta^*(\tau,\rho)$ there are exactly two solutions $\alpha_1(\tau,\rho,\beta^*(\tau,\rho)) < \alpha_3(\tau,\rho,\beta^*(\tau,\rho))$ with $\alpha_1(\tau,\rho,\beta^*(\tau,\rho))$ a "double" solution in the sense that

$$(4.11c)\qquad \begin{cases} \rho\,\alpha_1 = \beta^*\, f(\alpha_1 + \tau) \quad , \\ \rho = \beta^*\, f'(\alpha_1 + \tau) \ . \end{cases}$$

Moreover, (4.11b) holds. Further, if $\beta^*(\tau,\rho) < \beta$, then equation (4.9.ρ) has exactly one solution $\alpha_3(\tau,\rho,\beta)$ and this solution satisfies

$$\alpha_0 < \alpha_3 + \tau \ .$$

Finally, if $\rho_1 < \rho_2$ and there is a value $\beta_0 > 0$ such that <u>both</u> equation $(4.9.\rho_1)$ and equation $(4.9.\rho_2)$ each have exactly three solutions $\alpha_j(\tau,\rho_k,\beta_0)$ $j = 1,2,3,$ $k = 1,2$, then

$$(4.12) \qquad \begin{aligned} &\alpha_1(\tau,\rho_2,\beta_0) < \alpha_1(\tau,\rho_1,\beta_0) < \alpha_2(\tau,\rho_1,\beta_0) \\ &\qquad < \alpha_2(\tau,\rho_2,\beta_0) < \alpha_3(\tau,\rho_2,\beta_0) < \alpha_3(\tau,\rho_1,\beta_0) \ , \end{aligned}$$

and

$$(4.13) \qquad \begin{cases} \beta^*(\tau,\rho_1) < \beta^*(\tau,\rho_2) \\ \beta^*(\tau,\rho_1) < \beta^*(\tau,\rho_2) \ . \end{cases}$$

<u>Theorem 4.1.</u> Suppose $\tau > 0$. Suppose there is a value $\beta_0 > 0$ such that both equation (4.9.4) and equation (4.9.4e) have at least two solutions That is, suppose β_0 satisfies

$$(4.14) \qquad \beta_*(\tau,4e) \le \beta_0 \le \beta^*(\tau,4) \ .$$

Then there are at least three solutions of problem (1.1) (with $\beta = \beta_0$). Moreover, the minimal solution is stable. Indeed, for all β satisfying

$$(4.14') \qquad \beta \le \beta^*(\tau,4) \ ,$$

the minimal solution is stable.

<u>Proof</u>: Let $\rho_1 = 4$, $\rho_2 = 4e$. Let $\alpha_j(\tau,\rho_k,\beta)$, $j = 1,2,3;$ $k = 1,2$ denote the roots described above. If equality holds in (4.14) we have coalescence of two of these roots and $\alpha_2\ (\tau,\rho_k,\beta)$ does not appear. In any case, we have

$$0 < \alpha_1(\tau,4,\beta) < \alpha_3(\tau,4e,\beta) < \alpha_3(\tau,4,\beta) \quad . \tag{4.15}$$

Thus applying lemma 4.3 and lemma 3.2 we see that there is a solution $\bar{u}(x)$ which satisfies

$$0 \le \bar{u} \le \alpha_1(\tau,4,\beta)\,(1-x^2) \quad . \tag{4.16}$$

Moreover, since (4.11b) holds we may apply lemma 3.7 and corollary 3.7.1 and observe that there is only one solution $\bar{u}(x)$ which also satisfies (4.16). Indeed, this is true as long as (4.14') holds. Thus, we have a minimal stable solution.

Let $V_0(x,\alpha_3(\tau,4e,\beta))$ be given by equation (4.4). Let $V_1(x,\alpha_3(\tau,4e,\beta))$ be given by (4.5). Applying lemma 4.2 we see that $V_1(x,\alpha_3(\tau,4e,\beta))$ is a lower solution. Moreover, using the upper solution (3.10b); or, for all α with $\alpha > \alpha_3(\tau,4,\beta)$ the function $U_0(x,\alpha)$ given by (4.8) is an upper solution. Hence, there is an $\alpha > \alpha_3(\tau,4,\beta)$ such that

$$V_1\Big(x,\tau,\alpha_3(\tau,4e,\beta)\Big) \le \alpha(1-x^2) \quad .$$

And, the function $\alpha(1-x^2)$ is an upper solution. Hence, applying lemma 3.2 we see that there exists a solution $\bar{u}(x)$ which satisfies

$$V_1\Big(\cdot,\tau,\alpha_3(\tau,4e,\beta)\Big) < \bar{u} < U_0(\cdot,\alpha) \ .$$

Finally, since $\alpha_1(\tau,4,B) < \alpha_3(\tau,4e,\beta)$ we may apply lemma 3.3 and discover that there is a third solution satisfying

$$\alpha_1(\tau,4,\beta) < \bar{u}(0) < V_1\Big(0,\alpha_3(\tau,4e,\beta)\Big) \quad . \tag{4.17}$$

Theorem 4.2: Suppose $\tau = 0$. Suppose

(4.18) $$f'(0) = 0 .$$

Suppose β_0 satisfies (4.14). Then there are at least three solutions of problem (1.1).

Proof: This case differs from the case above in that $\bar{u}(x) \equiv 0$ is a solution. Moreover,

$$\alpha_1(0,4,\beta) = \alpha_1(0,4e,\beta) = 0 \quad ,$$

and one does not necessarily have an upper solution $\tilde{u}(x)$ above $\bar{u}(x) \equiv 0$. A look back at lemma 3.3 will show that this is necessary in the argument therein. Thus, we turn our attention to the construction of a sufficiently small upper solution $\tilde{u}(x)$. Let Λ_0 and Ψ_0 be the smallest eigenvalue and the associated eigenfunction of the linear eigenvalue problem

$$\begin{cases} L\Psi + \Lambda\Psi = 0 , \\ \Psi'(0) = \Psi(1) = 0 . \end{cases}$$

Then $0 < \Psi_0$ and we consider

$$\varphi(x;\epsilon) = \epsilon\Psi_0(x) .$$

We have

$$L\varphi + \beta f(\varphi) = \beta\epsilon\Psi_0[f'(\xi) - \Lambda_0/\beta]$$

where

$$0 \le \xi(x) \le \epsilon\Psi_0(x) \quad .$$

Using (4.18), we see that $\varphi(x;\varepsilon)$ is a nonnegative upper solution if ε is small enough.

The proof now follows immediately as in the case of theorem 4.1.

Let us now apply this result to the special case of interest.

Theorem 4.3. Let $\tau \in [0,1/4]$. Let

(4.19a) $$B^*(\tau) = 2\left\{(1-2\tau) - \sqrt{1-4\tau}\right\}\exp\left\{\frac{2}{1-\sqrt{1-4\tau}}\right\}$$

(4.19b) $$B_*(\tau) = 2e\left\{(1-2\tau) + \sqrt{1-4\tau}\right\}\exp\left\{\frac{2}{1+\sqrt{1-4\tau}}\right\} .$$

Let $f(\alpha) = f_0(\alpha)$ be given by (1.3). Then, if

(4.20) $$B_*(\tau) \le \beta \le B^*(\tau) \quad ,$$

there are at least three solutions of problem (1.1). Moreover, the minimal solution is stable.

Proof: Apply theorem 4.1 and theorem 4.2. Compute $\beta^*(\tau,4)$ and $\beta_*(\tau,4e)$.

Remark: Computational results show that

$$\beta_*(\tau,4e) < \beta^*(\tau,4) \ , \ 0 \le \tau < 0.15904 \quad ,$$

i.e.,

$$B_*(\tau) < B^*(\tau) \quad .$$

5. Uniqueness - Other Estimates

Lemma 5.1. Let Λ_0 be the first eigenvalue of the linear eigenvalue problem

$$(5.1) \qquad \begin{cases} L\psi + \lambda\psi = 0 , \\ \psi'(0) = \psi(1) = 0 , \end{cases}$$

and, let ψ_0 be the associated eigenfunction. Then

$$(5.2a) \qquad \Lambda_0 = r_0^2 \doteq (2.4048255)^2 = 5.78305$$

and

$$(5.2b) \qquad \psi_0(x) = J_0(r_0 x)$$

where $J_0(x)$ is the Bessel function of zero order and r_0 is its first positive zero. Moreover, if $\psi(x) \in C^1[0,1]$ with $\psi'(0) = \psi(1) = 0$, then either $\psi(x) \equiv 0$ or

$$(5.3a) \qquad \int_0^1 \left[\psi'(x)\right]^2 x\,dx \geq \Lambda_0 \int_0^1 \left[\psi(x)\right]^2 x\,dx \quad .$$

And, if equality holds in (5.3a), then

$$(5.3b) \qquad \psi(x) = \gamma J_0(r_0 x)$$

where γ is some constant.

Proof: These facts are well known [1], [6].

Remark: We normalize $J_0(r_0x)$ so that $J_0(0) = 1$.

Also we observe that $J_0(x)$ is concave in the interval $[0,r_0]$.

Lemma 5.2: Let

$$\sigma_0 = f'(\alpha_0) = \max \{f'(\alpha), \quad 0 \le \alpha < \infty \} . \tag{5.4}$$

If

$$0 \le \beta \le \Lambda_0/\sigma_0 \tag{5.5}$$

the solution of problem (1.1) is unique.

Proof: Suppose (5.5) holds and there are two nonnegative solutions $u_1(x)$, $u_2(x)$. Let $W(x) = u_1(x) - u_2(x)$. Then, after writing the equation satisfied by W and multiplying by xW we have

$$\begin{cases} xW \ LW + \beta x f'(\eta+\tau) W^2 = 0 , \\ \quad W'(0) = W(1) = 0 . \end{cases}$$

Integrating over $[0,1]$ we obtain

$$\int_0^1 \left[W'(x)\right]^2 x\,dx = \beta \int_0^1 x f'(\eta+\tau)\left[W(x)\right]^2 dx$$

$$\le \beta\sigma_0 \int_0^1 x\left[W(x)\right]^2 dx .$$

Thus, either $W(x) \equiv 0$ or $W(x)$ is a multiple of $J_0(r_0x)$ and

$$\sigma_0 \equiv f'(\eta(x) + \tau)$$

which is impossible (unless $\eta(x) \equiv 0$, $\tau = \alpha_0$).

Corollary 5.2.1. In the special case when $f(\alpha) = f_0(\alpha)$ is given by equation (1.3), then condition (5.5) becomes

(5.5')
$$0 \leq \beta \leq \frac{\Lambda_0}{4} e^2 .$$

Lemma 5.3. There is a smallest value τ_0 with

(5.6a)
$$0 \leq \tau_0 \leq \alpha_0$$

such that, for all $\tau \geq \tau_0$ and all $\alpha \geq 0$,

(5.6b)
$$\alpha f'(\alpha + \tau) \leq f(\alpha + \tau) .$$

Proof: Let $\alpha > 0$, $\alpha_1 \geq \alpha_0$, using (1.2b) we have

$$\alpha f'(\alpha+\alpha_1) \leq \int_{\alpha_1}^{\alpha_1+\alpha} f'(s)ds = f(\alpha+\alpha_1) - f(\alpha_1) < f(\alpha+\alpha_1) .$$

Thus, there is at least one value $\tau_0 = \alpha_0$ such that (5.6a), (5.6b) hold. Choose τ_0 as the smallest such value.

Corollary 5.3.1. Let $\tau \geq \tau_0$. The function

$$G(\alpha) = \frac{f(\alpha + \tau)}{\alpha}$$

is monotone nonincreasing in α.

Corollary 5.3.2. In the special case when $f(\alpha) = f_0(\alpha)$ is given by equation (1.3),

(5.7)
$$\tau_0 = 1/4 .$$

Theorem 5.1. Suppose $\beta > 0$ and

(5.8) $$0 < \tau_0 \le \tau .$$

Then the solution of problem (1.1) is unique.

Proof: Suppose the theorem is false. Let $u_1(x)$ be the minimal solution and let $u_2(x)$ be the maximal solution. Then

(5.9a) $$0 \prec u_1 \prec u_2 .$$

Using Corollary 5.3.1, we have

(5.9b) $$\frac{f(u_1 + \tau)}{u_1} \ge \frac{f(u_2 + \tau)}{u_2}$$

and

(5.9c) $$\frac{f(u_1 + \tau)}{u_1} \not\equiv \frac{f(u_2 + \tau)}{u_2}$$

On the other hand, using (5.9a) we see that β is the lowest eigenvalue (the Perron Root or the Krein Root) of the two linear eigenvalue problems.

$$\begin{cases} L\varphi + \lambda \left[\dfrac{f(u_k(x) + \tau)}{u_k(x)}\right]\varphi = 0, \quad k = 1,2 \\ \varphi'(0) = \varphi(1) = 0 . \end{cases}$$

However, the lowest eigenvalue is monotone decreasing in the weight function (see [15], [16]) and hence β cannot simultaneously be the lowest eigenvalue of both problems.

Corollary T.5.1.1. Suppose $u_1(x)$, $u_2(x)$ are two distinct solutions of problem (1.1). Suppose they "cross." That is, there is a value $x_0 \epsilon (0,1)$ such that

$$u_1(x_0) = u_2(x_0) = \tau_1 \quad .$$

Then

$$\tau_1 < \tau_0 \quad .$$

Proof: Note that $u_1(0) \neq u_2(0)$ because the initial value problem has a unique solution. Since the argument leading to Theorem 5.1 did not make use of the specific interval [0,1], we may apply the above argument to the restricted problem

$$(5.10) \qquad \begin{aligned} L\tilde{u} + \beta\, f(\tilde{u} + \tau_1) &= 0 \\ \tilde{u}'(0) = \tilde{u}(x_0) &= 0 \end{aligned}$$

Both $u_1(x)$ and $u_2(x)$ are solutions of equations (5.10). But, this problem has a unique solution for $\tau_0 \le \tau_1$.

Corollary T.5.1.2. Suppose $u_1(x)$ and $u_2(x)$ are solutions of problem (1.1) with

$$(5.11) \qquad u_2(0) < u_1(0) \quad .$$

Suppose either:

(i) There is an upper solution $\tilde{u}(x)$ such that

$$\tilde{u}_n \downarrow u_2 \qquad \text{as } n \to \infty \quad .$$

or

(ii) there is a lower solution $\underset{\sim}{u}(x)$ such that

$$\underset{\sim}{u}_n \uparrow u_2 \qquad \text{as } n \to \infty \ .$$

Then

$$u_2 \prec u_1 \ . \tag{5.12}$$

In particular, two stable solutions cannot cross. Moreover, the result (3.12) of lemma 3.3 can be strengthened to read

$$\tilde{W} \prec u \prec \underset{\sim}{w} \ . \tag{3.12'}$$

Proof: Suppose (5.12) is false. Then $u_1(x)$ and $u_2(x)$ cross. Thus there are points $0 < x_0 < x_1 \le 1$ such that

$$u_1(x_0) = u_2(x_0), \quad u_1(x_1) = u_2(x_1)$$

and, in the interval (x_0,x_1)

$$0 \le u_1(x) < u_2(x) \le \tau_0 \ . \tag{5.13}$$

Thus, we are dealing with a convex problem. If (i) holds, $\tilde{u}(x)$ restricted to $[x_0,x_1]$ is an upper solution and thus $u_2(x)$ is the minimal solution of the restricted problem according to corollary 3.7.1 and the remark following that corollary. This contradicts (5.13). Similiarly, if (ii) holds, $\underset{\sim}{u}(x)$ restricted to $[x_0,x_1]$ is a lower solution and once more we see that $u_2(x)$ is the minimal solution.

Lemma 5.4. Let $u(x)$ be a solution of problem (1.1). Then

$$4\,u(0) - \beta f\big(u(0) + \tau\big) < 0 \quad . \tag{5.14}$$

Proof: We have the Hammerstein equation (1.4) and $u(0) \geq u(x)$. Thus

$$u(0) = \beta \int_0^1 k(0,t)\, f(u(t) + \tau)\, dt \quad .$$

Thus

$$u(0) < \beta \int_0^1 k(0,t)\, f(u(0) + \tau)\, dt \quad . \tag{5.15}$$

However, we can evaluate the right-hand-side of (5.15). Let

$$Q(x) = \beta \int_0^1 k(x,t)\, f(u(0) + \tau)\, dt \quad .$$

Then $Q(x)$ satisfies

$$\begin{cases} L\,Q + \beta f(u(0) + \tau) = 0 \,, \\ Q'(0) = Q(1) = 0 \quad . \end{cases}$$

Now, $Q(0)$ = Right-Hand-Side of (5.15). And

$$Q(x) = \beta/4\, f(u(0) + \tau)(1 - x^2)$$

which proves (5.14).

Remark: Using the notation of Section 4, if $\alpha_1(\tau, \beta, 4)$, $\alpha_2(\tau, \beta, 4)$ and $\alpha_3(\tau, \beta, 4)$ are the roots of equation (4.9.4), then lemma 5.4 can be rephrased as

$$u(0) \in (0, \alpha_1) \cup (\alpha_2, \alpha_3) \quad . \tag{5.14'}$$

This remark leads immediately to the following uniqueness theorem.

Theorem 5.2: Let

$$0 \le \beta \le \beta_*(\tau, 4) \quad . \tag{5.16}$$

Then problem (1.1) has a unique solution.

Proof: Applying lemma 4.4(ii) we see that

$$0 \le u(0) \le \alpha_1(\tau, \beta, 4) < \alpha_0 - \tau \quad .$$

Thus, both the maximal and the minimal solutions lie entirely in the convex region. However, corollary 3.7.1 implies that both are the unique minimal solution.

Corollary T.5.2.1. In the particular case where $f(\alpha) = f_0(\alpha)$ is given by equation (1.3), this result reads: Let $\tau \in [0, 1/4)$, let

$$0 \le \beta \le 2\left\{(1-2\tau) + \sqrt{1-4\tau}\right\} \exp\left[\frac{2}{1+\sqrt{1-4\tau}}\right] \quad ,$$

then the solution of problem (1.1) is unique.

Corollary T.5.2.2. Let

$$0 \le \beta \le \max\left\{\Lambda_0/\sigma_0 \ , \ \beta_*(\tau, 4)\right\} \quad .$$

Then the solution of problem (1.1) is unique.

Remark: Thus we have obtained a lower bound for the "quenching" curve.

We now turn our attention to an upper curve $\tilde{\beta}(\tau)$ with the property that

$$\tilde{\beta}(\tau) \le \beta$$

implies that problem (1.1) has a unique solution. That is, $\tilde{\beta}(\tau)$ will be an upper bound for the explosion limit curve.

Lemma 5.5. Let Λ_0 be as in lemma 5.1. Let

(5.18) $$\beta \ge \beta^*(\tau, \Lambda_0) \quad .$$

Let $u(x)$ be a solution of problem (1.1). Then

(5.19a) $$\alpha_0 - \tau \le \alpha_3(\tau, \beta, \Lambda_0) \le u(0) \quad .$$

Moreover,

(5.19b) $$\alpha_3(\tau, \beta, \Lambda_0)\, J_0(r_0 x) \prec u(x) \quad .$$

Indeed, for any $\sigma > 0$, let $\Phi(x\,;\sigma)$ be given by

(5.20) $$\Phi(\cdot\,;\sigma) = A\big(\sigma J_0(r_0 x)\big) \quad .$$

If $0 < \sigma \le \alpha_3(\tau, \beta, \Lambda_0)$ then $\Phi(x\,;\sigma)$ is a lower solution and

(5.21) $$\Phi(\cdot\,;\sigma) \prec u \quad .$$

Proof: If $0 \le \sigma \le \alpha_3(\tau, \beta, \Lambda_0)$ then $\sigma J_0(r_0 x)$ is a lower solution because

$$L\big(\sigma J_0(r_0 x)\big) + \beta f\big(\sigma J_0(r_0 x) + \tau\big) = \beta f\big(\sigma J_0(r_0 x) + \tau\big) - \Lambda_0\, \sigma J_0(r_0 x) \ge 0 \, .$$

Suppose $u(x)$ is a solution and (5.21) is false. Then there is a $\sigma \ge 0$ such that

(5.22) $$\sigma J_0(r_0 x) \prec u \quad .$$

If (5.22) holds for all $\sigma \in [0, \alpha_3(\tau, \beta, \Lambda_0)]$ then (5.21) holds by lemma 3.2. Thus we may choose a σ_1 such that

(5.23a) $$0 < \sigma_1 \le \alpha_3(\tau, \beta, \Lambda_0)$$

(5.23b) $$\sigma_1 J_0(r_0 x) \le u(x)$$

(5.23c) $$\sigma_1 J_0(r_0 x) \not\prec u(x)$$

(5.23d) $$\sigma_1 J_0(r_0 x) \not\equiv u(x) \quad .$$

Applying lemma 3.2 again, we have

$$\sigma_1 J_0(r_0 x) \prec A\big(\sigma_1 J_0(r_0 x)\big) \prec u$$

which violates (5.23c).

Lemma 5.6. Let $\tau \in (0, \tau_0)$. Let $x_0(\beta, \tau)$ denote the unique solution of

(5.24) $$\alpha_3(\tau, \beta, \Lambda_0)\, J_0(r_0 x_0) = \alpha_0 - \tau \quad .$$

Then $x_0(\beta, \tau)$ is an increasing function of τ for fixed β. Moreover,

(5.25) $$0 < 1 - x_0(\beta, \tau) < \frac{\alpha_0 - \tau}{\alpha_3(\tau, \beta, \Lambda_0)} \quad .$$

Proof: The function

$$(\alpha_0 - \tau)/\alpha_3(\tau, \beta, \Lambda_0)$$

is a decreasing function of τ. Moreover, (5.25) follows from the concavity of $J_0(r_0 x)$ in $[0, 1]$.

Lemma 5.7. Let $\tau \in (0, \tau_0)$ and let (5.18) hold. Let $x_1(\beta, \tau)$ denote the unique solution of

(5.26a) $$\beta/4\,(1 - x_1^2) = \alpha_0 - \tau_0 \quad .$$

That is

(5.26b) $$x_1(\beta, \tau) = \left[1 - \frac{4}{\beta}\,(\alpha_0 - \tau_0)\right]^{1/2} \quad .$$

Let

$$c_0 = c_0(\tau, \beta) = \frac{\alpha_0 - \big(1 - x_0(\beta, \tau)\big)\big(\alpha_3(\tau, \beta, \Lambda_0) + \tau\big)}{x_0(\beta, \tau)} \quad , \tag{5.27a}$$

$$c_1 = c_1(\tau, \beta) = \frac{\left[\alpha_3(\tau, \beta, \Lambda_0) + \tau - \alpha_0\right]}{x_0(\beta, \tau)} \quad . \tag{5.27b}$$

Let

$$Y(x; \beta, \tau) = \begin{cases} c_0 + c_1(1-x) & ; \ 0 \le x \le x_0(\beta, \tau) \\ \alpha_0 & ; \ x_0(\beta, \tau) \le x \le x_1(\beta, \tau) \\ \frac{\beta}{4}(1-x)^2 + \tau_0 & ; \ x_1(\beta, \tau) \le x \le 1 \end{cases} \quad . \tag{5.28}$$

Let $u(x)$ be any solution of problem (1.1). Then the following inequalities hold.

$$\alpha_0 \le Y(x; \beta, \tau) < u(x) + \tau \quad , \quad 0 \le x \le x_0(\beta, \tau) \quad , \tag{5.29a}$$

$$0 \le u(x) + \tau \le Y(x; \beta, \tau) \le \alpha_0 \quad , \quad x_1(\beta, \tau) \le x \le 1 \quad , \tag{5.29b}$$

$$f'\big(Y(x; \beta, \tau)\big) \ge f'\big(u(x) + \tau\big) \quad . \tag{5.29c}$$

Moreover, if $0 < \tau_1 < \tau_2 < \tau_0$, then

$$f'\big(Y(x; \beta, \tau_2)\big) \le f'\big(Y(x; \beta, \tau_1)\big) \quad . \tag{5.30}$$

Proof: One can easily verify that $Y(x; \beta, \tau)$ is the straight line between $\alpha_3(\tau, \beta, \Lambda_0) + \tau$ and α_0 on the interval $[0, x_0(\beta, \tau)]$. Inequality (5.29a) follows from lemma 5.5 and the concavity of $J_0(r_0 x)$. Similarly on the interval $[x_1(\beta, \tau), 1]$ the function $Y(x; \beta, \tau)$ is a quadratic upper solution running from α_0 to $\tau_0 > \tau$. Thus (5.29b) follows. Moreover, (5.29c) follows from (1.2b). Finally, (5.30) follows from lemma 5.6.

Lemma 5.8. Let (5.18) hold. Let $x_0(\beta,\tau)$, $x_1(\beta,\tau)$ and $Y(x;\beta,\tau)$ be as in lemma 5.7. Let $\tau \in (0,\tau_0)$ be fixed. Suppose $f(\alpha)$ satisfies (1.2a)-(1.2c) and there are positive constants $\delta > 0$, $k_0 > 0$ such that

(5.31) $$0 \le f'(\alpha) \le \frac{k_0}{\alpha^{1+\delta}} \quad , \quad \alpha_0 \le \alpha \quad .$$

Then there is a value $\tilde{\beta}(\tau)$ such that: if

(5.32a) $$\tilde{\beta}(\tau) < \beta \quad .$$

Then

(5.32b) $$\beta \int_0^1 k(0,s)\, f'\Big(Y(s);\beta,\tau\Big)ds = I(\beta,\tau) \le 1 \quad .$$

Moreover, $\tilde{\beta}(\tau)$ can be chosen so that: if

(5.33a) $$0 < \tau_1 < \tau_2 < \tau_0$$

then

(5.33b) $$\tilde{\beta}(\tau_2) \le \tilde{\beta}(\tau_1) \quad .$$

Proof: In order to prove the first part of the lemma we need only prove (5.32b) holds for large β. Assuming this has been done, the monotone character of $\tilde{\beta}(\tau)$ follows from lemma 5.6 and inequality (5.30). We proceed to the proof of (5.32b). Let

(5.34) $$\theta = \frac{2}{2+\delta} \quad , \quad \sigma = \frac{\delta}{2+\delta} \quad .$$

Since

$$\alpha_3 = \alpha_3(\tau,\beta,\Lambda_0) \ge \frac{\beta}{\Lambda_0} f(\alpha_0)$$

we may choose β so large that

$$(5.35) \qquad \begin{cases} c_1(\tau,\beta) \geq \frac{\beta}{2\Lambda_0} f(\alpha_0) \\ \beta > 1 \\ \alpha_0 \leq c_1^{\theta} \leq \alpha_3 \quad . \end{cases}$$

Let $\bar{x} \in \left(0, x_0(\beta,\tau)\right)$ be chosen as the unique solution of

$$(5.36) \qquad Y(\bar{x};\beta,\tau) = c_1^{\theta} \quad .$$

Recall that

$$k(0,s) = -s \ln s \leq (1-s) \quad .$$

We write $I(\beta,\tau)$ in four parts. Let

$$I_1 = \beta \int_0^{\bar{x}} |s \ln s| \; f'\left(Y(s;\beta,\tau)\right) ds \quad ,$$

$$I_2 = \beta \int_{\bar{x}}^{x_0(\beta,\tau)} |s \ln s| \; f'\left(Y(s;\beta,\tau)\right) ds \quad ,$$

$$I_3 = \beta \int_{x_0(\beta,\tau)}^{x_1(\beta,\tau)} |s \ln s| \; f'\left(Y(s;\beta,\tau)\right) ds \quad ,$$

$$I_4 = \beta \int_{x_1(\beta,\tau)}^{1} |s \ln s| \; f'\left(Y(s;\beta,\tau)\right) ds \quad .$$

Then, using (5.31) and (5.36) we have

$$I_1 \leq \frac{\beta k_0}{4\left[Y(\bar{x};\beta,\tau)\right]^{1+\delta}} \leq \frac{\beta k_0}{4\, c_1^{\theta(1+\delta)}} \quad .$$

Using (5.35) we have

$$I_1 \le \frac{k_0}{4}\left[\frac{f(\alpha_0)}{2\Lambda_0}\right]^{\theta(1+\delta)} \beta^{-\sigma} \quad .$$

Thus,

$$(5.37) \qquad I_1 \le \frac{1}{4} \quad , \quad \beta \ge k_0^{1/\sigma}\left[\frac{f(\alpha_0)}{2\Lambda_0}\right]^{\frac{2(1+\delta)}{\delta}} \quad .$$

A direct integration gives

$$I_2 \le \frac{\beta}{c_1^2}\, Y(\overline{x})\;\; f\Big(Y(\overline{x};\beta,\tau)\Big) \;\le \frac{\beta}{c_1^{2-\theta}} \quad ,$$

$$I_2 \le \left[\frac{f(\alpha_0)}{2\Lambda_0}\right]^{2-\theta} \beta^{-\sigma} \quad .$$

Thus,

$$(5.38) \qquad I_2 \le \frac{1}{4} \quad , \quad \beta \ge 4^{1/\sigma}\left[\frac{f(\alpha_0)}{2\Lambda_0}\right]^{\frac{2(1+\delta)}{\delta}} \quad .$$

Moreover,

$$(5.39a) \qquad I_3 + I_4 \le \beta\, f'(\alpha_0) \int_{x_0(\beta,\tau)}^{1} (1-s)ds = \beta\,\frac{f'(\alpha_0)}{2}(1-x_0)^2 \quad .$$

Since

$$(1-x_0) \le \frac{\alpha_0}{\alpha_3} \le \frac{\alpha_0\Lambda_0}{\beta\, f(\alpha_3+\tau)}$$

we obtain

$$I_3 + I_4 \le \frac{1}{2}\,(\alpha_0)^2\;\frac{\Lambda_0^2}{[f(\alpha_3+\tau)]^2}\;\frac{f'(\alpha_0)}{\beta} \quad .$$

Thus

$$I_3 + I_4 \le \frac{1}{2} \quad , \quad \beta \ge \left[\frac{\alpha_0 \Lambda_0}{f(\alpha_3 + \tau)}\right]^2 f'(\alpha_0) \quad . \tag{5.39b}$$

Thus, the lemma is proven.

Theorem 5.3. Under the hypothesis of lemma 5.8, suppose

$$\tilde{\beta}(\tau) \le \beta \quad .$$

Then problem (1.1) has a unique solution.

Proof: Suppose there are two solutions. Let $u_1(x)$ denote the minimal solution and $u_2(x)$ any other solution. Let

$$W(x) = u_2(x) - u_1(x) \ge 0 \quad .$$

Then

$$W(x) = \beta \int_0^1 k(x, s)\, f'\big(\eta(s) + \tau\big)\, W(s)\, ds \tag{5.40}$$

where

$$u_1(s) \le \eta(s) \le u_2(s) \quad . \tag{5.41}$$

Moreover,

$$0 \le W(s) \le W(0) \quad . \tag{5.42}$$

Dividing by $W(0)$ we obtain

$$1 \le \beta \int_0^1 k(0, s)\, f'\big(\eta(s) + \tau\big)\, ds \quad . \tag{5.43}$$

Using (5.41) and lemma 5.7 together with lemma 5.8 we have

$$1 < \beta \int_0^1 k(0, s)\, f'\big(Y(s; \beta, \tau)\big)\, ds = I(\beta, \tau) \le 1 \quad .$$

Thus, the theorem is proven.

Remark: The careful reader will find many ways in which the estimates of lemma 5.8 could be refined. However, the difficulty of estimating the resulting integrals is much greater.

We now turn to the special case where $f(\alpha) = f_0(\alpha)$ is given by (1.3). In this case we have

$$(5.44) \qquad \begin{cases} \theta = 2/3 \ , \ \sigma = 1/3 \ , \ \alpha_0 = 1/2 \ , \ \tau_0 = 1/4 \\ k_0 = 1 \ , \quad f(\alpha_0) = e^{-2} \ , \quad f'(\alpha_0) = 4\, e^{-2} \end{cases}$$

$$(5.45) \qquad \beta^*(\tau, \Lambda_0) = \frac{\Lambda_0}{2} \left\{ (1 - 2\tau) - \sqrt{1 - 4\tau} \right\} \exp\left\{ \frac{2}{1 - \sqrt{1 - 4\tau}} \right\}$$

is monotone decreasing in $(0, 1/4)$ and approaches $+\infty$ as $\tau \to 0+$.

There is a value $\bar{\tau}_0$ such that

$$(5.46a) \qquad \beta^*(\bar{\tau}_0, \Lambda_0) = \Lambda_0\, e \quad .$$

Computational results give

$$\bar{\tau}_0 \doteq 0.20363219 \quad .$$

For $\tau < \bar{\tau}_0$ we consider the function

$$H(\alpha) = \Lambda \alpha - \beta^*(\tau, \Lambda_0)\, f_0(\alpha + \tau) \quad .$$

We find

$$H(1) = \Lambda_0 \left(1 - e^{\frac{\tau}{1+\tau}} \right) < 0 \quad .$$

Thus, $0 \le \tau \le \bar{\tau}_0$ and $\beta \ge \beta^*(\tau, \Lambda_0)$ implies that

$$\alpha_3(\tau, \beta, \Lambda_0) \ge 1 \quad .$$

Elementary computations verify that we may apply Theorem 5.3 in the range above. It is easy to see that I_1 and I_2 are each less than 1/8 in the range above. We now turn to (5.39a), the estimate for $I_3 + I_4$.

Since $J_0(r_0 x)$ is concave for $0 \le x \le 1$, and $x_0(\beta, \tau)$ increases with β and τ we obtain the following estimates. Let

$$\bar{x}_0 = x_0(\Lambda_0 e, 0) \quad .$$

Then, $\bar{x}_0 \le x \le 1$ we have $[\alpha_3(0, \Lambda_0 e, \Lambda_0) = 1]$

$$\frac{1-x}{2(1-\bar{x}_0)} \le J_0(r_0 x) \quad .$$

Thus, if $0 \le \tau \le \tau_0$ and (5.18) holds, then $x_0(\beta^*(\tau, \Lambda_0), \tau) > \bar{x}_0$, and

(5.46) $$1 - x_0(\beta, \tau) \le \frac{(1-\bar{x}_0)}{\alpha_3(\beta, \tau, \Lambda_0)} \quad .$$

Hence

$$I_3 + I_4 \le 2\beta \cdot e^{-2} \frac{(1-\bar{x}_0)^2}{[\alpha_3]^2} = \frac{2\beta e^{-2} \Lambda_0^2}{[\beta f(\alpha_3 + \tau)]^2} (1-\bar{x}_0)^2$$

$$I_3 + I_4 \le \frac{2\Lambda_0}{e} (1-\bar{x}_0)^2 \quad .$$

A glance at tables of $J_0(x)$ (see [1]) and elementary computations give

$$\bar{x}_0 > 0.623$$

$$(1-\bar{x}_0)^2 < (0.1422) \quad .$$

Thus, we obtain

$$I_3 + I_4 \le \frac{5.78305}{e} (0.2844) \doteq 0.60511 \quad .$$

Thus we have obtained the following estimate for the explosion limit curve.

Theorem 5.4. Let $f(\alpha) = f_0(\alpha)$ be given by (1.3). There is a value $\bar{\tau} \in (\bar{\tau}_0, 1/4)$ such that: if

$$\beta \geq \begin{cases} \beta^*(\tau, \Lambda_0) & ; \quad 0 < \tau \leq \bar{\tau} \\ \beta^*(\bar{\tau}, \Lambda_0) & ; \quad \bar{\tau} \leq \tau \leq 1/4 \end{cases} \tag{5.46}$$

then problem (1.1) has a unique solution.

Remark: The "looseness" in our estimates in lemma 5.7 is reflected in the "looseness" in the upper estimate for the explosion limit curve near $\tau = 1/4$.

We observe that, using the monotonicity of all eigenvalues of (1.4), we have the following results:

(i) If $\beta \leq \Lambda_1 f'(\alpha_0)$ then there can be no "crossing" of solutions of problem (1.1).

(ii) If $\bar{\beta} \leq \Lambda_1 f'(\alpha_0)$ and if $(\bar{\beta}, \bar{u})$ is a "bifurcation" point, then $\bar{\beta}$ is the lowest eigenvalue (the Perron root or the Krein root) of the problem

$$\begin{cases} L\varphi + \bar{\beta} f'(\bar{u} + \tau)\varphi = 0 \; ; \\ \varphi'(0) = \varphi(1) = 0 \; . \end{cases}$$

In this context Λ_1 is the second eigenvalue of the linear eigenvalue problem (5.1) and

$$\Lambda_1 = r_1^2 \doteq (5.520078)^2 \; . \tag{5.47}$$

These remarks can be applied to the special case of interest to obtain the following result.

Theorem 5.5: Let $f(\alpha) = f_0(\alpha)$ be given by (1.3). Let

$$\beta \leq 4\,\Lambda_1 e^2 \; . \tag{5.48a}$$

Then there is at most one non-trivial solution $\bar{u}(x)$ of problem (1.1) which also satisfies

(5.48b) $$0 \leq \bar{u}(x) \leq 1 \quad .$$

<u>Proof</u>: Suppose there are two solutions $u_1(x)$ and $u_2(x)$ both satisfying

$$0 \prec u_k \quad .$$

Then, by the remarks above, we can assume one is greater than the other. Assume

(5.49) $$u_1(x) \leq u_2(x) \quad .$$

But, as in the proof of lemma 3.5 we have

$$L\, u_k + \beta \left[\frac{f(u_k)}{u_k}\right] u_k = 0 \quad .$$

But using (5.49) we see that

$$\frac{f(u_2)}{u_2} \geq \frac{f(u_1)}{u_1} \quad .$$

Thus, because of the monotone decreasing nature of the lowest eigenvalue, we have a contradiction.

6. Concluding Remarks.

We have studied the general problem (1.1). Our results are consistent with the computational results of Paul Stein and Myron Stein. Indeed, in the special case where $f(\alpha) = f_0(\alpha)$ is given by (1.3) we have obtained

(I) A region R' inside the region R and inside R', there are at least three solutions of problem (1.1).

(II) A region R'' which includes the region R such that outside of R'' one has a unique solution.

Moreover, for $\tau \leq 0.20$ the upper boundary of R'' is described by

$$(6.1) \qquad \beta^*(\tau, \Lambda_0) = \frac{\Lambda_0}{2}\left[(1-2\tau) - \sqrt{1-4\tau}\right] \exp\left[\frac{2}{1-\sqrt{1-4\tau}}\right]$$

whereas the computational curve is described by

$$(6.2) \qquad \tilde{\beta}(\tau) = \frac{1}{2}\, s(\tau) \left[(1-2\tau) - \sqrt{1-4\tau}\right] \exp\left[\frac{2}{1-\sqrt{1-4\tau}}\right]$$

where $s(\tau)$ is a slowly varying monotone ~~in~~creasing function rather close to Λ_0.

The upper boundary of R' is described by

$$(6.3) \qquad \beta^*(\tau, 4) = \frac{4}{2}\left[(1-2\tau) - \sqrt{1-4\tau}\right] \exp\left[\frac{2}{1-\sqrt{1-4\tau}}\right] .$$

Unfortunately, we have not established that there are at most three solutions. Such a result would be extremely desirable. Indeed, the computational results make it very clear that such is the case.

Appendix

We turn now to the proof of Lemma 4.4.

For $\beta = 0$ then $\alpha = 0$ is the only solution of equation $(4.9.\rho)$. For $\beta > 0$ the right-hand-side of $(4.9.\rho)$ is a monotone increasing function which ranges from $\beta f(\tau)$ to β. The left-hand-side is a monotone increasing function which ranges from 0 to $+\infty$. Thus there is at least one solution $\alpha_1 = \alpha_1(\tau, \beta, \rho)$. Indeed, since $\beta f(\tau) > 0$ there is a first solution and

(A.1) $$\rho \alpha < \beta f(\alpha + \tau) \quad , \quad 0 \le \alpha < \alpha_1 \quad .$$

Hence

(A.2) $$\rho \ge \beta f'(\alpha_1 + \tau) \quad .$$

If $\alpha_1 + \tau \ge \alpha_0$ then $f'(\alpha + \tau)$ decreases as α increases beyond α_1 and α_1 is the only root of equation $(4.9.\rho)$.

Suppose $\alpha_1 + \tau < \alpha_0$. There are two cases.

Case I $$\rho = \beta f'(\alpha_1 + \tau) \quad .$$

Expanding in a Taylor series about $\alpha = \alpha_1$ we find

(A.3) $$\rho\alpha - \beta f(\alpha + \tau) = -\beta f''(\xi + \tau) \frac{(\alpha - \alpha_1)^2}{2}$$

where ξ lies between α and α_1. Since $\alpha_1 + \tau < \alpha_0$, for $0 < \alpha < \alpha_1 + \epsilon$ we have $\xi + \tau < \alpha_0$ and inequality (A.1) may be strengthened to read

(A.1′) $$\rho\alpha < \beta f(\alpha + \tau) \quad , \quad 0 \le \alpha < \alpha_1 + \epsilon , \quad \alpha \ne \alpha_1 \quad .$$

Considering the behavior as $\alpha \to \infty$ we see that there is a second root $\alpha_2 = \alpha_2(\tau, \beta, \rho)$ and we have, using (A.1′) in the interval (α_1, α_2),

(A.4) $$\rho \ge \beta f'(\alpha_2 + \tau) \quad .$$

Since there is an $\epsilon > 0$ such that

$$\rho < \beta f'(\alpha + \tau) \quad , \quad \alpha_1 < \alpha < \epsilon$$

we see that

(A.5) $$\alpha_2 + \tau > \alpha_0 \quad .$$

Thus, for $\alpha > \alpha_2$

$$\rho > \beta f'(\alpha + \tau)$$

and thus

(A.6) $$\rho\alpha > \beta f(\alpha + \tau) \quad , \quad \alpha > \alpha_2 \quad .$$

Hence there are exactly two roots and we have the situation described in (iii) of lemma 4.4.

Case II $$\rho > \beta f'(\alpha_1 + \tau) \quad .$$

In this case for $\alpha > \alpha_1$ and α near α_1 we have

(A.7) $$\rho\alpha > \beta f(\alpha + \tau) \quad , \quad \alpha_1 < \alpha < \alpha_1 + \epsilon \quad .$$

Moreover, this inequality holds until there is a second root α_2. There may not be a second root. Suppose however there is a second root $\alpha_2 = \alpha_2(\tau, \beta, \rho)$. Reasoning as above, we find

(A.8) $$\rho \le \beta f'(\alpha_2 + \tau) \quad .$$

Case II.1 $$\rho = \beta f'(\alpha_2 + \tau) \quad .$$

Expanding in a Taylor series about α_2 gives

$$\rho\alpha - \beta f(\alpha + \tau) = \frac{-(\alpha - \alpha_2)^2}{2} \beta f''(\xi + \tau) \quad .$$

Let $\alpha < \alpha_2$ then $f''(\xi + \tau) \le 0$. Thus

(A.9) $$\alpha_2 + \tau > \xi + \tau > \alpha_0 \quad .$$

Thus in all cases $\xi + \tau > \alpha_0$ and

$$\rho\alpha - \beta f(\alpha + \tau) > 0 \quad , \quad \alpha > \alpha_2 \quad .$$

Hence, in this case there are only two roots with α_2 a double root.

<u>Case II.2</u> $\qquad \rho < \beta f'(\alpha_2 + \tau) \quad .$

For $\alpha > \alpha_2$ and α near α_2 we have

$$\rho\alpha < \beta f(\alpha + \tau) \quad .$$

And, again, the behavior as $\alpha \to +\infty$ implies that there is at least one more root, $\alpha_3 = \alpha_3(\tau, \beta, \rho)$. Moreover, arguing as before, we find

$$\rho \geq \beta f'(\alpha_3 + \tau) \quad . \tag{A.10}$$

Comparison of (A.2), (A.4), and (A.10) shows that

$$\alpha_3 + \tau > \alpha_0 \quad .$$

Thus, for $\alpha > \alpha_3$

$$\rho > \beta f'(\alpha + \tau)$$

and for $\alpha > \alpha_3$

$$\rho\alpha > \beta f(\alpha + \tau) \quad . \tag{A.11}$$

Thus, equation (4.9.ρ) has exactly three roots.

The remaining statements of lemma 4.4 are easily checked from a careful study of the above arguments.

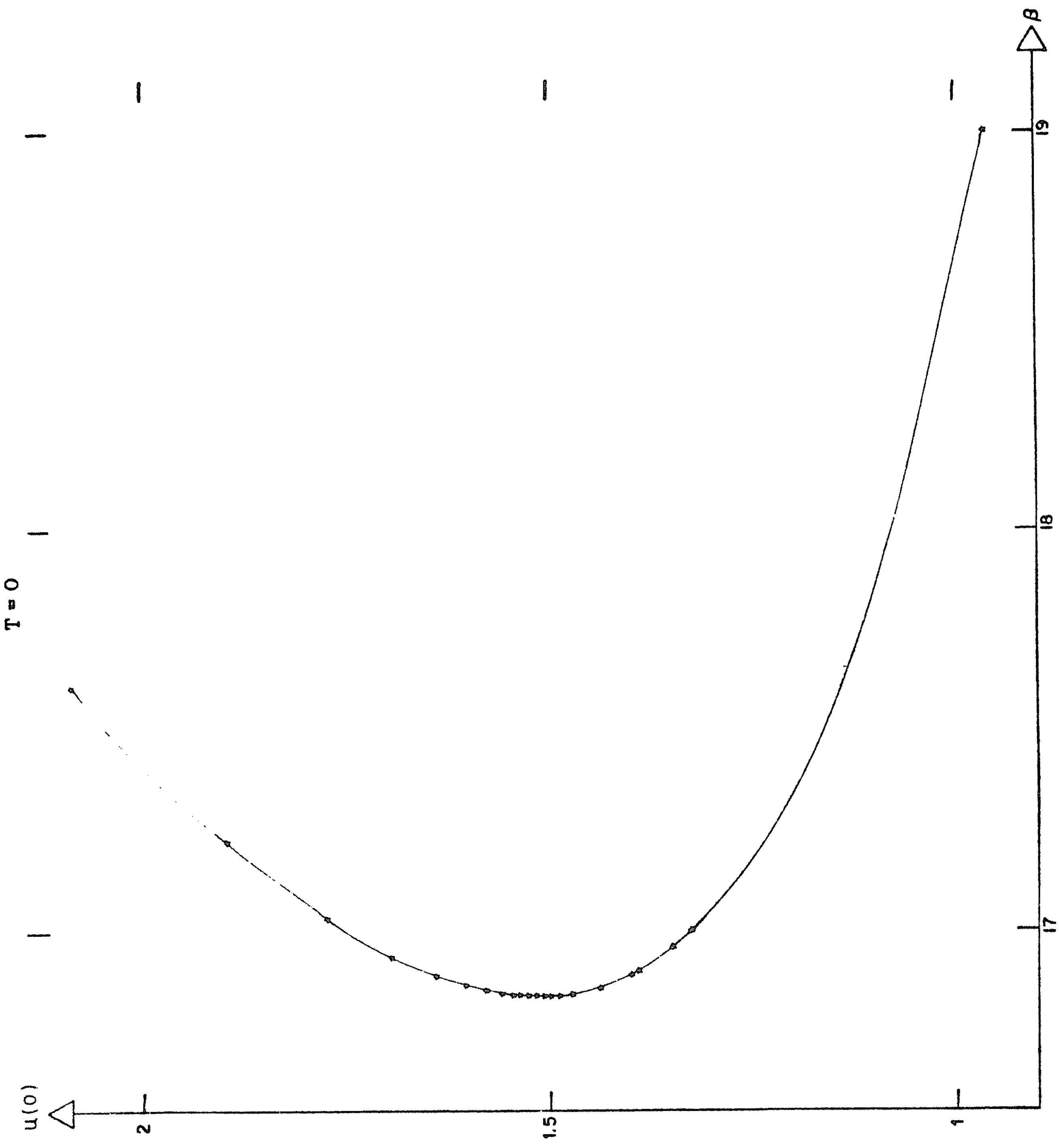

Figure 1.

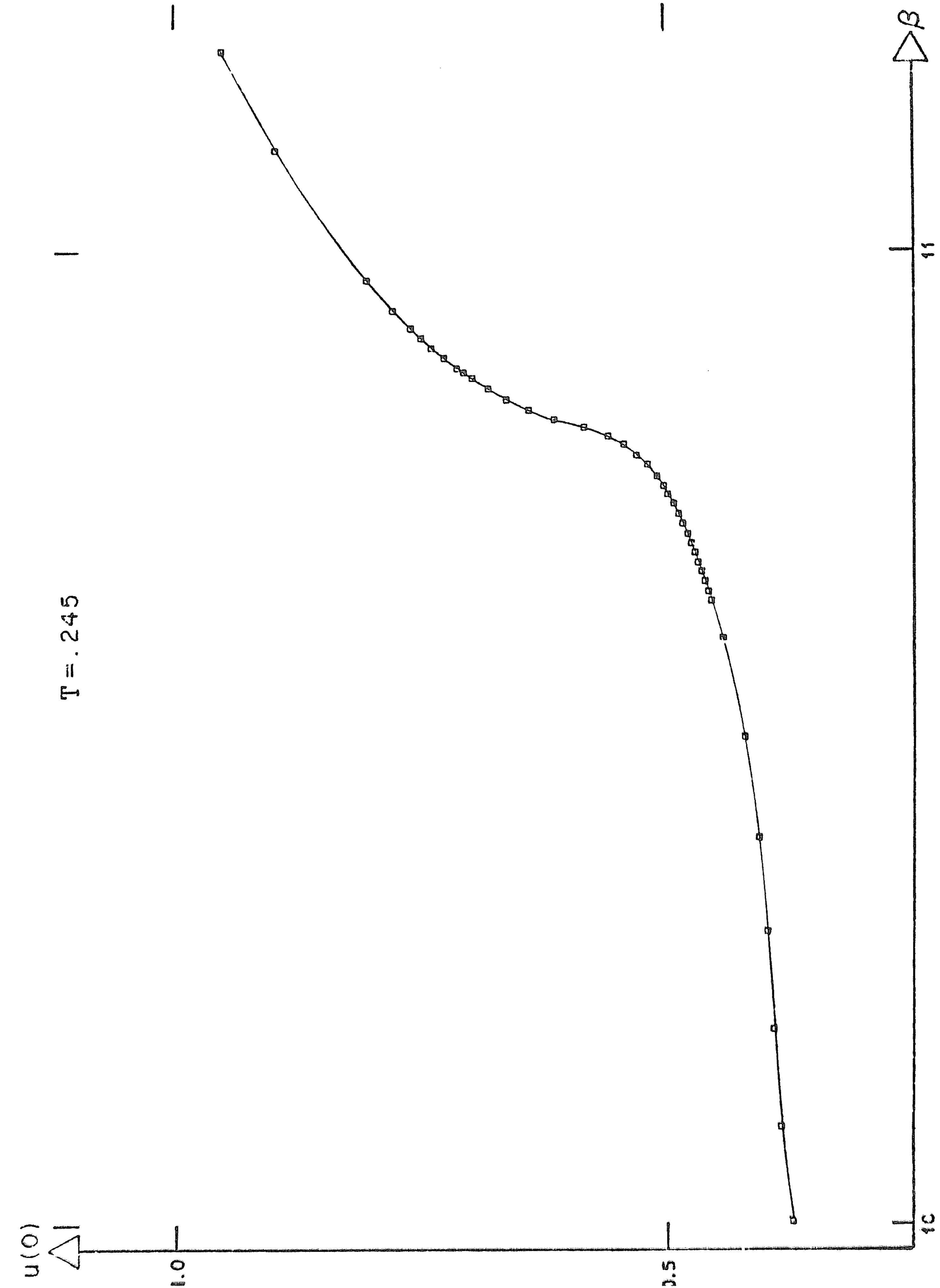

Figure 2.

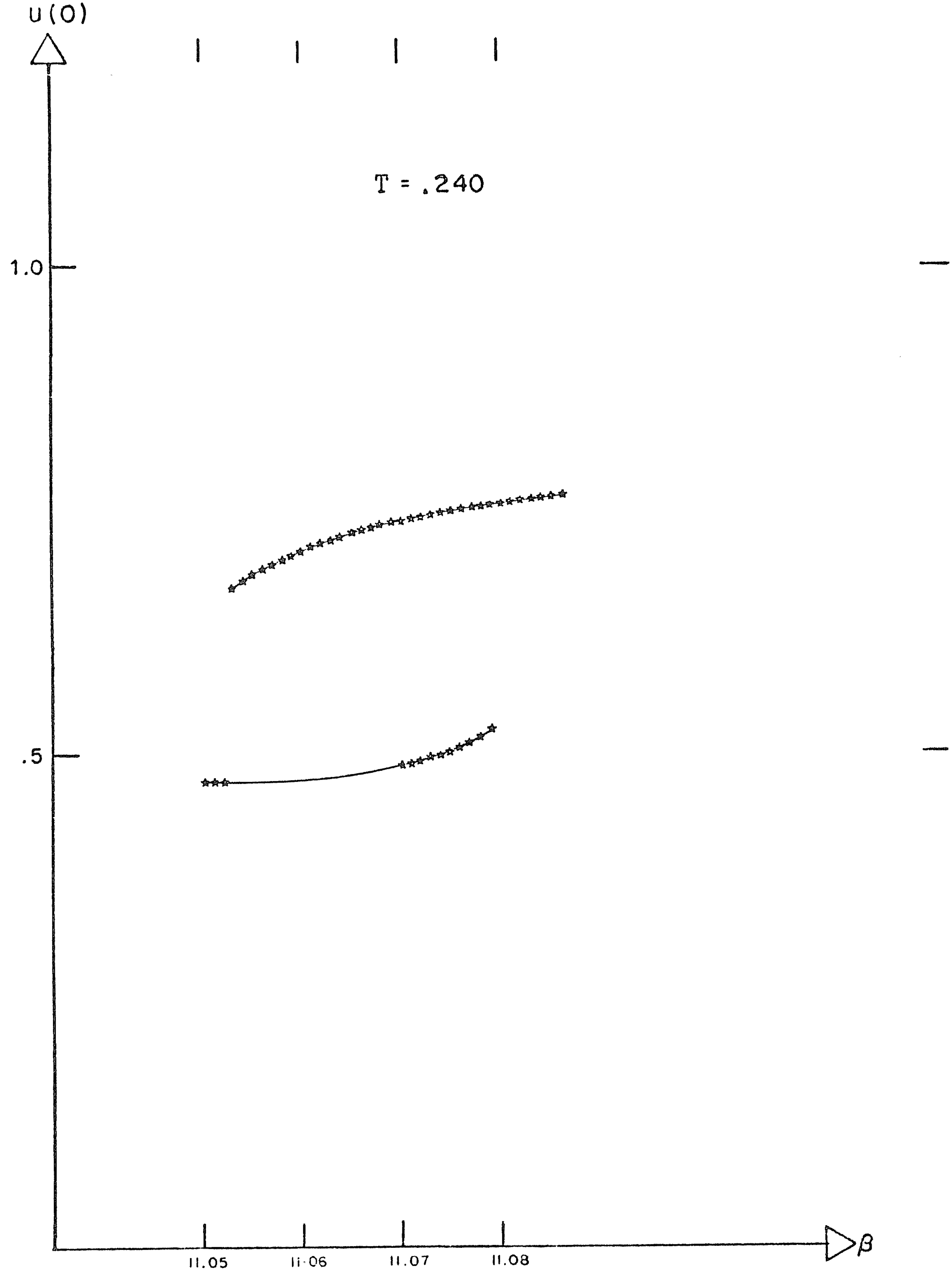

Figure 3

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