DISCRETE BARS, CONDUCTIVE HEAT TRANSFER, AND ELASTICITY

by

Donald Greenspan

Appendix: FORTRAN Program for Discrete Conductive Heat Transfer, by Sandle Turner Jones

Technical Report #164

November 1972

Received October 24, 1972
DISCRETE BARS, CONDUCTIVE HEAT TRANSFER, AND ELASTICITY

1. Introduction

Though the study of bars, or rods, is basic in structural analysis, heat transfer theory, and elasticity theory (see, e.g., refs. [1,3,4, 9-14] and the numerous references contained therein), most of the related models have been continuous and/or linear in nature. The intent of the present paper is to initiate a general computer oriented model which is discrete and nonlinear. For simplicity only, we will restrict attention to two dimensions, and, for convenience, we will describe an arithmetic, energy conserving n-body interaction model first.

2. Discrete n-Body Interaction

For positive time step $\Delta t$, let $t_k = k\Delta t$, $k = 0, 1, 2, \ldots$. At time $t_k$ let particle $P_i$ of mass $m_i$ be located at $x_{i,k} = (x_{i,k}, y_{i,k})$, have velocity $v_{i,k} = (v_{i,k,x}, v_{i,k,y})$, and have acceleration $a_{i,k} = (a_{i,k,x}, a_{i,k,y})$, for $i = 1, 2, \ldots n$. Position, velocity, and acceleration are assumed to be related by the typical, discrete formulas [5,6]:

\begin{align}
(2.1) & \quad \frac{v_{i,k+1} + v_{i,k}}{2} = \frac{x_{i,k+1} - x_{i,k}}{\Delta t} \\
(2.2) & \quad a_{i,k} = \frac{v_{i,k+1} - v_{i,k}}{\Delta t}.
\end{align}
If \( \vec{F}_{i,k} = (F_{i,k,x}, F_{i,k,y}) \) is the force acting on \( P_i \) at time \( t_k \), then force and acceleration are assumed to be related by the discrete dynamical equation

\[
(2.3) \quad \vec{F}_{i,k} = m_i \vec{a}_{1,k}.
\]

In particular, we will choose \( \vec{F}_{i,k} \) to have a component of attraction which behaves like \( \frac{p}{r^\alpha} \) and a component of repulsion which behaves like \( \frac{q}{r^\beta} \), where \( p, q, \alpha \) and \( \beta \) are non-negative parameters with \( \alpha \geq 2, \beta \geq 2 \), and where \( r \) is the distance between a given pair of particles. For this purpose, let \( r_{ij,k} \) be the distance between \( P_i \) and \( P_j \) at \( t_k \). Then \( \vec{F}_{i,k} \), the force exerted on \( P_i \) by the remaining particles, is defined by

\[
(2.4) \quad \vec{F}_{i,k} = m_i \sum_{j=1}^{n} \left\{ m_j \left( -\frac{p}{\sum_{\xi=0}^{\alpha-2} \frac{\alpha-\xi-2}{(r_{ij,k}^{\alpha-\xi}) r_{ij,k+1}^{\alpha-\xi}}} + \frac{q}{\sum_{\xi=0}^{\beta-2} \frac{\beta-\xi-2}{(r_{ij,k}^{\beta-\xi}) r_{ij,k+1}^{\beta-\xi}}} \right) \begin{pmatrix} x_{i,k+1} + x_{i,k} - x_{j,k+1} - x_{j,k} \end{pmatrix} \right\}
\]

The particular value of \( (2.4) \) lies in the observation that if one defines system work \( W \) from \( t_0 \) to \( t_N \) by
\[ W = \sum_{i=1}^{n} \sum_{k=0}^{N} \left( (\mathbf{x}_{i,k+1} - \mathbf{x}_{i,k}) \cdot \mathbf{F}_{i,k} \right), \]

system kinetic energy \( K_k \) at time \( t_k \) by

\[ K_k = \sum_{i=0}^{n} \left( \frac{1}{2} m_i (v_{i,k,x}^2 + v_{i,k,y}^2) \right), \]

and system potential energy \( V_k \) at time \( t_k \) by

\[ V_k = \sum_{i,j=1}^{n} \left\{ -\left( \frac{p}{r_{ij,k}^{\alpha-1}} + \frac{q}{r_{ij,k}^{\beta-1}} \right) m_i m_j \right\}, \]

then, as in [5] and [6],

\[ K_N + V_N = K_0 + V_0, \quad N = 0, 1, 2, \ldots, \]

which is the classical law of conservation of energy.

3. The Solid State Building Block

In modeling a solid, we will attempt to simulate contemporary physical thought [2, 8], in which molecules and atoms exhibit small vibrations within the solid. For this purpose, consider first a system of only two particles, \( P_1 \) and \( P_2 \), of equal mass, which interact according to (2.4). Assume that the force between the particles is zero. Then, from (2.4),
\[
\begin{align*}
(3.1) \quad & \frac{-p}{r_{ij,k}^{\alpha-1} - p} \sum_{\xi=0}^{\alpha-2} (r_{ij,k} r_{ij,k+1}^{\alpha-\xi-2}) + \frac{q}{r_{ij,k}^{\beta-1} - q} \sum_{\xi=0}^{\beta-2} (r_{ij,k} r_{ij,k+1}^{\beta-\xi-2}) = 0.
\end{align*}
\]

But, if there is zero force between the two particles, then \( r_{ij,k} = r_{ij,k+1} \), so set \( r_{ij,k} = r_{ij,k+1} = r \) in (3.1) to yield
\[
\begin{align*}
(3.2) \quad & \frac{-p}{r^{2\alpha-2}} \sum_{\xi=0}^{\alpha-2} r^{\alpha-2} + \frac{q}{r^{2\beta-2}} \sum_{\xi=0}^{\beta-2} r^{\beta-2} = 0.
\end{align*}
\]

Thus, for \( \beta \geq \alpha \),
\[
- pr^{\alpha-1} + qr^{\beta-1} = 0,
\]
or, finally,
\[
(3.3) \quad r^{\beta-\alpha} = \sqrt{\frac{q(\beta-1)}{p(\alpha-1)}}.
\]

Consider next a system of only three particles, \( P_1, P_2 \) and \( P_3 \), of equal masses, whose mutual distances apart are given by (3.3). Since no force acts between any two of the particles, it follows that there is no force acting upon any one of the three. Such a configuration of particles is therefore exceptionally stable and will be called a triangular building block.
When considering a solid we will decompose it into triangular building blocks. In this fashion, the force on any particular particle due to its nearby neighbors will be zero. By an appropriate choice of parameters, the force on any particle due to more distant particles will be made small, thus achieving the small vibrations desired.

To illustrate, let the six particles $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ be located at the vertices of the four triangular building blocks of the triangular region OAB, shown in Figure 3.1. Assume that $m_i = 1$, $p = q = 1$, $\alpha = 7$, and $\beta = 10$, so that $r = \frac{3}{\sqrt{15}}$. The particles' initial positions are, then,
\[ P_1: (1.14471, 1.98270) \]
\[ P_2: (0.57236, 0.99135) \]
\[ P_3: (1.71707, 0.99135) \]
\[ P_4: (0, 0) \]
\[ P_5: (1.14471, 0) \]
\[ P_6: (2.28943, 0) \]

Assign to each particle a initial velocity. Finally, let particles \( P_4 \) and \( P_6 \) be fixed and allow the remaining particles to move under force law (2.4). For \( \Delta t = 0.05 \) and for 2500 time steps, the motions of \( P_1, P_2, P_3 \) and \( P_5 \) were generated from (2.1)-(2.4). \( P_1 \) and \( P_5 \) exhibited small oscillations in the vertical directly only, while \( P_2 \) and \( P_3 \) exhibited small two dimensional oscillations. The maximum distance, for example, that \( P_1 \) moved from its initial position was approximately 0.02, and this occurred at approximately every one hundred time steps. The running time on the UNIVAC 1108 was 4 minutes. The basic computer program used, which is also typical of all examples which follow, was that of Jones [7].

Note that the magnitudes of the oscillations described above can be controlled completely by the appropriate choices of \( p, q, \alpha, \) and \( \beta \).

4. Flow of Heat in a Bar

Let us now develop the basic concepts of discrete conductive heat
transfer by concentrating on the prototype problem of heat flow in a bar. Physically, the problem is formulated as follows. Let the region bounded by rectangle $0ABC$, as shown in Figure 3.1, represent a bar. Let $|0A| = a$, $|0C| = c$. A section of the boundary of the bar is heated. The problem is to describe the flow of heat through the bar.

Our discrete approach to the problem proceeds as follows. First, subdivide the given region into triangular building blocks, one such possible subdivision of which is shown in Figure 3.2 for the parameter choices $m_1 = 1$, $p = q = 1$, $\alpha = 7$, $\beta = 10$, $a \sim 11$, $c \sim 2$. Note that from (3.3), $r \sim 1.1447142426$.

Now, by heating a section of the boundary of the bar, we will mean increasing the velocity, and hence the potential energy, of some of the particles whose centers are on $0ABC$. By the temperature $T_{i,k}$ of particle $P_i$ at time $t_k$, we will mean the following. Let $M$ be a fixed positive integer and let $K_{i,k}$ be the kinetic energy of $P_i$ at $t_k$. Then $T_{i,k}$ is defined by

$$T_{i,k} = \frac{1}{M} \sum_{j=k-M+1}^{k} K_{i,j'}$$

which is, of course, the arithmetic mean of $P_i$'s kinetic energies at $M$ consecutive time steps. By the flow of heat through the bar we will mean the transfer to other particles of the bar of the kinetic energy added at the boundary. Finally, to follow the flow of heat through the
bar one need only follow the motion of each particle and, at each time step, record its temperature.

To illustrate, consider the bar shown in Figure 3.2 with the parameter choices given above, that is, \( m_1 = 1, p = q = 1, \alpha = 7, \beta = 10, \) \( a \sim 11, c \sim 2. \) Assume that a strong heat source is placed above \( P_6, \) and then removed, in such a fashion that \( \vec{v}_{5,0} = (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}), \) \( \vec{v}_{6,0} = (0, -1), \) \( \vec{v}_{7,0} = (\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}), \) while all other initial velocities are \( \vec{0}. \) With regard to temperature calculation, assume that the velocities of all particles prior to \( t_0 \) were \( \vec{0}. \) As regards the choice of \( M, \) which is a difficult choice to make, one would usually wish to choose it relatively large, since the use of an average is, generally, more meaningful when the number of quantities being averaged is relatively large. We shall arbitrarily set \( M = 20. \) From the resulting calculations with \( \Delta t = 0.025, \) Figures 3.3 - 3.7 show the constant temperature contours \( T = 0.1, 0.06, 0.025, 0.002 \) at \( t_5, t_{10}, t_{15}, t_{20}, t_{25}, \) respectively. The resulting wave motion is clear and Figure 3.7 exhibits wave reflection. It is interesting, also, to note that the temperature at \( P_6 \) increases, until \( t_{20}, \) at which time it is a maximum, and only then does it proceed to decrease. Figures 3.8 - 3.12 show the constant kinetic energy contours \( K = 0.1, 0.05, 0.01, 0.001 \) at each of the times \( t_5, t_{10}, t_{15}, t_{20}, t_{25}, \) respectively, and indicate the magnitude of the particle velocities at these time steps.
Other heat transfer concepts can be defined now in the same spirit as above, as follows. A side of the bar is **insulated** means that the bar particles cannot transfer energy across this side of the bar to particles outside the bar, while **melting** is the result of adding a sufficient quantity of heat so that various particle velocities attain sufficient magnitude so as to break the bonding effect of (2, 4).
5. Oscillation of an Elastic Bar

Next, let us develop the basic concepts of discrete elasticity by concentrating on the vibration of an elastic bar. The problem is formulated physically as follows. Let the region bounded by rectangle 0ABC, as shown in Figure 3.1, represent a bar which can be deformed, and which, after deformation, tends to return to its original shape. The problem is to describe the motion of such a bar after the external force, which has deformed the bar, is removed. Equivalently, the problem is to describe the motion of an elastic bar after release from a position of tension.

Our discrete approach proceeds as follows. The given region is first subdivided into triangular building blocks. Then, deformation results in the compression of certain particles and the stretching apart of others. Release from a position of deformation, or tension, results, by (2.4), in repulsion between each pair of particles which have been compressed and attraction between each pair which have been stretched, the net effect being the motion of the bar.

As a particular example, let \( m_1 = 1, \alpha = 7, \beta = 10, p = 425, q = 1000, \) and \( \Delta t = .025. \) From (3.3), \( r = 1.52254. \) Consider, for variety, the thirty particle bar which results by deleting \( P_{11} \) and \( P_{32} \) from the configuration of Figure 3.2. The particles \( P_1, P_{12}, \) and \( P_{22}, \) whose respective coordinates are \( (0, 2.63711), (0.75127, 1.31855), \) and \( (0, 0), \)
are to be held fixed throughout. In order to obtain an initial position of tension like that shown in Figure 4.1a, first set \( P_{13}', P_{14}', P_{15}', P_{16}', P_{17}', P_{18}', P_{19}', P_{20} \) and \( P_{21} \) at \((2.28357, 1.29198), (3.80588, 1.26541), (5.32632, 1.18573), (6.84052, 1.02658), (8.33992, .76219), (9.81058, .36813), (11.23199, -.17750), (12.57631, -.89228), \) and \((13.80807, -1.78721)\), respectively. Any two consecutive points \( P_k, P_{k+1}, k = 13, 14, \ldots, 20, \) are positioned \( r \) units apart. The points \( P_2 - P_{10} \) and \( P_{23} - P_{31} \) are then positioned as follows: \( P_{k-10} \) and \( P_{k+11} \) are the two points which are \( r \) units from both \( P_k \) and \( P_{k+1} \) for each of \( k = 12, 13, \ldots, 20. \) Each consecutive pair of points in the \( P_2 - P_{10} \) set is then separated by a distance greater than \( r, \) while each consecutive pair of points in the \( P_{23} - P_{31} \) set is separated by a distance less than \( r. \) Thus, the points \( P_2 - P_{10} \) are in a stretched position, while the points \( P_{23} - P_{31} \) are compressed.

From the initial position of tension shown in Figure 4.1a, the oscillatory motion of the bar is determined from (2.1)-(2.4) with all initial velocities set as \( \vec{0}. \) The upward swing of the bar was plotted automatically at every ten time steps and is shown in Figure 4.1a-w from \( t_0 \) to \( t_{220}. \) It is of interest to note that as the bar moves, each row of particles exhibits wave oscillation and reflection.
Fig. 4.1

(a) $t_0$

(b) $t_{10}$

(c) $t_{20}$

(d) $t_{30}$
Fig. 4.1 continued

(e) \( t_{40} \)

(f) \( t_{50} \)

(g) \( t_{60} \)

(h) \( t_{70} \)
Fig. 4.1 cont.

(i) \( t_{30} \)

(j) \( t_{90} \)

(k) \( t_{100} \)

(l) \( t_{110} \)
Fig. 4.1 cont.

(m) $t_{120}$

(n) $t_{130}$

(o) $t_{140}$

(p) $t_{150}$
Fig. 4.1 cont.

(q) $t_{160}$

(r) $t_{170}$

(s) $t_{180}$

(t) $t_{190}$
Fig. 4.1 completed.

(u) $t_{200}$

(v) $t_{210}$

(w) $t_{220}$
6. Remarks

A limited number of other examples were run, and these indicated that square building blocks were less stable than triangular ones, while the choices \( \alpha = 2, \beta = 5 \) and \( \alpha = 7, \beta = 13 \) were less viable than \( \alpha = 7, \beta = 10 \). Generally speaking, any choice \( p > q \) resulted in increased oscillations so that, for example, for the elastic bar model of Section 4, the choice \( p = 3, q = 1 \) required a refinement of time step to \( \Delta t = 10^{-3} \) in order to study the resulting oscillations. The major handicap in all the computer examples run was the lack of adequate funding to enable the study of models with large numbers of particles.

Finally, it should be noted that the writer feels that varying \( \alpha, \beta, p \) and \( q \) in computer models with large numbers of particles will enable the researcher to produce viable computer models and to derive insight into the actual parameter values for various physical solids.
References


APPENDIX: FORTRAN Program for Discrete Conductive Heat Transfer
by S. T. Jones

FORTRAN PROGRAM FOR DISCRETE CONDUCTIVE HEAT TRANSFER

INDEX TO PROGRAM VARIABLES

A = ALFA
ANGLF(T) = ANGLE (IN DEGREES) OF INITIAL VELOCITY VECTOR WITH
RFSPECT TO POSITIVE X-AXIS
B = BETA
DB(T) = DISTANCE OF PARTICLE T FROM ITS INITIAL POSITION
DMAX = MAXIMUM ALLOWABLE DISTANCE OF ANY PARTICLE FROM ITS INITIAL
POSITION
DT = TIME INCREMENT
EPS = CONVERGENCE CRITERION FOR NEWTON'S METHOD
FX(T) = FORCE COMPONENT ON PARTICLE T IN X-DIRECTION
FY(T) = FORCE COMPONENT ON PARTICLE T IN Y-DIRECTION
TXTS(T) = 0 IF LEFTMOST PARTICLE IN ROW T IS ON Y-AXIS
       = 1 IF LEFTMOST PARTICLE TO BE SHIFTED RIGHT OR LEFT,
       RESPECTIVELY
IEND = 0 IF ANOTHER DATA CASE FOLLOWS
       = 1 IF END OF RUN
IMAX = MAXIMUM NUMBER OF ITERATIONS PER TIMESTEP FOR NEWTON'S
METHOD
ISPRT = PRINT-STEP INCREMENT
IFUNCH = PUNCH-STEP INCREMENT
TRGW(T) = NUMBER OF PARTICLES IN ROW T
ISTART = 0 IF NEW DATA CASE
       = 1 IF RESTART
IVEL(I) = NUMBER OF PARTICLE TO BE GIVEN AN INITIAL VELOCITY
JFUNCH = 0 IF NO PUNCH REQUIRED, PUNCH OTHERWISE
MT(I) = MASS OF PARTICLE I
N = NUMBER OF PARTICLES IN SYSTEM
MAXTS = NUMBER OF ROW ON X-AXIS
NFIX = TOTAL NUMBER OF PARTICLES TO BE FIXED
NMAX = MAXIMUM NUMBER OF TIMESTEPS THIS DATA CASE
NFIX(I) = NUMBER OF PARTICLE TO BE FIXED
NROW = NUMBER OF ROWS IN SYSTEM
NSTEP = TIMESTEP NUMBER
NVEL = TOTAL NUMBER OF PARTICLES TO BE GIVEN AN INITIAL VELOCITY
OMEGA = SUCCESSIVE OVER-RELAXATION FACTOR FOR NEWTON'S METHOD
P = ATTRACTION PARAMETER
Q = REPULSION PARAMETER
RI(T,J+1) = DISTANCE BETWEEN PARTICLES T AND J AT PREVIOUS TIMESTEP
RJ(T,J+1) = DISTANCE BETWEEN PARTICLES I AND J AT CURRENT TIMESTEP
SK(T) = SUM OF KINETIC ENERGIES FOR PARTICLE T OVER ALL TIMESTEPS
TEMP(T) = MEASURE OF TEMPERATURE OF PARTICLE T AT CURRENT TIMESTEP
VEL(T) = MAGNITUDE OF INITIAL VELOCITY VECTOR
VX(T,I) = X-COMPONENT OF VELOCITY OF PARTICLE I, PREVIOUS TIMESTEP
VX(T,J) = SAME AS ABOVE, CURRENT TIMESTEP, PREVIOUS ITERATION
VX(T,J) = SAME AS ABOVE, CURRENT TIMESTEP, CURRENT ITERATION
VX(T,J) = X-COMPONENT OF INITIAL VELOCITY, PARTICLE I
VY(T,I) = Y-COMPONENT OF VELOCITY OF PARTICLE I,
VY(T,J) = SAME DEFINITIONS
VY(T,J) = AS VX(T,J), ABOVE
VY(T,J) = Y-COMPONENT OF INITIAL VELOCITY, PARTICLE I
C \( x(i+1) = x\text{-component of position of particle i} \)
C \( x(t+2) = \) same definitions
C \( x(t+3) = \) \( as \ v_x(i,j) \)
C \( x_l(i) = x\text{-component of initial position, particle i} \)
C \( x_kf(i) = \) kinetic energy of particle i at each timestep
C \( y(i+1) = y\text{-component of position of particle i} \)
C \( y(t+2) = \) same definitions
C \( y(t+3) = \) \( as \ v_y(i,j) \)
C \( y_l(i) = \) y-component of initial position, particle i

IMPLICIT DOUBLE PRECISION (A-H, I-M, O-Z)
DIMENSION XO(100), YO(100), VXO(100), VYO(100), X(100, 3), Y(100, 3),
       VX(100, 3), VY(100, 3), FX(100), FY(100), DD(100), XK(100), NP(100),
       G(100, 2), N(100)
DIMENSION IRW(15), TAXTS(5), TVF(5), VFL(5), ANGLE(5)
DIMENSION SKF(40), TEMP(50)

10,1 FORMATTED(8D14.6)
10,2 FORMATTED(16TS)
10,3 FORMATTED(4D10.6, 'E')
10,4 FORMATTED(5E19.10)
70,0 FORMATTED(1H1)
70,1 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,2 FORMATTED(5E19.6, 'F', 100, 'E', 10, 19)
70,3 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,4 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,5 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,6 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,7 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,8 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,9 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,10 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,11 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,12 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,13 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,14 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,15 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,16 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,17 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,18 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,19 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,20 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,21 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,22 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,23 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,24 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,25 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,26 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,27 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,28 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,29 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,30 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,31 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,32 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,33 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,34 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,35 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,36 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,37 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,38 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,39 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,40 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,41 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,42 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,43 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,44 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,45 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,46 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,47 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,48 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,49 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)
70,50 FORMATTED(5E19.6, 'DMECA', 'F', 100, 'E', 10, 19)

C PRINT 20000
A = 0.0
READ 10001, OM, ECA, FFS, PFS, DMAX, DT
READ 10002, NMAX, IMAX, IPRINT, TRUNCH, JPUNCH, TSTART
READ 10003, NRU, NFX, NAXTS, NVEL
READ 10004, A, B, F, S, TEND
PRINT 20, 1, N
XKF(I) = 0.0
DO 20 I = 1, N
   XKF(I) = 0.0
   TEMP(I) = 0.0
20 CONTINUE
IF (TSTART .LT. 1) GO TO 1
C PFSTART
READ 10002, NSTEP
READ 2E9, (x(1:100), i = 1, N)
READ 2E9, (y(1:100), i = 1, N)
READ 2E9, (x_kf(1:100), i = 1, N)
GO TO 6
C NEW CASE -- CALCULATE POSITIONS
1 BASEX = (I + (A - 1) / (B - 4)) * (1, 0.5 * BASEX) * 2
BASEY = SQRT(BASEX ** 2 - (0.5 * BASEX) ** 2)
READ 10002, (IROW(I), IAXIS(I), I = 1, NROW)
% EQUATIONS OF MOTION
TL=1
IU=1
NO 3 T=1*NRGW
IU=IU+TRDW(I)
XSHIFT=I*ES*TA*X(T)*BASEX
YSHIFT=WAXIS-I
DO 2 J=IL+IU
X(U(J))=(J-IL)*BASEX*XSHIFT
Y(U(J))=YSHIFT*BASEY
2 CONTINUE
IL=IU+1
3 CONTINUE

C NEW CASE -- CALCULATE VELOCITIES
10 DO 4 I=1*N
VXU(I)=0.0
VYU(I)=0.0
4 CONTINUE
IF(NVEL*F*E.1) GO TO 6
READ 1004* (VFL(I),VFL(I)+ANGLE(I)+I=1*NFL)
F1=3.14159265358979324D+00
RAD=FT/180.0
DO 5 T=1*NFL
THFTA=ANGLE(T)*RAD
VX(U(J))=VFL(T)+COS(THFTA)
VYU(J)=VFL(T)+SIN(THFTA)
5 CONTINUE
6 DO 7 T=1*N
X(T)=1.0
7 CONTINUE
OMEGA=1.0
TFK(NFIX*FL=0) GO TO 11
READ 1002* (NFL(I),I=1*NFL)
11 PRINT 2002*A*P*F*U*DT
T=0.0
DTZ=DT/2.0
PRINT 2003*NSF
DO 20 I=1*N
PRINT 2004*M(T)*X0(T)*Y0(T)*DD(T)*XXFT(T)
20 CONTINUE
C SPECIFY INITIAL GUESS FOR NEWTON'S ITERATION AT FIRST TIMESTEP
DO 40 T=1,N
X(T,0)=X0(T)
VX(T,0)=VX0(T)
Y(T,0)=Y0(T)
40 VX(T,3)=VY0(T)
CALL RCALC

C UPDATE POSITIONS, VELOCITIES, DISTANCES FOR ALL TIMESTEPS
45 NSF=NSF+1
T=T+DT
DO EU T=1,N
X(T+1)=X(T,3)
VX(T,1)=VX(T,3)
Y(T+1)=Y(T,3)


```plaintext
VY(I,1)=VY(I,3)
DO 50 J=1,N
50 RT(I,J,1)=R(I,J,2)
GO TO CONTINUE
C BEGIN ITERATION LOOP
DO 90 K=1,TMAX
C UPDATE ALL VARIABLES, CURRENT TIMESTEP, PREVIOUS ITERATION
DO 70 T=1,N
X(I,T+2)=X(I,T)
VX(I,T+2)=VX(I,T)
Y(I,T+2)=Y(I,T)
VY(I,T+2)=VY(I,T)
70 CONTINUE
C UPDATE POSITIONS, CURRENT TIMESTEP, CURRENT ITERATION
DO 73 T=1,N
IF(NF(T,N,F,0)) GO TO 72
DO 71 J=1,NFX
IF(I,F,F,J) GO TO 72
71 CONTINUE
72 X(I,T+1)=OM*X(I,T)+OMFAR(DT+VX(I,T)+VX(I,T+1)+Y(I,T+1))
Y(I,T+1)=OM*Y(I,T)+OMFAR(DT+VY(I,T)+VY(I,T+1)+Y(I,T+1))
72 CONTINUE
CALL RECALC
CALL FCALC
C UPDATE VELOCITIES, CURRENT TIMESTEP, CURRENT ITERATION
DO 80 I=1,N
IF(NF(I,N,F,0)) GO TO 78
DO 74 J=1,NFX
IF(I,F,F,J) GO TO 80
74 CONTINUE
75 VX(I,T+1)=OM*VX(I,T)+OMFAR(DT*FX(I)+VX(I,T+1))
VY(I,T+1)=OM*VY(I,T)+OMFAR(DT*FY(I)+VY(I,T+1))
80 CONTINUE
C TEST FOR CONVERGENCE
DO 85 I=1,N
IF(ABS(VX(I,T+2)-VX(I,T+1)) GT FPS) GO TO 85
IF(ABS(VY(I,T+2)-VY(I,T+1)) GT FPS) GO TO 85
IF(ABS(VX(I,T+2)-VX(I,T+1)) GT FPS) GO TO 85
IF(ABS(VY(I,T+2)-VY(I,T+1)) GT FPS) GO TO 85
85 CONTINUE
GO TO 95
C CONTINUE
C PRINT 200N+K,NSTF
GO TO 110
95 CALL DTCALC
DO 100 I=1,N
IF(DD(I) GT DMAX) GO TO 105
100 CONTINUE
STEPS=NSTF
DO 700 TT=1,N
XKFF(TT)=E*A*(TT)*(VX(TT,T)+VX(TT,T+2)+VY(TT,T)+VY(TT,T+2))
SKF(TT)=SKF(TT)+XKFF(TT)
TEMP(TT)=SKF(TT)/STEPS
700 CONTINUE
```
IF (MOD(NSTP,100) .EQ. 0) GO TO 103
IF (J*PUNCH .EQ. 0) GO TO 102
IF (MOD(NSTP,1FUNCH) .EQ. 0) GO TO 102
WRITE(1,1002) NSTP
DO 151 I=1,N
WRITE(1,2501)X(I,3),Y(I,3)
151 CONTINUE
DO 152 I=1,N
WRITE(1,2501)X(I,7),Y(I,3)
152 CONTINUE
WRITE(1,2501)SKF(I),II=1,N)
102 CALL OUTF
103 IF (NSTP .EQ. NMAX) GO TO 110
GO TO 10
105 FRINT Z(i) .EQ. NMAX
CALL OUTF
110 IF (I(ENDD .EQ. 0) GO TO 10
STOP
C INTERNAL SUBROUTINE TO COMPUTE DISTANCES BETWEEN PARTICLES
SUBROUTINE RCALC
DO 210 T=I.N
IF I=I+1
DO 200 J=I.N
R(IJ,XZ)=(X(I,3)-X(J,3))**2+(Y(I,3)-Y(J,3))**2
R(IJ+2)=R(IJ+2)
200 CONTINUE
210 CONTINUE
RETURN
C INTERNAL SUBROUTINE TO COMPUTE DISTANCE OF PARTICLE FROM ITS INITIAL POSITION
SUBROUTINE OCALC
DO 300 T=I.N
DO(I)=SGRT((X(I,3)-X(I))**2+(Y(I,3)-Y(I))**2)
300 CONTINUE
RETURN
C INTERNAL SUBROUTINE TO COMPUTE FORCES
SUBROUTINE FCALC
TA=A-1
TA=B-1
DO 400 I=1,N
IF (NFTX .EQ. 0) GO TO 450
DO 400 K=1,NFTX
IF (I .EQ. NP(KK)) GO TO 450
400 CONTINUE
450 SUMX=0.O
SUMP=0.O
DO 460 J=1,N
460 SUMY=0.O
IF (J .EQ. JJ) GO TO 550
SUMP=0.O
SUMP=SUMP+(R(IJJ,1)**2*(I2-1)**2+(R(IJJ,2)**2)*A-(I2-1)**2))
550 CONTINUE
DO 501 IZ=1,TP
SUMG=SUMG+(R(II, JJ+1)**(T2-1))*(R(II, JJ+2)**(A-(T2-1)-2))
501 CONTINUE
PD=R(II, JJ+1)**(A-1)*R(II, JJ+2)**(A-1)*RTJ
SUMF=F*SUMF/PD
UD=R(II, JJ+1)**(B-1)*R(II, JJ+2)**(B-1)*RTJ
SUMG=G*SUMG/UD
SUMX=(SUMD-SUMF)*M(JJ)*(X(II, ZZ)+X(II, 1)-X(JJ, 0)-X(JJ, 1))*SUMX
SUMY=(SUMD-SUMF)*M(JJ)*(Y(II, ZZ)+Y(II, 1)-Y(JJ, 0)-Y(JJ, 1))*SUMY
500 CONTINUE
FX(II)=SUMX
FY(II)=SUMY
600 CONTINUE
RETURN
C INTERNAL PRINT SUBROUTINE
SUBROUTINE OUTF
3001 FORMAT(2X,6F15.10)
3002 FORMAT(1X,6F15.10)
PRINT 3001,NSTF
DO 800 II=1,N
PRINT 3002,X(II, ZZ),Y(II, ZZ),C0(II),XKF(II),SKF(IT),TEMP(II)
800 CONTINUE
RETURN
END
**Title and Subtitle**

Discrete Bars, Conductive Heat Transfer, and Elasticity

**Author(s)**

Donald Greenspan

**Performing Organization Name and Address**

Computer Sciences Department  
The University of Wisconsin  
1210 West Dayton Street  
Madison, Wisconsin 53706

**Abstract**

Heat transfer and elasticity are modelled from a discrete point of view. Computer examples using bars are given to show the feasibility of the models. The dynamical equations are discrete and energy conserving.

**Key Words and Document Analysis. 17a. Descriptors**

Elasticity, heat transfer, numerical analysis.

**Availability Statement**

Available to the public.

**Security Class (This Report)**

UNCLASSIFIED

<table>
<thead>
<tr>
<th>Security Class (This Page)</th>
<th>No. of Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNCLASSIFIED</td>
<td>30</td>
</tr>
</tbody>
</table>

**Price**

UNCLASSIFIED