Computer Sciences Department
1210 West Dayton Street
Madison, Wisconsin 53706

THE SOLUTION OF THE DIRICHLET PROBLEM FOR LAPLACE'S EQUATION WHEN THE BOUNDARY DATA IS DISCONTINUOUS AND THE DOMAIN HAS A BOUNDARY WHICH IS OF BOUNDED ROTATION BY MEANS OF THE LEBESGUE-STIELTJES INTEGRAL EQUATION FOR THE DOUBLE LAYER POTENTIAL.

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Colin W. Cryer

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1. Introduction.

Let C be a rectifiable Jordan curve in the xy-plane which separates the plane into a bounded simple-connected domain $\mathcal{R} = \mathcal{R}_+$ and an unbounded simply-connected domain $\mathcal{R}_-$ (see Figure 1.1).

Let C have the parametric representation

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq S, \tag{1.1}$$

where S is the length of C and s denotes arc length measured in the anti-clockwise direction from a fixed point A.

The point $(x(\sigma), y(\sigma)) \in C$ will be denoted by $P(\sigma)$, and the length of the shorter of the two arcs joining $P(s)$ and $P(\sigma)$ will be denoted by $\|s - \sigma\|$ so that

$$\|s - \sigma\| = \min_{k=-1,0,1} |s - \sigma + kS| .$$

If $\|s - \sigma\| < S/2$, we write $P(s) < P(\sigma)$ if $P(s)$ is "before" or "to the left" of $P(\sigma)$ and $P(s) > P(\sigma)$ if $P(s)$ is "after" or "to the right" of $P(\sigma)$. The distance between $P(s)$ and $P(\sigma)$ is denoted by $|P(s) - P(\sigma)|$.

Let $g = g(s)$ be defined on C and let $u = u(x, y)$ be the solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathcal{R}, \tag{1.2}$$

$$u = g, \quad (x, y) \in C. \tag{1.3}$$

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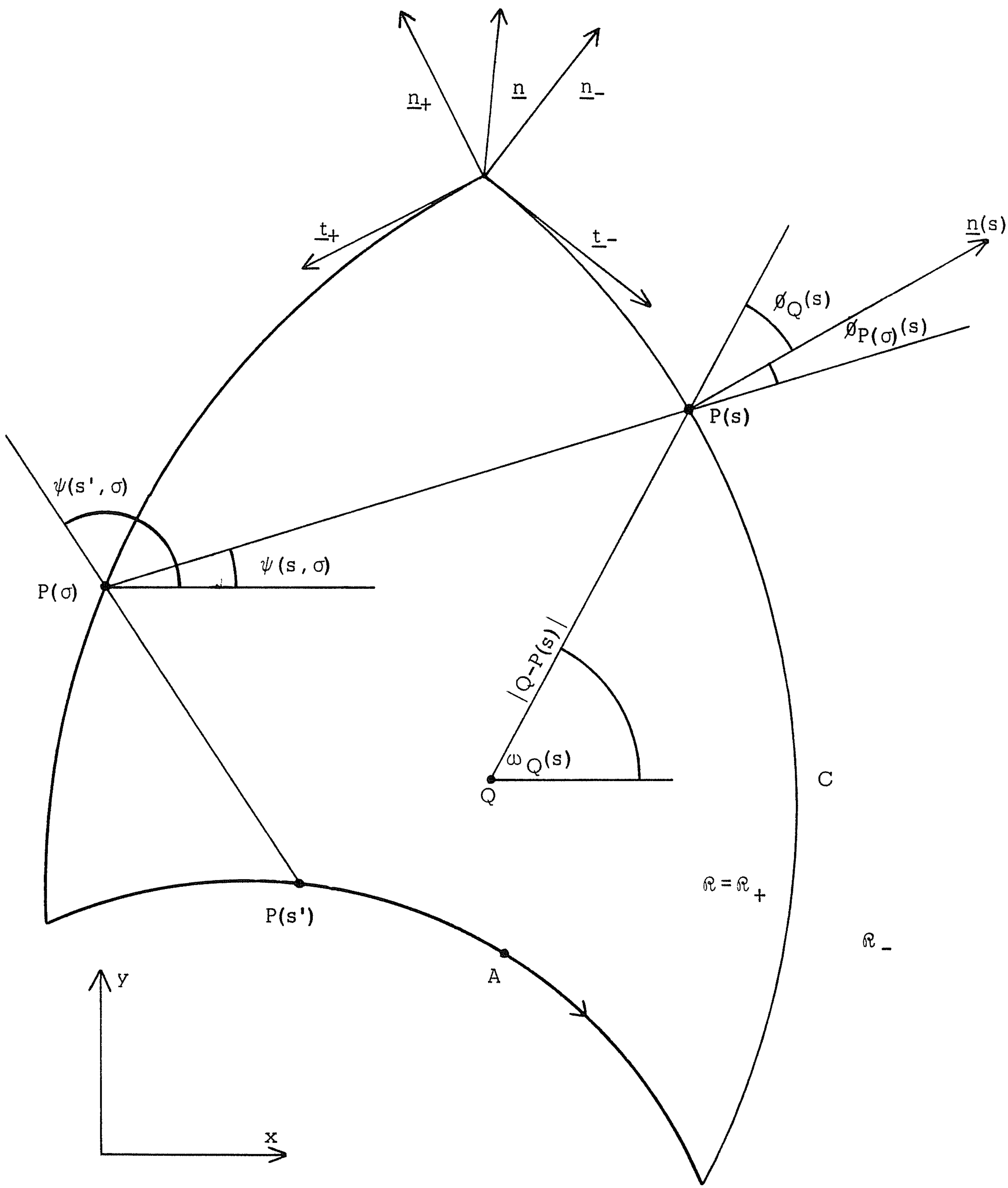

Figure 1.1

The curve C.

If C and g are smooth then it is well-known (Kantorowitsch and Krylow [55, p. 115]) that u is equal to the potential corresponding to a double-layer of density $\varphi = \varphi(s)$ on C. That is, for $Q \in \mathcal{R}$,

$$u(Q) = \int_C \varphi(\sigma) \frac{\partial}{\partial \underline{n}(\sigma)} \log |Q - P(\sigma)| d\sigma , \tag{1.4}$$

where $\underline{n}(\sigma)$ is the unit outward normal to C at $P(\sigma)$ and $|Q - P(\sigma)|$ is the distance from Q to $P(\sigma)$ (see Figure 1.1). The density φ satisfies the integral equation.

$$\varphi(s) + \int_0^S \varphi(\sigma) K(s,\sigma) d\sigma = g(s)/\pi , \quad 0 \leq s \leq S , \tag{1.5}$$

where $K(s,\sigma)$ is a known smooth function which depends only upon C. (In fact,

$$K(s,\sigma) = \frac{1}{\pi} \frac{d}{d\sigma} \text{ arc tan } \left[\frac{y(\sigma) - y(s)}{x(\sigma) - x(s)} \right].)$$

When C has corners the above theory is no longer adequate. Carleman [24] considered this case, but a more satisfactory theory was developed by Radon [110]. C is said to be of <u>bounded rotation</u> if

$$\left. \begin{aligned} x(s) &= x(0) + \int_0^s \cos[\vartheta(\sigma)]\, d\sigma , \\ y(s) &= y(0) + \int_0^s \sin[\vartheta(\sigma)]\, d\sigma , \\ &\text{for } 0 \leq s \leq S \end{aligned} \right\} \tag{1.6}$$

where ϑ is of bounded variation on $[0,S]$, that is,

$$\int_0^S |d\vartheta(\sigma)| < \infty . \tag{1.7}$$

(This concept is due to Radon who generalized earlier work on convex curves by Study [124, p. 103]). Radon showed that if g is continuous and C is of bounded rotation then the Dirichlet problem (1.2), (1.3) is solved by the double-layer potential (1.4) provided that φ satisfies the Stieltjes integral equation

$$\varphi(s) + \int_0^S \varphi(\sigma)\, d_\sigma \psi(s,\sigma) = g(s)/\pi, \qquad 0 \leq s \leq S, \tag{1.8}$$

where ψ is as in Figure 1.1 (an analytic definition of ψ is given in section 2). Recently, Radon's results have generalized by Arbenz [7], Burago et al [19, 20, 21], Maz'ja and Sapoznikova [88], and Kral [58,139]; for further references see Appendix A .

So far as the author is aware, previous workers have considered (1.8) under the assumption that

$$\left.\begin{array}{ll}\text{(a)} & g \text{ is continuous,} \\ \text{(b)} & C \text{ has no cusps.}\end{array}\right\} \tag{1.9}$$

When considering (1.8) as a means of obtaining numerical solutions for the Dirichlet problem, assumption (1.9a) is undesirable since in practical problems g may be discontinuous or, even if g is continuous, approximate solutions to (1.8) may be computed by approximating φ by piecewise continuous functions. For example, Bruhn and Wendland [16, p. 142] assert that the theory for (1.8) can be extended to the case when φ is Borel measurable. Benveniste [12] also encountered difficulties when working in the space of continuous functions.

The main purpose of the present paper is to extend the theory for (1.8) to the case when C is of bounded rotation and g is bounded and measurable. The question as to whether the Dirichlet problem (1.2), (1.3) is meaningful in this case is answered affirmatively in section 5. In section 7 it is shown that the Dirichlet problem (1.2), (1.3) is solved by the double layer potential (1.4) provided that φ satisfies the Lebesgue-Stieltjes integral equation

$$(I + T)\varphi = g/\pi, \tag{1.10}$$

where I denotes the identity operator, the operator T is defined by

$$(T\varphi)(s) = \int_C \varphi(\sigma)\, \Pi_s(d\sigma), \tag{1.11}$$

and Π_s is the Lebesgue-Stieltjes measure determined by the function $\psi(s, \sigma)$ (Dunford and Schwartz [26, p. 142]). In section 9 it is proved that (1.10) can be uniquely solved for φ if C has no cusps.

For a long time the author believed that assumption (1.9b) was not essential and could be relaxed, perhaps by permitting C to have outward-pointing cusps. However, in section 10 it is shown that if C has cusps then $(I + T)^{-1}$ is unbounded, so that (1.9b) is essential within the present framework.

Certain other results are also obtained: in sections 3 and 4 some new properties of curves of bounded rotation are proved; and in section 8 some properties of T are derived. The author believes that these results will be of value when studying the numerical solution of (1.10).

In conclusion, two questions arising from the present work may perhaps be mentioned. Firstly, it would be of interest to determine whether the case of cusps could be handled by working in the space of absolutely integrable functions. Secondly, it would be of interest to study the theory of (1.10), (1.11), for the case when the measure Π_s is not determined by a function such as $\psi(s, \sigma)$.

Acknowledgement

The author is indebted to Professor Ben Noble for bringing several important references to the author's attention.

2. Curves of bounded rotation: basic properties

In this section the geometrical results obtained by Radon [110] for curves of bounded rotation are summarized. It should be mentioned that Radon's paper contains several minor misprints, some of which are corrected in the Russian translation [111]. A detailed reworking of parts of Radon's paper will be found in Appendix B.

Let C be a curve of bounded rotation so that (1.6) and (1.7) hold. Then it can be shown that C has a tangent at all except a denumerable number of points, the corners of C, at which points forward and backward tangents exist. It is convenient to assume (and involves no loss of generality) that $P(0) = P(S)$ is is not a corner and that $\vartheta(s)$ is continuous and differentiable for $s = 0$ and $s = S$.

Since C is smooth at $P(0) = P(S)$ we shall sometimes allow ourselves the liberty of abusing notation in order to avoid having to treat $P(0)$ or $P(S)$ separately. For example, strictly speaking, $\vartheta(S+)$ is not defined. However, since ϑ is continuous for $s = S$, $\vartheta(S+)$ should clearly be interpreted as meaning $\vartheta(S)$.

Theorem 2.1.

To every point $P(s)$ there corresponds an $\epsilon(s) > 0$ such that if $B(s;\rho)$ is the circle with center $P(s)$ and radius $\rho \leq \epsilon(s)$ then

(a) $B(s;\rho)$ cuts C in precisely two points, $P'(s;\rho)$ and $P''(s;\rho)$ say, where $P'(s;\rho) < P(s)$ and $P''(s;\rho) > P(s)$. Hence $B(s;\rho)$ splits C into two subarcs, an open subarc $C_0 = C_0(s;\rho)$ containing $P(s)$, and a closed subarc $C_1 = C_1(s;\rho)$.

(b) Let $B_+(s;\rho) = \bar{\mathcal{R}}_+ \cap B(s;\rho)$, $B_-(s;\rho) = \bar{\mathcal{R}}_- \cap B(s;\rho)$, and denote by $W_+(s;\rho)$ and $W_-(s;\rho)$ the angles subtended at $P(s)$ by $B_+(s;\rho)$ and $B_-(s;\rho)$, respectively. Then as $\rho \to 0$, $W_+(s;\rho)$ and $W_-(s;\rho)$ tend to limits, $W_+(s)$ and $W_-(s)$ say, which will be called the <u>interior</u> and <u>exterior angles</u> at $P(s)$.

(c) If $P(\sigma)$ lies inside or on $B(s;\epsilon(s))$ then

$$\frac{1}{2}\|s-\sigma\| \leq |P(s)-P(\sigma)| \leq \|s-\sigma\|.$$

(d) If one moves from $P'(s;\rho)$ to $P''(s;\rho)$ along $B_-(s;\rho)$ and back to $P'(s;\rho)$ along $B_+(s;\rho)$, then $B(s;\rho)$ has been traversed in the positive (anti-clockwise) direction.

The function ψ shown in Figure 1.1 will now be defined. Let

$$\mathcal{S} = \{(s,\sigma):\ 0 \leq s < \sigma \leq S \ \text{ and } \ \sigma - s < S\}\ .$$

Then

$$\left.\begin{aligned}
&\cos\psi(s,\sigma) = [x(s)-x(\sigma)]/|P(s)-P(\sigma)|,\\
&\sin\psi(s,\sigma) = [y(s)-y(\sigma)]/|P(s)-P(\sigma)|,\\
&\qquad\qquad \text{for } (s,\sigma)\in\mathcal{S},\\
&\psi(s,\sigma) = \psi(\sigma,s), \ \text{ for } (\sigma,s)\in\mathcal{S}.
\end{aligned}\right\}\tag{2.1}$$

ψ is required to be continuous on $\mathcal{S}$ so that (2.1) determines ψ up to a multiple of 2π. It can be shown that ψ can be extended by continuity to all the points of the square $[0,S]\times[0,S]$ with the exception of points of the form (s,s)

where P(s) is a corner point. At such points the limits

$$\psi(s, s\pm) = \lim_{\sigma \to s\pm 0} \psi(s, \sigma)$$

can be shown to exist, and setting

$$\psi(s, s) = \psi(s, s+), \tag{2.2}$$

the definition of ψ is complete. It should be noted that (2.2) holds for all s, not just s for which P(s) is a corner.

At this point it is convenient to redefine ϑ:

$$\vartheta(s) = \psi(s, s) + \pi, \quad 0 \leq s \leq S . \tag{2.3}$$

It can be shown that (1.6) and (1.7) remain valid and that, in addition,

$$\vartheta(s+) = \vartheta(s), \tag{2.4}$$

$$\vartheta(s+) - \vartheta(s-) = W_-(s) - \pi = \pi - W_+(s), \tag{2.5}$$

$$\left.\begin{aligned} \vartheta(s+) &= \psi(s, s+) + \pi, \\ \vartheta(s-) &= \psi(s, s-) + \pi . \end{aligned}\right\} \tag{2.6}$$

From (2.5) it follows that

$$|\vartheta(s+) - \vartheta(s-)| \leq \pi, \tag{2.7}$$

and

$$\vartheta(s+) - \vartheta(s-) = \begin{cases} +\pi, & \text{iff } P(s) \text{ is a cusp which extrudes from } \mathfrak{R}, \\ -\pi, & \text{iff } P(s) \text{ is a cusp which intrudes into } \mathfrak{R}. \end{cases} \tag{2.8}$$

Next, let $\omega_Q(s)$ be as in Figure 1.1. That is,

$$\left.\begin{aligned} \cos\omega_Q(s) &= [x(s) - x_Q]/|P(s) - Q|, \\ \sin\omega_Q(s) &= [y(s) - y_Q]/|P(s) - Q|, \\ & 0 \le s \le S, \quad Q \in \mathfrak{R}_+ \cup \mathfrak{R}_- . \end{aligned}\right\} \tag{2.9}$$

ω_Q is required to be continuous in s and Q and is therefore determined up to a multiple of 2π in both $\mathfrak{R}_+$ and $\mathfrak{R}_-$.

There are three useful and geometrically obvious relations:

$$\left.\begin{aligned} \vartheta(S) - \vartheta(0) &= 2\pi, \\ \psi(s, S) - \psi(s, 0) &= \pi, \\ \omega_Q(S) - \omega_Q(0) &= 2\pi . \end{aligned}\right\} \tag{2.10}$$

Finally,

<u>Theorem 2.2.</u>

For fixed Q, $\omega_Q(\sigma)$ is of bounded variation in σ and

$$\int_C |d\omega_Q(\sigma)| \le \int_C |d\vartheta(\sigma)| .$$

Theorem 2.3.

For fixed s, $\psi(s, \sigma)$ is of bounded variation in σ and

$$\int_C |d_\sigma \psi(s, \sigma)| \leq \int_C |d\vartheta(\sigma)| - \pi .$$

3. Curves of bounded rotation: further properties

Lemma 3.1.

For fixed $\xi \in [0, S]$ let

$$\Psi(s) = \int_0^{\xi} \psi(s, \sigma)d\sigma \quad .$$

Then Ψ is a continuous function on $[0, S]$.

Proof: For $\epsilon \geq 0$ let

$$\mathcal{J}_{\epsilon} = \{(s, \sigma): \ 0 \leq s, \sigma \leq S \ \text{ and } \ |s - \sigma| \geq \epsilon \},$$

and let

$$m = \sup_{\mathcal{J}_0} |\psi(s, \sigma)| \ .$$

Now choose $\epsilon_0 > 0$. Since ψ is continuous on $\mathcal{J}_{\epsilon_0}$ there is a δ_0, $0 < \delta_0 \leq \epsilon_0$, such that

$$|\psi(s', \sigma) - \psi(s'', \sigma)| < \epsilon_0$$

provided that $|s' - s''| \leq \delta_0$ and $|s' - \sigma| \geq 2\epsilon_0$.

Hence, if $|s' - s''| \leq \delta_0$,

$$|\Psi(s') - \Psi(s'')|$$

$$\leq \int_C |\psi(s', \sigma) - \psi(s'', \sigma)|d\sigma,$$

$$\leq \int_{\substack{C \\ |s'-\sigma| \geq 2\epsilon_0}} |\psi(s', \sigma) - \psi(s'', \sigma)|d\sigma + \int_{\substack{C \\ |s'-\sigma| \leq 2\epsilon_0}} |\psi(s', \sigma) - \psi(s'',\sigma)|d\sigma,$$

$$\leq [S + 4m]\epsilon_0 \ .$$

Q.E.D.

Lemma 3.2.

Let C_1 be a rectifiable arc of length L. Let σ denote the arc length on C_1. Let P_1 and P_2 be two points not on C_1. For $i = 1, 2$ let $\alpha_i(\sigma)$ be a continuous function of σ which is equal, modulo 2π, to the angle between the x-axis and the line joining P_i to $P(\sigma) \in C_1$.

Then

(i) $\int_{C_1} |d[\alpha_1(\sigma) - \alpha_2(\sigma)]| \leq L\,|P_1 - P_2|/d^2$,

where

$$d = \min_{i=1,2} \{\text{dist.}\,[P_i , C_1]\} .$$

(ii) $\int_{C_1} |d\,\alpha_1(\sigma)| \leq L/d_1$,

where

$$d_1 = \text{dist}\,[P_1 , C_1].$$

Proof: First consider the case when C_1 is a line segment joining two points, P_3 and P_4 say. Let z_1 through z_4 be the complex numbers corresponding to P_1 through P_4. Let $z = z(\sigma)$ be the complex number corresponding to $P(\sigma) \in C_1$ so that $\sigma = |z - z_3|$. Finally, let $\beta(\sigma) = \alpha_1(\sigma) - \alpha_2(\sigma)$.

Then,

$$\begin{aligned}\int_{C_1} |d[\alpha_1(\sigma) - \alpha_2(\sigma)]| &= \int_{C_1} |d\beta(\sigma)| \quad ,\\ &= \int_{C_1} \left|\frac{d\beta}{d\sigma}\right| d\sigma \ ,\\ &\leq L \max_{C_1} \left|\frac{d\beta}{d\sigma}\right| .\end{aligned}$$

Similarly,

$$\int_{C_1} |d\,\alpha_1(\sigma)| \leq L \max_{C_1} \left|\frac{d\alpha_1}{d\sigma}\right| .$$

However,

$$\beta(\sigma) = \text{Imag}[\text{Log}(z - z_1) - \text{Log}(z - z_2)], \ (\text{mod } 2\pi),$$

so that,

$$\left|\frac{d\beta}{d\sigma}\right| \leq \left|\frac{d}{dz}(\text{Log}(z - z_1) - \text{Log}(z - z_2))\right| ,$$

$$= \left|\frac{1}{z - z_1} - \frac{1}{z - z_2}\right| ,$$

$$= \left|\frac{(z_1 - z_2)}{(z - z_1)(z - z_2)}\right| ,$$

$$\leq |P_1 - P_2| / d^2 .$$

Similarly,

$$\left|\frac{d\alpha_1}{d\sigma}\right| \leq 1/d_1 .$$

Therefore, the lemma holds for the special case when C_1 is a line segment.

Now consider the general case. Let π be any partition,

$$0 = \sigma_0 < \sigma_1 < \ldots < \sigma_n = L ,$$

with $\|\pi\| = \max_{1 \leq i \leq n} |\sigma_i - \sigma_{i-1}| < d$. Then every line segment $P(\sigma_{i-1})P(\sigma_i)$ is at a distance greater than or equal to $(d - \|\pi\|)$ from P_1 and P_2. Hence,

$$\sum_{i=1}^{n} |[\alpha_1(\sigma_i) - \alpha_2(\sigma_i)] - [\alpha_1(\sigma_{i-1}) - \alpha_2(\sigma_{i-1})]|$$

$$\leq \sum_{i=1}^{n} |P(\sigma_i) - P(\sigma_{i-1})| |P_1 - P_2| / [d - \|\pi\|]^2$$

$$\leq L |P_1 - P_2| / [d - \|\pi\|]^2 .$$

Similarly,

$$\sum_{i=1}^{n} |\alpha_1(\sigma_i) - \alpha_1(\sigma_{i-1})| \leq L / [d_1 - \|\pi\|] .$$

The lemma follows by letting $\|\pi\| \to 0$.

Corollary 3.3.

If $Q_1, Q_2 \in \mathfrak{R}$ then

$$\int_C |d[\omega_{Q_1}(\sigma) - \omega_{Q_2}(\sigma)]| \leq S\,|Q_1 - Q_2| / d^2$$

where $d = \min_i \{\text{dist.}[Q_i, C]\}$.

Corollary 3.4.

Let C_1 be a subarc of C. Let $P(s_1), P(s_2) \in C - C_1$. Then

$$\int_{C_1} |d_\sigma[\psi(s_1, \sigma) - \psi(s_2, \sigma)]| \leq S\,|P(s_1) - P(s_2)| / d^2 ,$$

where $d = \min_i \{\text{dist.}[P(s_i), C_1]\}$.

Lemma 3.5.

Let

$$\bar{\psi}(s,\sigma) = \begin{cases} \psi(s,\sigma), & 0 \leq \sigma < s, \\ \psi(s, s-), & \sigma = s, \\ \psi(s,\sigma) - [\psi(s,s+) - \psi(s,s-)], & s < \sigma \leq S. \end{cases}$$

Then, for fixed s, $\bar{\psi}(s,\sigma)$ is A C (absolutely continuous) as a function of σ on $[0,S]$.

Proof: Throughout the proof it is assumed that s is fixed. For clarity, set

$$f(\sigma) = \bar{\psi}(s,\sigma), \qquad 0 \leq \sigma \leq S .$$

Clearly f is continuous. Hence, it suffices to prove that f is A C on each of the intervals $[0,s]$ and $[s,S]$. The two cases are essentially the same and only the interval $[0,s]$ will be considered.

For $0 < \epsilon < s$ set $I(\epsilon) = [0, s-\epsilon]$. We assert that f is A C on $I(\epsilon)$. To prove this we begin by observing that, from Theorem 2.3, f is of bounded variation on $[0,s]$. Since f is also continuous the function

$$\int_0^\sigma |df(\tau)|$$

is continuous for $\sigma \in [0,s]$ (see Natanson [92, p. 226]). Hence there is a partition of $I(\epsilon)$,

$$0 = a_0 < a_1 < \dots < a_n = s - \epsilon$$

such that the variation of f on each subinterval $I_i = [a_{i-1}, a_i]$ is less than $\pi/8$.

Let c_i be the smallest integer such that

$$|f([a_i + a_{i-1}]/2) - c_i \pi/2| \le \pi/4 .$$

Set

$$g_i(\sigma) = \begin{cases} (c_i - 1)\pi/2 + \arccos\left[\dfrac{x(\sigma) - x(s)}{|P(\sigma) - P(s)|}\right], & \text{if } c_i \text{ is odd} \\ c_i \pi/2 + \arcsin\left[\dfrac{y(\sigma) - y(s)}{|P(\sigma) - P(s)|}\right], & \text{if } c_i \text{ is even.} \end{cases}$$

Remembering that the variation of f on I_i is less than $\pi/8$, and that $\sigma < s$, it follows from (2.1) that $f(\sigma) = g_i(\sigma)$ for $\sigma \in I_i$. But, from (1.6), $[x(\sigma) - x(s)]$ and $[y(\sigma) - y(s)]$ are A C. Since arc cos and arc sin are Lipschitz continuous, g_i is also A C (see Natanson [92, p. 247]). Hence, f is A C on each I_i, and thus A C on $I(\epsilon)$, as previously asserted.

Next, we observe from (2.1) and (1.6) that

$$\begin{aligned} &\cos[f(\sigma) - \vartheta(s-)] \\ &= \cos[\psi(\sigma,s) - \vartheta(s-)] \\ &= \cos\psi(\sigma,s)\cos\vartheta(s-) + \sin\psi(\sigma,s)\sin\vartheta(s-), \\ &= |P(\sigma) - P(s)|^{-1}\{[x(\sigma) - x(s)]\cos\vartheta(s-) + [y(\sigma) - y(s)]\sin\vartheta(s-)\}, \\ &= |P(\sigma) - P(s)|^{-1}\int_s^\sigma \{\cos\vartheta(\tau)\cos\vartheta(s-) + \sin\vartheta(\tau)\sin\vartheta(s-)\}\,d\tau \\ &= |P(\sigma) - P(s)|^{-1}\int_s^\sigma \cos[\vartheta(\tau) - \vartheta(s-)]\,d\tau . \end{aligned}$$

Similarly,

$$\sin[f(\sigma) - \vartheta(s-)] = |P(\sigma) - P(s)|^{-1}\int_s^\sigma \sin[\vartheta(\tau) - \vartheta(s-)]\,d\tau .$$

Now let $J(\epsilon) = [s - \epsilon, s]$ where $0 < \epsilon < s$, and assume that ϵ is so small that the variation of f on $J(\epsilon)$ is less than $\pi/8$. Then it follows from the above that on $J(\epsilon)$,

$$f(\sigma) = \vartheta(s-) + c\pi + F(\sigma),$$

where c is an integer and

$$F(\sigma) = \begin{cases} \arctan \dfrac{\int_s^\sigma \sin[\vartheta(\tau) - \vartheta(s-)]\, d\tau}{\int_s^\sigma \cos[\vartheta(\tau) - \vartheta(s-)]\, d\tau}, & \sigma < s, \\ 0, & \sigma = s. \end{cases}$$

If it is further assumed that ϵ is so small that $|\cos[\vartheta(\sigma) - \vartheta(s-)] \geq 1/2$ on $J(\epsilon)$, then, for $\sigma \in J(\epsilon)$,

$$|F(\sigma)| \leq \arctan\Big[2 \sup_{s-\epsilon \leq \sigma < s} |\vartheta(\sigma) - \vartheta(s-)|\Big]$$

We can now prove that f is A C on $[0, s]$. To do so, we recall that f is continuous and of bounded variation. Therefore, by a theorem of Banach (Natanson [92, p. 252]) it suffices to prove that f maps sets of measure zero into sets of measure zero.

Let $E \subset [0, s]$ be of measure zero, that is $m(E) = 0$. Let ϵ be small, and set $E' = E \cap I(\epsilon)$ and $E'' = E \cap J(\epsilon)$. Since f is A C on $I(\epsilon)$, we have (Natanson [92, p. 251]),

$$m[f(E')] = 0,$$

while by the preceding results,

$$\begin{aligned} m[f(E'')] &= m[\vartheta(s-) + c\pi + F(E'')] \\ &\leq 2 \arctan\Big[2 \sup_{s-\epsilon \leq \sigma < s} |\vartheta(\sigma) - \vartheta(s-)|\Big]. \end{aligned}$$

Letting ϵ tend to zero, it follows that $m[f(E)] = 0$ so that f is A C on $[0, s]$.

Lemma 3.6.

$\omega_Q(\sigma)$ is AC (absolutely continuous) as a function of σ on $[0,S]$.

Proof: Using arguments similar to those employed in Lemma 3.5, it can be proved that the interval $[0,S]$ can be split up into subintervals I_i such that throughout each I_i $\omega_Q(\sigma)$ can be represented by one of the following formulas:

$$\omega_Q(\sigma) = c_i\,\pi/2 + \text{arc cos}\left[\frac{x(\sigma) - x_Q}{|P(\sigma) - Q|}\right]$$

$$\omega_Q(\sigma) = c_i\,\pi/2 + \text{arc sin}\left[\frac{y(\sigma) - y_Q}{|P(\sigma) - Q|}\right],$$

where the c_i are integers. Since the functions arc sin and arc cos are Lipschitz continuous, and since, by (1.6), $[x(\sigma) - x_Q]$ and $[y(\sigma) - y_Q]$ are AC, it follows that $\omega_Q(\sigma)$ is AC on each I_i and hence AC on $[0,S]$.

The final two lemmas concern certain geometrically obvious properties of curves of bounded rotation.

The unit outward normal to C at P(s) is defined to be $\underline{n}(s) = (n_1(s), n_2(s))$ where

$$\left.\begin{aligned} n_1(s) &= \sin[\{\vartheta(s+) + \vartheta(s-)\}/2], \\ n_2(s) &= -\cos[\{\vartheta(s+) + \vartheta(s-)\}/2]. \end{aligned}\right\} \qquad (3.1)$$

If C has a tangent at P(s) then $\underline{n}(s)$ is the usual normal. If C does not have a tangent at P(s), then, since $\vartheta(s+)$ and $\vartheta(s-)$ exist, C does have forward and backward tangents $\underline{t}_+$ and $\underline{t}_-$ and hence forward and backward normals $\underline{n}_+$ and $\underline{n}_-$ (see Figure 1.1); obviously $\underline{n}(s)$ bisects $\underline{n}_+$ and $\underline{n}_-$. If C has a cusp at P(s) then it follows from (2.8) that $\underline{n}(s)$ does indeed point out from C. Hence, in all cases $\underline{n}(s)$ corresponds to the outward normal as understood geometrically.

Lemma 3.7.

Using the notation of Theorem 2.1 let

$$C_+(s;\rho) = \{P \in C_0(s;\rho)\ ;\ P(\sigma) > P(s)\},$$

$$C_-(s;\rho) = \{P \in C_0(s;\rho)\ ;\ P(\sigma) < P(s)\}.$$

There exist $\epsilon_1(s) \leq \epsilon(s)$, and a function $a(s;\rho)$ such that

(i) $a(s;\rho) \downarrow 0$ as $\rho \to 0$.

(ii) If $\rho \leq \epsilon_1(s)$, $P(\sigma_1)$, $P(\sigma_2) \in C_{\pm}(s;\rho)$, and $0 < s < S$, then

(a) $x(\sigma_1) - x(\sigma_2) - (\sigma_1 - \sigma_2)\cos \vartheta(s\pm) = \theta_x\, a(s;\rho)\|\sigma_1 - \sigma_2\|$,

(b) $y(\sigma_1) - y(\sigma_2) - (\sigma_1 - \sigma_2)\sin \vartheta(s\pm) = \theta_y\, a(s;\rho)\|\sigma_1 - \sigma_2\|$,

(c) $(1 - a(s;\rho))\|\sigma_1 - \sigma_2\| \leq |P(\sigma_1) - P(\sigma_2)| \leq \|\sigma_1 - \sigma_2\|$,

where $|\theta_x|$, $|\theta_y| \leq 1$.

Similar results hold when $s = 0$ or $s = S$.

(iii) If $P(\sigma) \in C_{\pm}(s;\epsilon_1(s))$ and $\rho \leq \epsilon_1(s) - \|\sigma - s\|$, then $B(\sigma;\rho)$ cuts $C_{\pm}(s;\epsilon_1(s))$ in precisely two points.

Proof: Set

$$a_1(s;\rho) = \max\left\{\sup_{P(\sigma)\in C_+(s;\rho)} |\vartheta(\sigma) - \vartheta(s+)|,\ \sup_{P(\sigma)\in C_-(s;\rho)} |\vartheta(\sigma) - \vartheta(s-)|\right\},$$

$$a(s;\rho) = \max\left\{2a_1(s;\rho),\ 8[a_1(s;\rho)]^2\right\}.$$

Let $\epsilon_1(s)$ be any number less than $\epsilon(s)$ which satisfies $a(s;\epsilon_1(s)) \leq 1$. Condition (i) clearly is satisfied. To prove (ii), assume that $\rho \leq \epsilon_1(s)$ and that $P(\sigma_1)$, $P(\sigma_2) \in C_{\pm}(s;\rho)$. Then

$$|x(\sigma_1) - x(\sigma_2) - (\sigma_1 - \sigma_2)\cos \vartheta(s\pm)|$$

$$= |\int_{\sigma_1}^{\sigma_2} [\cos \vartheta(\tau) - \cos \vartheta(s\pm)]\, d\tau|$$

$$= |2\int_{\sigma_1}^{\sigma_2} \sin[(\vartheta(\tau) - \vartheta(s\pm))/2]\, \sin[(\vartheta(\tau) + \vartheta(s\pm))/2]\, d\tau| ,$$

$$\leq 2|\sigma_1 - \sigma_2|\; a_1(s;\rho),$$

which implies (iia). Similar arguments yield (iib).

Next, note that if $P(\sigma_1)$, $P(\sigma_2) \in C_{\pm}(s;\rho)$ with $\rho \leq \epsilon(s)$ then

$$|P(\sigma_1) - P(\sigma_2)|^2$$

$$= [\int_{\sigma_1}^{\sigma_2} \sin(\vartheta(\tau))d\tau]^2 + [\int_{\sigma_1}^{\sigma_2} \cos(\vartheta(\tau))d\tau]^2 ,$$

$$= \int_{\sigma_1}^{\sigma_2}\int_{\sigma_1}^{\sigma_2} \cos(\vartheta(\tau) - \vartheta(\nu))\, d\tau\, d\nu .$$

Therefore,

$$|P(\sigma_1) - P(\sigma_2)|^2 - |\sigma_1 - \sigma_2|^2$$

$$= \int_{\sigma_1}^{\sigma_2}\int_{\sigma_1}^{\sigma_2} [\cos(\vartheta(\tau) - \vartheta(\nu)) - 1]\, d\tau\, d\nu ,$$

$$= -2\int_{\sigma_1}^{\sigma_2}\int_{\sigma_1}^{\sigma_2} [\sin(\vartheta(\tau) - \vartheta(\nu))]^2\, d\tau\, d\nu ,$$

$$\geq -2\,|\sigma_1 - \sigma_2|^2\,[2a_1(s;\rho)]^2 \geq -a(s;\rho)\,|\sigma_1 - \sigma_2|^2,$$

which implies (iic).

To prove (iii) it suffices to prove that if $P(\sigma_1)$, $P(\sigma_2) \in C_{\pm}(s;\rho)$ and $\rho \leq \epsilon(s)$ then, for σ_1 fixed, $|P(\sigma_1) - P(\sigma_2)|$ is a continuous, strictly monotone, function of σ_2 for $\sigma_2 > \sigma_1$ and for $\sigma_2 < \sigma_1$. That this is so follows from the observation that if $(\sigma_3 - \sigma_1)(\sigma_2 - \sigma_1) \geq 0$ then

$$|[\sigma_2 - \sigma_1]^2 - [\sigma_3 - \sigma_1]^2| \quad \cos[2a_1(s;\rho)]$$

$$\leq |(\int_{\sigma_1}^{\sigma_2}\int_{\sigma_1}^{\sigma_2} - \int_{\sigma_1}^{\sigma_3}\int_{\sigma_1}^{\sigma_3}) \cos[\vartheta(\tau) - \vartheta(\nu)]\, d\tau\, d\nu| ,$$

$$= \left| |P(\sigma_1) - P(\sigma_2)|^2 - |P(\sigma_1) - P(\sigma_3)|^2 \right|$$

$$\leq |[\sigma_2 - \sigma_1]^2 - [\sigma_3 - \sigma_1]^2| .$$

<u>Lemma 3.8.</u>

Let $U(s,\beta)$ denote the open cone with axis $-\underline{n}(s)$, vertex $P(s)$, and vertex angle 2β. Assume that $P(s)$ is not an extruding cusp. Then there exist $\alpha = \alpha(s) > 0$ and $\rho_1 = \rho_1(s) > 0$ such that if $\|\sigma - s\| \leq \rho_1$ then $P(\sigma) \not\in U(s, \alpha)$.

<u>Proof:</u> Since $P(s)$ is not an extruding come, it follows from (2.7) and (2.8) that there is an α, $0 < \alpha < \pi/2$, such that

$$-\pi/2 \leq [\vartheta(s+) - \vartheta(s-)]/2 \leq \pi/2 - \alpha .$$

Since ϑ is of bounded variation, there exists $\delta > 0$ such that if $\|s - \sigma\| \leq \delta$ then

$$|\vartheta(\sigma) - \vartheta(s-)| \leq \alpha/4 , \quad \text{if } P(\sigma) < P(s) ,$$
$$|\vartheta(\sigma) - \vartheta(s+)| \leq \alpha/4 , \quad \text{if } P(\sigma) > P(s) .$$

Throughout the remainder of the argument it will be assumed that $0 < \|s - \sigma\| \leq \delta$.

First, note that

$$|P(\sigma) - P(s)|^2$$

$$= [\int_s^\sigma \sin \vartheta(\tau) d\tau]^2 + [\int_s^\sigma \cos \vartheta(\tau)\, d\tau]^2 ,$$

$$= \int_s^\sigma \int_s^\sigma \cos[\vartheta(\tau) - \vartheta(\nu)]\, d\tau\, d\nu ,$$

$$\geq \int_s^\sigma \int_s^\sigma \cos[\alpha/2] d\tau\, d\nu ,$$

$$= \|s - \sigma\|^2 \cos[\alpha/2].$$

Next, note that

$$-\pi/2 - \alpha/2 \leq \vartheta(\sigma) - [\vartheta(s+) + \vartheta(s-)]/2 \leq \pi/2 - \alpha/2, \text{ if } P(\sigma) > P(s),$$

$$-\pi/2 + \alpha/2 \leq \vartheta(\sigma) - [\vartheta(s+) + \vartheta(s-)]/2 \leq \pi/2 + \alpha/2, \text{ if } P(\sigma) < P(s).$$

Consequently,

$$[-\underline{n}(s)] \cdot [P(\sigma) - P(s)]$$

$$= -\sin([\vartheta(s+) + \vartheta(s-)]/2) \int_s^\sigma \cos \vartheta(\tau)\, d\tau +$$

$$+ \cos([\vartheta(s+) + \vartheta(s-)]/2) \int_s^\sigma \sin \vartheta(\tau) d\tau ,$$

$$= \int_s^\sigma \sin[\vartheta(\tau) - \{\vartheta(s+) + \vartheta(s-)\}/2]\, d\tau ,$$

$$\leq \|\sigma - s\| \quad \sin[\pi/2 - \alpha/2],$$

$$= \|\sigma - s\| \cos(\alpha/2),$$

$$\leq |P(\sigma) - P(s)| [\cos(\alpha/2)]^{\frac{1}{2}} .$$

Setting

$$\beta = \operatorname{arc\,cos} [[\cos(\alpha/2)]^{\frac{1}{2}}]$$

the lemma follows.

4. Curves of bounded rotation: still further properties.

Up to the present point Stieltjes integrals have sufficed, but it now becomes necessary to use Lebesgue-Stieltjes (Radon) integrals (Radon [108]).

Consider the functions $\omega_Q(\sigma)$, $\psi(s,\sigma)$, and $\vartheta(\sigma)$. By Theorems 2.2 and 2.3, these functions are of bounded variation on $[0,S]$; since $P(0) = P(S)$ is not a corner, they are continuous for $\sigma = 0$ and $\sigma = S$; by (2.2) and (2.4) they are continuous on the right at every point in $[0,S]$. Hence (Radon [110, p. 1143], Dunford and Schwartz [26, p. 141]) the functions $\omega_Q(\sigma)$, $\psi(s,\sigma)$, and $\vartheta(\sigma)$, determine Lebesgue-Stieltjes measures Π_Q, Π_s, and $\textcircled{\sim}$, on $[0,S]$.

Certain elementary properties of Π_Q, Π_s, and $\textcircled{\sim}$, are discussed below. To avoid unnecessary repetition, let $f(\beta)$ denote a function of bounded variation on $[0,S]$ which is continuous except possibly at the corners of C where it is continuous on the right, and let $\sum$ denote the measure determined by f; in particular, $\sum$ may be either Π_Q, Π_s, or $\textcircled{\sim}$. Then:

Property (a). Since $P(0) = P(S)$ is not a corner of C, $\sum$ may be regarded as a measure on C. This point of view will often be adopted since it obviates the need to treat the points $s = 0$ and $s = S$ differently from the other points of $[0,S]$.

Property (b). If E is Borel-measurable then it is $\sum$-measurable. In particular, open, closed, and semi-open intervals are $\sum$-measurable. (Dunford and Schwartz [26, p. 142].)

Property (c). If E is $\sum$-measurable then, as usual, the variation of $\sum$ over E will be denoted by

$$\int_E |\textstyle\sum|(d\sigma).$$

Property (d). If C_1 is a closed subarc of C, and if f is continuous at the end points of C_1, then

$$\int_{C_1} |\textstyle\sum| (d\sigma) = \int_{C_1} |df(\sigma)| .$$

(Dunford and Schwartz [26 ; pages 98, 137, 138 , and 141].)

Property (e). Let C and $\widetilde{C}$ be two curves which have an open subarc C_1 in common. Let $f = \widetilde{f}$ on C_1. Then $\sum$ and $\widetilde{\sum}$ agree on C_1.

Property (f). Let $P(s) \neq P(s_1)$. Set $E = \{P(s)\}$. Then, by direct computation,

$$\int_E |\odot|(d\sigma) = \int_E |\Pi_s|(d\sigma) = |\psi(s,s+) - \psi(s,s-)| = |\vartheta(s+) - \vartheta(s-)| ,$$

$$\int_E |\Pi_Q|(d\sigma) = \int_E |\Pi_{s_1}|(d\sigma) = 0 .$$

Lemma 4.1.

Let C_1 be an open subarc of C with closure $\bar{C}_1$. Then

$$\int_{\bar{C}_1} |\Pi_Q|(d\sigma) \leq \pi + \int_{C_1} |\odot| (d\sigma).$$

Proof: This lemma is essentially due to Radon [110, p. 1147].

Let C_1 be the subarc $a < s < b$, and let $\widetilde{C}$ be the closed curve obtained by doubling $\bar{C}_1$.

Applying Theorem 2.2, and remembering that $\widetilde{C}$ has cusps at $P(a)$ and $P(b)$, it follows that, with an obvious notation,

$$2 \int_{\overline{C}_1} |\mathrm{II}_Q| (d\,\sigma)$$

$$= \int_{\widetilde{C}} |\widetilde{\mathrm{II}}_Q| (d\,\sigma) \leq \int_{\widetilde{C}} |\widetilde{\odot}| (d\,\sigma) ,$$

$$= \int_{\widetilde{C} - \{P(a), P(b)\}} |\widetilde{\odot}| (d\sigma) + \int_{\{P(a), P(b)\}} |\widetilde{\odot}| (d\sigma) ,$$

$$= 2 \int_{C_1} |\odot| (d\sigma) + 2\,\pi .$$

<u>Lemma 4.2.</u>

Let C_2 be an open, closed, or semi-open subarc of C with closure $\overline{C}_2$. Let $P(s) \in \overline{C}_2$. Then

$$\int_{C_2} |\mathrm{II}_s| (d\,\sigma) \leq \int_{C_2} |\odot| (d\,\sigma) .$$

<u>Proof:</u> The lemma will first be proved for the special case when C_2 is an open arc with endpoint $P(s)$.

Construct a closed arc $\widetilde{C}$ by extending $\overline{C}_2$ smoothly beyond its endpoints by means of line segments. The lemma will follow if it can be shown that, with an obvious notation,

$$\int_{\widetilde{C}} |\widetilde{\mathrm{II}}_s| (d\sigma) \leq \int |\widetilde{\odot}| (d\sigma) ,$$

or, equivalently,

$$\int_{\widetilde{C}} |d_\sigma \widetilde{\psi}(s, \sigma)| \leq \int_{\widetilde{C}} |d\, \widetilde{\mathfrak{F}}(\sigma)| .$$

For the remainder of the argument the tildas will be omitted. Therefore, we wish to show that if C is a closed arc, $P(s)$ is an interior point of C, and $P(s)$ is not a corner, then,

$$\int_C |d_\sigma \psi(s,\sigma)| \leq \int_C |d\,\vartheta(\sigma)| .$$

Choose $\mu > 0$. Then there is a partition π of C, consisting of points $P_i = P(\sigma_i)$, $0 \leq i \leq n$, such that

$$\int_C |d_\sigma \psi(s,\sigma)| \leq \sum_{i=1}^{n} |\psi(s,\sigma_i) - \psi(s,\sigma_{i-1})| + \mu .$$

By refining π if necessary it may be assumed that

(i) For some k, $0 < k < n$,

$$P(\sigma_k) = P(s) ,$$

(ii) $|P(\sigma_{k-1}) - P(s)| = |P(s) - P(\sigma_{k+1})| = \rho < \epsilon(s)$,

where $\epsilon(s)$ is as in Theorem 2.1.

Set $\|\pi\| = \max_i |P(\sigma_i) - P(\sigma_{i-1})|$.

Let $Q \notin C$. Applying Lemma 4.1,

$$\Big\{\sum_{\substack{i=1 \\ i\neq k,k+1}}^{n} |\omega_Q(P_i) - \omega_Q(P_{i-1})|\Big\} + |\omega_Q(P_k) - \omega_Q(P_{k-1})| + |\omega_Q(P_{k+1}) - \omega_Q(P_k)|$$

$$\leq \int_C |d\,\omega_Q(\sigma)| ,$$

$$\leq \pi + \int_C |d\,\vartheta(\sigma)| .$$

Letting Q tend to $P(s)$, and noting that

$$\lim_{Q\to P(s)} [\,|\omega_Q(P_{k+1}) - \omega_Q(P_k)| + |\omega_Q(P_k) - \omega_Q(P_{k-1})|\,] \equiv W_+(s;\rho) (\mathrm{mod}\ 2\pi),$$

where $W_+(s;\rho)$ is the interior angle subtended at $P(s)$ by P_{k-1} and P_{k+1}, it follows that

$$\sum_{\substack{i=1 \\ i \neq k, k+1}}^{n} |\psi(s,\sigma_i) - \psi(s,\sigma_{i-1})| + W_+(s;\rho)$$

$$\leq \pi + \int_C |d\,\vartheta(\sigma)| .$$

Hence,

$$\int_C |d_\sigma\, \psi(s,\sigma)|$$

$$\leq \sum_{i=1}^{n} |\psi(s,\sigma_i) - \psi(s,\sigma_{i-1})| + \mu ,$$

$$\leq \int_C |d\,\vartheta(\sigma)| + |\pi - W_+(s)| + |W_+(s) - W_+(s;\rho)|$$

$$+ |\psi(s,\sigma_{k-1}) - \psi(s,s)| + |\psi(s,\sigma_{k+1}) - \psi(s,s)| + \mu ,$$

where $W_+(s)$ is the interior angle at $P(s)$ (see Theorem 2.1). Since $P(s)$ is not a corner, $\psi(s,\sigma)$ is continuous for $\sigma = s$, and $W_+(s) = \pi$. Noting Theorem 2.1, the required result follows by letting $\|\pi\| \to 0$.

Now consider the general case. Let C_2' be the interior of C_2 and set $C_3 = C_2 - C_2'$. Then

$$C_2 = C_3 \cup [\,\{P(s)\} - [\bar{C}_2 - C_2]] \cup C_4 \cup C_5$$

where C_4 and C_5 are open, possibly empty, subarcs of C_2 with endpoint $P(s)$. Applying the results already proved together with Property (f), the lemma follows.

<u>Lemma 4.3.</u>

Let $C_2 = C_2(\delta)$

denote an open subarc of C with endpoint $P(s)$ and length δ. Then

$$\lim_{\delta\to 0} \int_{C_2(\delta)} |\circleddash|(d\sigma) = 0 .$$

Proof: Only the case when $C(\delta)$ is the subarc $s < \sigma < s+\delta$ will be considered.

For some $d > 0$ let $\widetilde{C}_2$ be the subarc, $s \leq \sigma \leq s+d$ of C. Set

$$\widetilde{\vartheta}(\sigma) \begin{cases} \vartheta(s+) , & \sigma = s , \\ \vartheta(\sigma) , & s < \sigma < s+d , \\ \vartheta((s+d)-) , & \sigma = s+d . \end{cases}$$

Since $\widetilde{\vartheta}(\sigma)$ is of bounded variation and continuous at $\sigma = s$, it follows (Natanson [92, p. 226]) that

$$\lim_{\delta\to 0} \int_s^{s+\delta} |d\,\widetilde{\vartheta}(\sigma)| = 0 .$$

Since C has only a denumerable number of corners there is a sequence $\{\delta_n\}$, such that $\delta_n > 0$, $\delta_n \to 0$ as $n \to \infty$, and $P(s+\delta_n)$ is not a corner. Noting Property (d),

$$\lim_{n\to\infty} \int_s^{s+\delta_n} |\widetilde{\circleddash}|(d\sigma) = 0 .$$

which implies that

$$\lim_{\delta\to 0} \int_s^{s+\delta} |\widetilde{\circleddash}|(d\sigma) = 0 .$$

Applying Property (e) the lemma follows.

Lemma 4.4.

Let $P(s) \in C$. Let $\bar{\psi}(s,\sigma)$ be as in Lemma 3.5 and let $\bar{\Pi}_s$ be the corresponding measure.

Then, for any $\epsilon > 0$ there exists $\eta(s) > 0$ such that

$$\int_C |\Pi_t - \bar{\Pi}_s|(d\sigma) \leq |\vartheta(s+) - \vartheta(s-)| + \epsilon ,$$

for all t satisfying $\|s-t\| \leq \eta(s)$.

Proof: Let $B(s;\rho)$, $C_0 = C_0(s;\rho)$, $C_1 = C_1(s;\rho)$, be as in Theorem 2.1. Set $C'_0 = C_0 - \{P(s)\}$. Assume that $\rho \le \epsilon(s)$ and that

$$\rho^4 \leq \rho/2 \ .$$

Using Corollary 3.4, Lemma 4.2 and Property (f), and remembering that Π_s and $\bar{\Pi}_s$ coincide on C_1, it follows that if $P(t)$ is such that

$$|P(s) - P(t)| \leq \rho^4$$

then

$$I(t) = \int_C |\Pi_t - \bar{\Pi}_s| \, (d\sigma) \ ,$$

$$\leq \int_{C_0} |\Pi_t| (d\sigma) + \int_{C_0} |\bar{\Pi}_s| (d\sigma) + \int_{C_1} |\Pi_t - \Pi_s| \, (d\sigma) \ ,$$

$$\leq \int_{C_0} |\odot| (d\sigma) + \int_{C_0} |\bar{\Pi}_s| (d\sigma) + \rho^4 S/(\rho/2)^2 \ ,$$

$$= |\vartheta(s+) - \vartheta(s-)| + I_1(\rho) + I_2(\rho) + 4 S \rho^2 \ ,$$

where

$$I_1(\rho) = \int_{C'_0} |\odot| (d\sigma) \ ,$$

$$I_2(\rho) = \int_{C_0} |\bar{\Pi}_s| \, (d\sigma) \ .$$

It follows from Lemma 4.3 that $I_1(\rho) \to 0$ as $\rho \to 0$. Since $\bar{\psi}(s,\sigma)$ is continuous and of bounded variation as a function of σ, it follows (Natanson [92, p. 226]) that $I_2(\rho) \to 0$ as $\rho \to 0$. Therefore, given $\epsilon > 0$ there exists $\rho_0 > 0$ such that if

$$|P(s) - P(t)| \leq \rho_0^4,$$

then

$$|I(t)| \leq |\vartheta(s+) - \vartheta(s-)| + \epsilon \ .$$

Setting $\eta < \rho_0^4$ the lemma follows .

5. The generalized Dirichlet Problem

When C and g are not smooth, the Dirichlet problem (1.2), (1.3) may be meaningless. In this section a more general problem is introduced and discussed. First, some notation is necessary.

For $P(s) \in C$,

$$\lim_{Q \to P_+(s)} u(Q)$$

will denote the limit, if it exists, of $u(Q)$ as $Q \in \mathcal{R}_+$ tends to $P(s)$.

For $P(s)$ not an extruding cusp,

$$\lim_{Q \to P_+(s)}(n)\ u(Q)$$

will denote the limit, if it exists, of $u(Q)$ as $Q \in \mathcal{R}_+$ tends to $P(s)$ along the normal to C at $P(s)$. The reason for excluding extruding cusps is that with such cusps certain topological problems can arise. As an example consider the case when C contains an extruding cusp with "arms"

$$y(x) = x^{10}[\sin(\tfrac{1}{x}) + 1/2],$$

$$y(x) = x^{10}[\sin(\tfrac{1}{x}) - 1/2], \quad 0 \leq x \leq 1 .$$

Then the normal at the cusp $(0, 0)$ coincides with the x-axis. However, there is no $\epsilon > 0$ such that the line segment joining $(0, 0)$ to $(\epsilon, 0)$ lies in $\mathcal{R}$.

The Generalized Dirichlet Problem:

Find a function $u = u(x, y)$ such that

$$\left.\begin{array}{ll} \text{(a)} & u \text{ is continuous and bounded in } \mathfrak{R}, \\ \text{(b)} & u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathfrak{R}, \\ \text{(c)} & \lim\limits_{Q \to P_+(s)}(n) \quad u(Q) = g(s), \text{ for almost all } s \in [0,S]. \end{array}\right\} \quad (5.1)$$

Theorem 5.1.

Let g be bounded and in $L_1[0, S]$, the space of Lebesgue integrable functions on $[0, S]$. Then

(a) The generalized Dirichlet problem has a unique solution, u.

(b) If g is continuous at s, then

$$\lim_{Q \to P_+(s)} u(Q) = g(s).$$

(c) If $g(s+)$ and $g(s-)$ exist, and if $P(s)$ is not an extruding cusp, then

$$\lim_{Q \to P_+(s)}(n) \quad u(Q) = [g(s+) + g(s-)]/2.$$

Proof: This theorem is based upon well-known facts but does not seem to be explicitly stated in the literature.

Let $\tilde{\mathfrak{R}}$ be the unit circle centered at the origin in the $\widetilde{xy}$-plane with boundary $\tilde{C}$. Let χ map $\mathfrak{R}$ conformally onto $\tilde{\mathfrak{R}}$. Since C is a Jordan curve, χ can be extended to C, and it will be assumed that this has been done. Then χ is a bicontinuous and one-to-one mapping of $\mathfrak{R} \cup C$ onto $\tilde{\mathfrak{R}} \cup \tilde{C}$. (Hille [47, p. 367]).

Let ϕ be the restriction of χ to C. Then ϕ: $[0, S] \to [0, 2\pi]$ and is continuous and one-to-one. Furthermore, both ϕ and ϕ^{-1} map measurable sets into measurable sets and sets of measure zero into sets of measure zero (Goluzin [36, p. 420 and p. 421]).

Now let $\tilde{u}$ and $\tilde{g}$ correspond to u and g under the mapping χ. Clearly $\tilde{g}$ is bounded. Moreover, $\tilde{g}$ is measurable since

$$\{\tilde{s}: \tilde{g}(\tilde{s}) \geq a\} = \phi\{s: g(s) \geq a\},$$

and hence $\tilde{g} \in L_1[0, 2\pi]$.

For $\tilde{P}(\tilde{s}) \in \tilde{C}$, let

$$\lim_{\tilde{Q} \to \tilde{P}_+(\tilde{s})}(\tilde{n}) \ \tilde{u}(\tilde{Q})$$

denote the limit, if it exists, of $\tilde{u}(\tilde{Q})$ as $\tilde{Q} \in \tilde{\mathfrak{R}}$, tends to $\tilde{P}(\tilde{s})$ along the path $\tilde{n}$ which is the image under χ of the normal to C at $P(\phi^{-1}(\tilde{s}))$.

Then, using the above-mentioned properties of χ and ϕ it follows that u is a solution of the generalized Dirichlet problem iff $\tilde{u}$ satisfies:

$$\left.\begin{aligned} &\text{(a)}\ \tilde{u} \text{ is continuous and bounded in } \tilde{\mathfrak{R}};\\ &\text{(b)}\ \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}_{\tilde{y}\tilde{y}} = 0, \quad (\tilde{x}, \tilde{y}) \in \tilde{\mathfrak{R}},\\ &\text{(c)}\ \lim_{\tilde{Q} \to \tilde{P}_+(\tilde{s})}(\tilde{n}) \ \tilde{u}(\tilde{Q}) = \tilde{g}(\tilde{s}), \text{ for almost all } \tilde{s} \in [0, 2\pi]. \end{aligned}\right\} \tag{5.2}$$

Since $\tilde{u}$ is bounded it can be represented as a Poisson integral (Goluzin [36, p. 391]). That is,

$$\tilde{u}(\tilde{r}, \tilde{s}) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{1 - \tilde{r}^2}{1 - 2\tilde{r}\cos(t-\tilde{s}) + \tilde{r}^2} \, dt, \; 0 \leq \tilde{r} < 1, \; 0 \leq \tilde{s} \leq 2\pi , \quad (5.3)$$

where $f \in L_2[0, 2\pi]$ and $(\tilde{r}, \tilde{s})$ are polar coordinates in $\tilde{C}$. Furthermore, (Goluzin [36, p. 384]),

$$\lim_{\tilde{Q} \to \tilde{P}_+(\tilde{s})} u(\tilde{Q}) = f(\tilde{s}), \quad (5.4)$$

for almost all $\tilde{s} \in [0, 2\pi]$, the limit being taken along all "non-tangential" paths.

Since χ is conformal at almost all points of C (Goluzin [36, p. 422]), χ maps the normal to C at $P(\phi^{-1}(\tilde{s}))$ into a non-tangental path at $\tilde{P}(\tilde{s})$ for almost all $\tilde{s}$. Hence it follows from (5.2c) and (5.4) that $f(\tilde{s}) = \tilde{g}(\tilde{s})$ for almost all $\tilde{s}$, so that, from (5.3),

$$\tilde{u}(\tilde{r}, \tilde{s}) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{g}(t) \frac{1 - \tilde{r}^2}{1 - 2\tilde{r}\cos(t - \tilde{s}) + \tilde{r}^2} \, dt . \quad (5.5)$$

Equations (5.4) and (5.5) imply the existence and uniqueness of $\tilde{u}$ satisfying (5.2) and hence the existence and uniqueness of u satisfying (5.1), so that Part (a) of the theorem has been proved.

Part (b) of the theorem follows from the fact that (Goluzin [36, p. 381]) if $\tilde{u}$ is given by (5.5) and $\tilde{g}$ is continuous at $\tilde{s}$ then

$$\lim_{\tilde{Q} \to \tilde{P}_+(\tilde{s})} u(\tilde{Q}) = \tilde{g}(\tilde{s})$$

where $\tilde{Q}$ tends to $\tilde{P}(\tilde{s})$ along any path in $\tilde{C}$.

To prove Part (c) of the theorem we follow Zygmund [138, p. 98]. Without loss of generality it may be assumed that $\tilde{s} = \phi(s) = 0$. Then $\tilde{g}(0+)$ and $\tilde{g}(2\pi-)$ exist and are equal, respectively, to $g(s+)$ and $g(s-)$. Set

$$d = \tilde{g}(0+) - \tilde{g}(2\pi-),$$

$$\tilde{\varphi}(\tilde{\theta}) = \begin{cases} \frac{1}{2}(\pi - \tilde{\theta}), & 0 < \tilde{\theta} < 2\pi, \\ 0, & \tilde{\theta} = 0 \text{ or } \tilde{\theta} = 2\pi, \end{cases}$$

$$\tilde{h}(\tilde{\theta}) = \begin{cases} \tilde{g}(\tilde{\theta}) - \frac{d}{\pi}\tilde{\varphi}(\tilde{\theta}), & 0 < \tilde{\theta} < 2\pi, \\ [\tilde{g}(0+) + \tilde{g}(2\pi-)]/2, & \tilde{\theta} = 0, 2\pi, \end{cases}$$

$$\tilde{v}(\tilde{r}, \tilde{\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(t) \frac{1 - \tilde{r}^2}{1 - 2\tilde{r}\cos(t - \tilde{\theta}) + \tilde{r}^2}\, dt .$$

Since $\tilde{h}$ is continuous at the point $\tilde{\theta} = 0, 2\pi$, it follows as in the proof of Part (b) that

$$\lim_{(\tilde{r}, \tilde{\theta}) \to (1,0)} \tilde{v}(\tilde{r}, \tilde{\theta}) = \tilde{h}(0) = [g(s+) + g(s-)]/2 .$$

Now, by direct computation,

$$\tilde{u}(\tilde{r}, \tilde{\theta}) = \tilde{v}(\tilde{r}, \tilde{\theta}) + \frac{d}{\pi} \arctan\left[\frac{\tilde{r}\sin\tilde{\theta}}{1-\tilde{r}\cos\tilde{\theta}}\right] .$$

Since the arctan in the above expression is equal to the angle between the negative x-axis and the line joining the point $(1, 0)$ to the point $(\tilde{r}, \tilde{\theta})$, it follows that $\tilde{u}(\tilde{r}, \tilde{\theta})$ tends to $[g(s+) + g(s-)]/2$ if $(\tilde{r}, \tilde{\theta})$ tends to $(1, 0)$ along a path which is tangential to the x-axis at $(1, 0)$. Since C has forward and backward tangents at $P(s)$, and since $P(s)$ is, by hypothesis, not an extruding cusp, it follows from the angle-preserving properties of conformal mappings at corners (Caratheodory [23, p. 91]) that χ maps the normal to C at $P(s)$ into a path which is tangential to the normal to $\tilde{C}$ at $(1, 0)$, that is, tangential to the x-axis at $(1, 0)$. Part (c) of the theorem follows.

Remark.

The hypothesis in Theorem 5.1 that u be bounded is necessary. To see this it suffices to consider the well-known example (Goluzin [36, p. 391]),

$$w(x, y) = \text{Real}\left(\frac{1+z}{1-z}\right), \quad z = x + iy.$$

w is harmonic in the unit disk. But w is equal to zero on the unit circle except at the point $x = 0$, $y = 1$, and w cannot be represented as a Poisson integral.

6. Some definitions and remarks.

In the remainder of the paper the solution of the generalized Dirichlet problem (5.1) by means of double-layer potentials will be considered. In the present section certain preliminaries are disposed of.

M will denote the space of bounded real-valued Lebesgue-measurable functions φ defined on $[0,S]$ with the maximum norm

$$\|\varphi\| = \sup_{0 \leq s \leq S} |\varphi(s)| . \tag{6.1}$$

M is a Banach space for, as is well-known (Halmos [42, p. 84]), if φ is the pointwise limit of a sequence of Lebesgue-measurable functions then φ is Lebesgue-measurable. Note that two functions φ_1, $\varphi_2 \in M$ are equal iff $\varphi_1(s) = \varphi_2(s)$ for all $s \in [0,S]$; therefore, M differs from the Lebesgue space $L_\infty[0,S]$ in which the elements are equivalence classes of functions.

The space M belongs to the class of spaces $B(s, \sum)$ discussed by Dunford and Schwartz [26, p. 240]. Indeed, $M = B([0,S], \mathfrak{L})$, where $\mathfrak{L}$ is the field of Lebesgue-measurable subsets of $[0,S]$. It should be noted that we can assert that $M = B([0,S], \mathfrak{L})$, instead of merely that $M \subset B([0,S], \mathfrak{L})$, because $\mathfrak{L}$ is a σ-field. Two basic papers on the properties of M are those of Vulich [132] and Fichtenholz and Kantorovitch [31]; further information will be found in Dunford and Schwartz [26, chapters IV and VI].

If $\varphi \in M$ then φ is integrable with respect to Π_Q and Π_s. For it is a consequence of Lemmas 3.5 and 3.6 that the σ-fields corresponding to Π_Q and Π_s coincide with the σ-field corresponding to Lebesgue measure. (see Dunford and Schwartz [26, p. 142]). Indeed, using the connection between Lebesgue-Stieltjes integrals and Lebesgue integrals implied by the Radon-Nikodym theorem (Riesz and Sz.-Nagy [113, p. 126], Dunford and Schwartz [26, pages 132, 180, and 181]), if $\varphi \in M$ then

$$\int_C \varphi(\sigma)\, \Pi_Q(d\sigma) = \int_C \varphi(\sigma) \frac{d}{d\sigma}[\omega_Q(\sigma)]\, d\sigma , \tag{6.2}$$

$$\left.\begin{aligned} &(T\varphi)(s) \\ &= \frac{1}{\pi} \int_C \varphi(\sigma)\Pi_s(d\sigma), \\ &= \frac{1}{\pi}[\psi(s,s+) - \psi(s,s-)]\,\varphi(s) + \frac{1}{\pi}\int_C \varphi(\sigma)\frac{d}{d\sigma}[\psi(s,\sigma)]\, d\sigma . \end{aligned}\right\} \tag{6.3}$$

As usual (Dunford and Schwartz [26, p. 218]) s is said to be a Lebesgue point of φ if

$$\lim_{|h| \to 0} \chi(h) = 0 , \tag{6.4}$$

where

$$\chi(h) = \frac{1}{h}\int_s^{s+h} |\varphi(s) - \varphi(\sigma)|\, d\sigma . \tag{6.5}$$

If $T: M \to M$, the Fredholm radius of T, Ω_T, is given by

$$\Omega_T = [\inf_{V \in \mathscr{V}} \|T - V\|]^{-1} ,$$

where $\mathscr{V}$ is the set of compact operators mapping M into M. Clearly,

$$\|T\|^{-1} \leq \Omega_T \leq \infty .$$

The concept of the Fredholm radius was introduced by Radon [109, p. 1114], and is discussed in Riesz and Sz.-Nagy [113]. It should be noted that Arbenz, to whose work [7] we often refer, defines the Fredholm radius of T to be $[\Omega_T]^{-1}$.

Several subspaces of M will be considered. Each of these subspaces is of course also equipped with the maximum norm and is a Banach space.

1. $\mathscr{B}$, the subspace of all bounded Borel-measurable functions. It should be noted (Halmos [42, p. 90]) that if $\varphi \in M$ then there exists $\varphi_1 \in \mathscr{B}$ such that $\varphi(s) = \varphi_1(s)$ except on a set of measure zero.

2. $\mathcal{C}$, the subspace of continuous functions.

3. The Baire spaces $\mathcal{B}_0, \mathcal{B}_1, \ldots$. These spaces are defined by induction:

   1. $\mathcal{B}_0 = \mathcal{C}$

   2. $\varphi \in \mathcal{B}_k$ iff there exists a sequence $\{\varphi_n\}$ such that

      a) $\varphi_n \in \mathcal{B}_{k-1}$, $n = 1, 2, \ldots$.

      b) $\lim\limits_{n \to \infty} \varphi_n(s) = \varphi(s)$, $0 \leq s \leq S$.

The most complete description of the properties of the Baire spaces of which we are aware is in the books of Hahn [40] and Natanson [92]. $\mathcal{B}_k$ is not separable for $k \geq 1$, and

$$\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k \ .$$

4. The subspace $\mathcal{D}$ which consists of functions $\varphi(s)$ such that

   (i) $\varphi(s+0)$ exists for $0 \leq s < S$,

   (ii) $\varphi(s-0)$ exists for $0 < s \leq S$,

   (iii) $f(s) = [f(s+0) + f(s-0)]/2$, for $0 < s < S$.

5. $\mathcal{S}$, the subspace of $\mathcal{D}$ consisting of step functions, that is functions $\varphi(s)$ such that there exist constants a_i and c_i for which

   $0 = a_0 < a_1 < \ldots < a_n = S$,

   $\varphi(s) = c_i$, for $s \in (a_{i-1}, a_i)$, $i \leq i \leq n$,

   and

   $\varphi(a_i) = (c_i + c_{i+1})/2$, $1 \leq i \leq n-1$.

The spaces $\mathcal{D}$ and $\mathcal{S}$ are considered by Hahn [40, 41]; see also Appendix C . $\mathcal{D}$ is of interest from the viewpoint of numerical analysis since $\mathcal{S}$ is dense in $\mathcal{D}$ or, to put it another way, $\mathcal{D}$ is the largest space whose elements can be approximated with arbitrary accuracy in the uniform norm by step functions.

7. Double-layer potentials: behaviour near the boundary.

If $u(Q)$ is the double-layer potential (1.4) then we set

$$\varphi_+(s) = \lim_{Q \to P_+(s)} u(Q),$$

and

$$\varphi_{(n)+}(s) = \lim_{Q \to P_+(s)} (n)\; u(Q),$$

if these limits exist.

Theorem 7.1.

Let $\varphi \in M$.

(i) If s is a point of continuity of φ then $\varphi_+(s)$ exists and

$$\varphi_+(s)/\pi = \varphi(s) + (T\varphi)(s).$$

(ii) If s is a Lebesgue point of φ and $P(s)$ is not an extruding cusp, then $\varphi_{(n)+}(s)$ exists and

$$\varphi_{(n)+}(s)/\pi = \varphi(s) + (T\varphi)(s).$$

Proof: Using (2.10),

$$u(Q) = \pi\,\varphi(s) + \pi\,(T\varphi)(s) + I(Q),$$

where

$$I(Q) = \int_C [\varphi(\sigma) - \varphi(s)]\,(\Pi_Q - \Pi_s)(d\sigma),$$

so that the theorem will follow if it can be shown that, in case (i),

$$\lim_{Q \to P_+(s)} I(Q) = 0,$$

and, in case (ii),

$$\lim_{Q \to P_+(s)} (n)\, I(Q) = 0.$$

Let $\epsilon(s)$, ρ, $B(s;\rho)$, $C_0 = C_0(s;\rho)$, and $C_1 = C_1(s;\rho)$ be as in Theorem 2.1. Set $\delta = |Q - P(s)|$.

Let $I_1(Q;\rho)$ and $I_0(Q;\rho)$ denote the splitting of $I(Q)$ corresponding to $C_1(s;\rho)$ and $C_0(s;\rho)$. Setting $P_1 = P(s)$ and $P_2 = Q$ in Lemma 3.2, it follows that if $\delta \le \rho/2$ then

$$|I_1(Q;\rho)| \le \int_{C_1} |\varphi(\sigma) - \varphi(s)|\,|d_\sigma[\omega_Q(\sigma) - \psi(s,\sigma)]| \;,$$

$$\le 8\,\|\varphi\|\, S\, |Q - P(s)| \,/\, \rho^2 \,. \tag{7.1}$$

The remainder of the proof differs for the two cases.

Case (i).

Using (7.1) and Theorems 2.1c, 2.2, 2.3,

$$|I(Q)|$$

$$\le \quad |I_1(Q;\delta^{\frac{1}{4}})| + |I_0(Q;\delta^{\frac{1}{4}})|,$$

$$\le \quad 8\|\varphi\|\; S\; \delta^{1/2} + \int_{C_0} |\varphi(\sigma) - \varphi(s)|(|\Pi_Q| + |\Pi_s|)(d\sigma)\,,$$

$$\le \quad 8\|\varphi\|\; S\; \delta^{1/2} + [\max_{\|s-\sigma\|\le 2\delta^{\frac{1}{4}}} |\varphi(\sigma) - \varphi(s)|] \int_C (|\Pi_Q| + |\Pi_s|)(d\sigma)\,,$$

$$\le \quad 8\,\|\varphi\|\; S\; \delta^{1/2} + 2[\max_{\|s-\sigma\|\le 2\delta^{\frac{1}{4}}} |\varphi(\sigma) - \varphi(s)|] \int_C |d\,\vartheta(\sigma)|\;.$$

Part (i) of the theorem follows.

Case (ii).

From (7.1),

$$\lim_{Q \to P_+(s)} (n) \quad I_1(Q; \epsilon(s)) = 0 ,$$

so that it suffices to prove that

$$\lim_{Q \to P_+(s)} (n) \ I_0 (Q ; \epsilon(s)) = 0 .$$

Since $P(s)$ is not an extruding cusp, Lemma 3.8 is applicable. Assume that

$$\delta \leq \min \{\rho_1/3, \ \epsilon(s)/3\}.$$

where ρ_1 is as in Lemma 3.8.

The circle $B(s; 2\delta)$ splits $C_0(s;\epsilon(s))$ into two parts, $C_2 = C_2(\delta)$ and $C_3 = C_3(\delta)$, where C_2 is an open subarc containing $P(s)$ and $C_3 = C_0 - C_2$. Let $I_2 (Q ; \delta)$ and $I_3 (Q ; \delta)$ denote the corresponding splitting of $I_0(Q; \epsilon(s))$.

Consider I_2. If $P(\sigma) \in C_2$ then it follows from Lemma 3.8 that

$$|Q - P(\sigma)| \geq k\delta$$

where k is a positive constant. Set $C_2' = C_2 - P(s)$. Then, using Lemmas 3.2 and 4.2,

$$|I_2 (Q ; \delta)|$$

$$\leq \int_{C_2'} |\varphi(\sigma) - \varphi(s)| \, |\Pi_s|(d\sigma) + \int_{C_2'} |\varphi(\sigma) - \varphi(s)| \, |\Pi_Q| (d\sigma) ,$$

$$\leq 2 \, \|\varphi\| \int_{C_2'} |\Pi_s|(d\sigma) + \frac{1}{k\delta} \int_{C_2} |\varphi(\sigma) - \varphi(s)| \, d\sigma ,$$

$$\leq 2 \, \|\varphi\| \int_{C_2'} |\Theta|(d\sigma) + \frac{4}{k}[\chi(4\delta) + \chi(-4\delta)] , \tag{7.2}$$

where χ is as in (6.5).

Now consider I_3. Let $P(\sigma) \epsilon C_3$ and consider the triangle $P(s)\,Q\,P(\sigma)$.

Then

$$
\begin{aligned}
|Q - P(\sigma)| &\geq |P(s) - P(\sigma)| - |Q - P(s)| , \\
&\geq |P(s) - P(\sigma)| / 2 .
\end{aligned}
$$

Therefore, using Lemma 3.2,

$|I_3 (Q ; \delta)|$

$$
\leq \int_{C_3} \frac{|\varphi(s) - \varphi(\sigma)| \quad |Q - P(s)|}{[\,|P(s) - P(\sigma)|/2]^2} \, d\sigma ,
$$

$$
\leq 16\delta \left| \int_{C_3} \frac{|\varphi(s) - \varphi(\sigma)|}{(s - \sigma)^2} \, d\sigma \right| ,
$$

$$
= 16\delta \left| \int_{C_3} \frac{d}{d\sigma} [(\sigma - s)\chi(\sigma - s)] \frac{d\sigma}{(\sigma - s)^2} \right| .
$$

Remembering that C_3 consists of two parts, one to the "left" and one to the "right" of $P(s)$,

$|I_3 (Q ; \delta)|$

$$
\leq 16\delta \left\{ \int_{-2\epsilon}^{-2\delta} + \int_{2\delta}^{2\epsilon} \frac{d}{dh} [h\chi(h)] \frac{dh}{h^2} \right\} ,
$$

$$
= 16\delta \int_{2\delta}^{2\epsilon} \frac{d}{dh} [h\{\chi(h) + \chi(-h)\}] \frac{dh}{h^2} ,
$$

$$
= 16\,\delta \{ [(\chi(h) + \chi(-h))/h\,]_{2\delta}^{2\epsilon} + 2\mu(\delta) \} ,
$$

$$
= 16\,\{ \delta\,[\chi(2\epsilon) + \chi(-2\epsilon)]/\,2\epsilon + [\chi(2\delta) + \chi(-2\delta)]\,/2 + 2\delta\,\mu(\delta) \} , \qquad (7.3)
$$

where

$$
\mu(\delta) = \int_{2\delta}^{2\epsilon} \frac{[\chi(h) + \chi(-h)]}{h^2} \, dh .
$$

From (7.2) and (7.3) it follows that

$$\lim_{Q \to P_+(s)} (n) \quad I_0(Q; \epsilon(s)) = 0$$

provided that

$$\lim_{\delta \to 0} \delta\, \mu(\delta) = 0 \,. \tag{7.4}$$

Equation (7.4) is easily established. If $\mu(\delta)$ is bounded as $\delta \to 0$ then (7.4) holds. On the other hand, if $\mu(\delta)$ is unbounded then, applying the rule of l'Hospital,

$$\lim_{\delta \to 0} \frac{\mu(\delta)}{(1/\delta)} = \lim_{\delta \to 0} \frac{-[\chi(2\delta) + \chi(-2\delta)]/2\delta^2}{-1/\delta^2} = 0 \,.$$

<u>Remarks</u>

1. The idea of splitting C_0 into two parts was suggested by the work of Günter [38, p. 52].
2. Similar results, for Lyapunov surfaces, were obtained by Fichera [30]; this work is reproduced by Günter [38, p. 108].
3. Related results have been obtained by Evans [27, 28].

8. Mapping properties of T.

In the present section the mapping properties of T are considered. It turns out that T has certain smoothing properties in that if $\varphi \in M$ then $T\varphi \in \mathcal{B}_1$, (Theorem 8.4) and $T\varphi$ is continuous except at the corners of C (Theorem 8.3). Without further assumptions no more can be said (Theorem 8.8). However, if φ and C have additional smoothness, then this is inherited by $T\varphi$ (Theorems 8.1, 8.5, 8.6, and 8.7).

Theorem 8.1.

T maps $\mathcal{C}$ into $\mathcal{C}$.

Proof: It is known (Riesz and Sz.-Nagy [113, p. 220]) that T maps $\mathcal{C}$ into $\mathcal{C}$ provided that the following conditions are satisfied:

(a) There is a constant m such that

$$\int_0^S |d_\sigma \psi(s, \sigma)| \leq m, \quad \text{for } s \in [0, S],$$

(b) For every $\xi \in [0, S]$,

$$\Psi(s) = \int_0^\xi \psi(s, \sigma) d\sigma$$

is a continuous function of s,

(c) $\psi(s, S)$ is a continuous function of s,

(d) $\psi(s, 0)$ is a continuous function of s.

(Condition (d) is necessary since to bring T into the form studied by Riesz and Sz.-Nagy, we set $\tau_s(\sigma) = \psi(s, \sigma) - \psi(s, 0)$, thereby ensuring that $\tau_s(0) = 0$.)

Conditions (a) and (b) follow from Theorem 2.3 and Lemma 3.1, respectively, while conditions (c) and (d) hold because $P(0) = P(S)$ is not a corner.

Remark.

Theorem 8.1 is due to Radon [110, p. 1142]. However, Radon's proof is somewhat indirect since he first shows that the double-layer potential (1.4) can be continuously extended to C, and then uses this fact to deduce that $T\varphi$ is continuous.

Let $\varphi \in M$, and $P(s) \in C$. In the next few theorems we generalize Theorem 8.1 by establishing conditions which ensure that the limit

$$\lim_{s_1 \to s} (T\varphi)(s_1)$$

exists. In order to avoid repetitions, we first prove an auxiliary lemma which states essentially that the behavior of $T\varphi$ near s depends only upon the properties of C and φ near s.

Lemma 8.2.

Let $\varphi \in M$ and $P(s)$ be a fixed point on C. Let

$$\rho = |P(s_1) - P(s)|^{1/4} < \epsilon(s) ,$$

where $\epsilon(s)$ is as in Theorem 2.1. Finally, let $B(s; \rho)$, $C_0 = C_0(s; \rho)$ and $C_1 = C_1(s; \rho)$ be as in Theorem 2.1.

Then

$$(T\varphi)(s_1) = (T\varphi)(s) + I(\varphi, s, s_1)/\pi + \xi(\varphi, s, s_1)/\pi$$

where

$$I(\varphi, s, s_1) = \int_{C_0} \varphi(\sigma)(\Pi_{s_1} - \Pi_s)(d\sigma),$$

and $\xi(\varphi, s, s_1) \to 0$ as $s_1 \to s$.

<u>Proof</u>: Straightforward manipulation shows that

$$(T\varphi)(s_1) = (T\varphi)(s) + I(\varphi, s, s_1)/\pi + \xi(\varphi, s, s_1)/\pi,$$

where

$$\xi(\varphi, s, s_1) = \int_{C_1} \varphi(\sigma)(\Pi_{s_1} - \Pi_s)(d\sigma) .$$

It may be assumed that s_1 is so close to s that

$$\rho^4 = |P(s_1) - P(s)| \leq \rho/2 .$$

Then, by Corollary 3.4,

$$\begin{aligned} &|\xi(\varphi, s, s_1)| \\ &\leq \|\varphi\| \int_{C_1} |d_\sigma[\psi(s_1, \sigma) - \psi(s, \sigma)]|, \\ &\leq \|\varphi\| S \rho^4/[\rho/2]^2 = 4\|\varphi\| S\rho^2 . \end{aligned}$$

The lemma follows.

<u>Theorem 8.3.</u>

If $\varphi \in M$ then $T\varphi$ is continuous except possibly at the corners of C.

<u>Proof</u>: Let $P(s) \in C$ be a point which is not a corner point. With the notation of Lemma 8.2, it is sufficient to prove that

$$\lim_{s_1 \to s} I(\varphi, s, s_1) = 0.$$

Using Lemma 4.2,

$$|I(\varphi, s, s_1)|$$

$$\leq \|\varphi\| \int_{C_0} (|\Pi_{s_1}| + |\Pi_s|)(d\sigma)$$

$$\leq 2\|\varphi\| \chi(\rho),$$

where $\chi(\rho) = \int_{C_0} |\Theta|(d\sigma)|$.

For s_1 close to s,

$$\chi(\rho) \leq \int_{s-2\rho}^{s+2\rho} |d\vartheta(\sigma)|,$$

so that, since ϑ is continuous at s, $\chi(\rho) \to 0$ as $\delta \to 0$ (Natanson [92, p. 226]). The theorem follows.

Theorem 8.4.

T maps M into $\mathscr{B}_1$.

Proof: Let $\varphi \in M$ and $g = T\varphi$.

Then g is bounded. Furthermore, it follows from Theorem 8.3 that g has only a denumerable number of points of discontinuity because C has only a denumerable number of corners. But it is known (Natanson [92, p. 407]) that any bounded function with only a denumerable number of points of discontinuity belongs to $\mathscr{B}_1$.

Theorem 8.5.

Let $\varphi \in M$. If $\varphi(s-)$ $\{\varphi(s+)\}$ exists then $(T\varphi)(s+)$ $\{T\varphi(s-)\}$ exists and satisfies:

$$(T\varphi)(s+) = (T\varphi)(s) + [\vartheta(s+) - \vartheta(s-)][\varphi(s-) - \varphi(s)]/\pi,$$

$$\{(T\varphi)(s-) = (T\varphi)(s) + [\vartheta(s+) - \vartheta(s-)][\varphi(s+) - \varphi(s)]/\pi\} .$$

Proof: Set $a = \varphi(s-)$. Then, with the notation of Lemma 8.2 it suffices to prove that

$$\lim_{s_1 \to s+} I(\varphi, s, s_1) = [\vartheta(s+) - \vartheta(s-)][a - \varphi(s)]$$

Set

$$C_+ = \{P(\sigma) \in C_0;\ \sigma > s\},$$

$$C_- = \{P(\sigma) \in C_0;\ \sigma < s\} .$$

Throughout the proof it will be assumed that $s_1 > s$.

Then,

$$I(\varphi, s, s_1) = \sum_{k=1}^{4} I_k(s_1) ,$$

where,

$$I_1(s_1) = \int_{C+} \varphi(\sigma)(\Pi_{s_1} - \Pi_s)(d\sigma) - \int_{C-} \varphi(\sigma)\Pi_s(d\sigma) ,$$

$$I_2(s_1) = \int_{\{P(s)\}} \varphi(\sigma)(\Pi_{s_1} - \Pi_s)(d\sigma) ,$$

$$I_3(s_1) = \int_{C-} a\Pi_{s_1}(d\sigma),$$

$$I_4(s_1) = \int_{C-} (\varphi(\sigma) - a)\Pi_{s_1}(d\sigma) .$$

Consider I_1. Using Lemma 4.2,

$$|I_1(s_1)| \leq 2\,\|\varphi\| \int_{C+} |\tilde{\Theta}|(d\sigma) + \|\varphi\| \int_{C-} |\Theta|(d\sigma) .$$

It follows from Lemma 4.3 that

$$\lim_{s_1 \to s+} |I_1(s_1)| = 0 .$$

Clearly,

$$I_2(s_1) = -[\vartheta(s+) - \vartheta(s-)]\varphi(s) .$$

Consider I_3.

$$\begin{aligned} I_3(s_1) &= a\int_{C-} \Pi_{s_1}(d\sigma) , \\ &= a \int_{C-} d_\sigma \psi(s_1, \sigma) , \\ &= a\,[\psi(s_1, s) - \psi(s_1, s_2)], \end{aligned}$$

where $P(s_2)$ is the "left" end of C_-. Since $|P(s_2) - P(s)| = \rho$ and $|P(s_2) - P(s_1)| = \rho^4$,

$$\lim_{s_1 \to s+} [\psi(s, s_2) - \psi(s_1, s_2)] = 0 .$$

Therefore

$$\lim_{s_1 \to s+} I_3(s_1) = a \lim_{s_1 \to s+} [\psi(s, s_1) - \psi(s, s_2) + \psi(s, s_2) - \psi(s_1, s_2)],$$

$$= a\, [\psi(s, s+) - \psi(s, s-)],$$

$$= a[\vartheta(s+) - \vartheta(s-)] .$$

Finally, consider I_4. Given $\epsilon > 0$ there is a ρ_1 such that if $\rho \leq \rho_1$ then $|\varphi(s_3) - a| \leq \epsilon$ for $P(s_3) \in C_-$. Thus, if $\rho \leq \rho_1$,

$$|I_4(s_1)| \leq \epsilon \int_{C-} |\Pi_{s_1}|(d\sigma) ,$$

so that

$$\lim_{s_1 \to s+} I_4(s_1) = 0.$$

Combining the above results, the theorem follows.

<u>Theorem 8.6.</u>

T maps $\mathcal{D}$ into $\mathcal{D}$.

<u>Proof:</u> Follows immediately from Theorem 8.5.

<u>Remark.</u>

Theorem 8.6 is due originally to Hahn [41]; see also Kaltenborn [52], and Hildebrandt [46].

<u>Theorem 8.7.</u>

Let $P(s) \in C$ be such that

(i) For some $\delta_1 > 0$, ϑ is continuously differentiable on $[s - \delta_1, s)$ $\{(s, s + \delta_1]\}$.

(ii) There is a constant φ_- $\{\varphi_+\}$ such that

$$\lim_{|h|\to 0} \chi_-(|h|) = 0 \qquad \{\lim_{|h|\to 0} \chi_+(|h|) = 0\},$$

where

$$\chi_-(h) = \left|\frac{1}{h}\int_s^{s-|h|} |\varphi(\tau) - \varphi_-|d\tau\right|$$

$$\{\chi_+(h) = \left|\frac{1}{h}\int_s^{s+|h|} |\varphi(\tau) - \varphi_+|d\tau\right|\}.$$

(iii) $P(s)$ is not a cusp.

Then $(T\varphi)(s+)$ $\{T\varphi(s-)\}$ exists and satisfies,

$$(T\varphi)(s+) = (T\varphi)(s) + [\vartheta(s+) - \vartheta(s-)][\varphi_- - \varphi(s)]/\pi,$$

$$\{(T\varphi)(s-) = (T\varphi)(s) + [\vartheta(s+) - \vartheta(s-)][\varphi_+ - \varphi(s)]/\pi\}.$$

Proof: Set $a = \varphi_-$. Then the proof is the same as the proof of Theorem 8.5 except for the arguments concerning I_4.

For sufficiently small ρ, ϑ is continuous on C_-. Hence (Natanson [92, p. 269])

$$I_4(s_1) = \int_{C_-} \beta_{s_1}(\sigma)d\alpha(\sigma),$$

where

$$\alpha(\sigma) = \int_s^\sigma [\varphi(\tau) - a]d\tau,$$

$$\beta_{s_1}(\sigma) = \frac{d}{d\sigma}\, \psi(s_1, \sigma)\,,$$

$$= \frac{\cos \phi_{P(s_1)}(\sigma)}{|P(s_1)-P(\sigma)|}\,, \quad \text{(see Figure 1.1)}$$

$$= \frac{[x(\sigma) - x(s_1)]\sin\vartheta(\sigma) - [y(\sigma) - y(s_1)]\cos\vartheta(\sigma)}{[x(\sigma) - x(s_1)]^2 + [y(\sigma) - y(s_1)]^2}\,.$$

Direct computation yields,

$$\left|\frac{d}{d\sigma}\, \beta_{s_1}(\sigma)\right|$$

$$= \left|\frac{[x(\sigma) - x(s_1)]\cos\vartheta(\sigma) + [y(\sigma) - y(s_1)]\sin\vartheta(\sigma)}{[x(\sigma) - x(s_1)]^2 + [y(\sigma) - y(s_1)]^2}\,[\vartheta'(\sigma) - 2\beta_{s_1}(\sigma)]\right|$$

$$\leq \frac{|\vartheta'(\sigma)| + 2|\beta_{s_1}(\sigma)|}{|P(\sigma) - P(s_1)|}\,.$$

Let $P(\sigma) \in C_-$, and let $\measuredangle\, P(\sigma)P(s)P(s_1)$ be the angle subtended at $P(s)$ by $P(\sigma)$ and $P(s_1)$. Since

$$x(\sigma) - x(s) = (\sigma - s)\cos\vartheta(s-) + o(|s - \sigma|),$$

$$y(\sigma) - y(s) = (\sigma - s)\sin\vartheta(s-) + o(|s - \sigma|),$$

$$x(s_1) - x(s) = (s_1 - s)\cos\vartheta(s+) + o(|s - \sigma|),$$

$$y(s_1) - y(s) = (s_1 - s)\sin\vartheta(s+) + o(|s - \sigma|)\,,$$

it follows that, for small ρ,

$$|\sin \measuredangle\, P(\sigma)P(s)P(s_1)| \geq \frac{1}{2} \sin|\vartheta(s+) - \vartheta(s-)|,$$

$$> 0 .$$

Considering the triangle $P(s_1)P(s)P(\sigma)$, one finds that

$$|P(\sigma) - P(s_1)| = \frac{|P(\sigma) - P(s)|}{\sin \measuredangle\, P(\sigma)P(s_1)P(s)} \cdot \sin \measuredangle\, P(\sigma)P(s)P(s_1) ,$$

$$\geq \frac{1}{2} \; |P(\sigma) - P(s)| \; \sin|\vartheta(s+ - \vartheta(s-)| ,$$

$$\geq A|s - \sigma|,$$

where $A = \sin[\vartheta(s+) - \vartheta(s-)]/4 > 0$.

Therefore,

$$|\beta_{s_1}(\sigma)| \leq \frac{1}{|P(s_1) - P(\sigma)|} \leq \frac{1}{A|s - \sigma|} ,$$

$$|\frac{d}{d\sigma} \; \beta_{s_1}(\sigma)| \leq \frac{|\vartheta'(\sigma)| + 2|\beta_{s_1}(\sigma)|}{A|s - \sigma|} .$$

Furthermore, by hypothesis,

$$|\alpha(\sigma)| \leq |s - \sigma|\chi(|s - \sigma|) \leq |s - \sigma|\chi(2\rho).$$

Since both β_{s_1} and α are continuous and of bounded variation we may integrate by parts to obtain (Dunford and Schwartz [26, p. 154]),

$$I_4(s_1) = [\alpha(\sigma)\beta_{s_1}(\sigma)]_{\sigma=s_2+}^{\sigma=s-} - \int_{C_-}\alpha(\sigma) \; \frac{d}{d\sigma}[\beta_{s_1}(\sigma)]d\sigma ,$$

where $P(s_2)$ is the "left" end of C_-. Consequently,

$$
\begin{aligned}
&|I_4(s_1)| \\
&\le \chi(2\rho)[1 + \int_{C-} [|\vartheta'(\sigma)| + 2|\beta_{s_1}(\sigma)|]d\sigma]/A , \\
&= \chi(2\rho)[1 + \int_{C-} |d\vartheta(\sigma)| + \int_{C-} |d_\sigma \psi(s_1, \sigma)|]/A,
\end{aligned}
$$

so that

$$\lim_{s_1 \to s+} I_4(s_1) = 0 .$$

The theorem follows.

Remarks.

1. The proof is modelled after the proof of a similar theorem given by Dunford and Schwartz [26, p. 219].

2. Condition (ii) of the theorem might perhaps be paraphrased as follows: s is a left {right} **Lebesgue** point of φ.

Theorem 8.8.

Let $\mathfrak{R}$ be the right-angled triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$.

Then T does not map $\mathcal{B}_1$ into $\mathcal{D}$.

Proof: To prove the theorem, a function $\varphi \in \mathcal{B}_1$ will be constructed such that $T\varphi \notin \mathcal{D}$.

Let d and D be constants such that

$$0 < d < 1 < D ,$$

and

$$\int_d^D \frac{dz}{1+z^2} = 9\,I\,/10,$$

where

$$I = \int_0^\infty \frac{dz}{1+z^2}\ .$$

Set

$$\delta = d/D,$$

$$a_k = d\delta^{k-1}, \quad k \geq 1,$$

$$g(y) = \begin{cases} 1, & \text{if } \delta^{2i+1} \leq y \leq \delta^{2i}, \quad i = 0, 1, 2, \ldots, \\ 0, & \text{if } \delta^{2i+2} < y < \delta^{2i+1}, \quad i = 0, 1, 2, \ldots, \\ 0, & \text{if } y = 0, \end{cases}$$

$$\varphi(P) = \begin{cases} g(y), & \text{if } P = (0, y) \text{ and } 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then φ has only a countable number of discontinuities so that (Natanson [92, p. 407]) $\varphi \in \mathscr{B}_1$.

Let $P(0) = (0, 0)$. (It is convenient to ignore the condition that $P(0)$ should not be a corner.) Then, for $0 < x \leq 1$, $P(x) = (x, 0)$ and

$$(T\varphi)(x)$$

$$= \int_C \varphi(\sigma) \frac{\cos \phi_{P(x)}(\sigma)}{|P(x) - P(\sigma)|} d\sigma ,$$

$$= \int_0^1 g(y) \frac{x}{x^2 + y^2} dy .$$

Setting $z = x/y$,

$$(T\varphi)(x) = \int_x^\infty g\left[\frac{x}{z}\right] \frac{1}{1+z^2} dz .$$

Therefore,

$$(T\varphi)(a_k)$$

$$= \int_{a_k}^\infty g\left[\frac{a_k}{z}\right] \frac{dz}{1+z^2} ,$$

$$= \int_{a_k}^d + \int_d^D + \int_D^\infty g\left[\frac{a_k}{z}\right] \frac{dz}{1+z^2} ,$$

$$= I_1(k) + I_2(k) + I_3(k) \quad \text{(say)}.$$

Since $|g| \leq 1$,

$$|I_1(k)| + |I_3(k)| \leq I/10.$$

On the other hand, since $a_k/d = \delta^{k-1}$ and $a_k/D = \delta^k$,

$$I_2(k) = \begin{cases} 9I/10, & \text{if } k = 2i+1, \\ 0, & \text{if } k = 2i+2. \end{cases}$$

Therefore,

$$(T\varphi)(a_k) \geq 8I/10 \text{ if } k \text{ is odd,}$$

and

$$(T\varphi)(a_k) \leq I/10 \text{ if } k \text{ is even.}$$

Consequently, $(T\varphi)(0+)$ does not exist and $\varphi \notin \mathcal{D}$.

9. The integral equation.

Theorem 9.1.

The generalized Dirichlet problem (5.1) is solved by the double-layer potential (1.4) provided that $\varphi \in M$ and

$$(I + T)\varphi = g/\pi. \tag{9.1}$$

Proof: Assume that $\varphi \in M$ and that (9.1) holds. Since $s \in [0,S]$ is a Lebesgue point of φ for almost all s (Natanson [92, p. 258]) it follows from (9.1) and Theorem 7.1 that, for almost all s, $\varphi_{(n)+}(s)$ exists and satisfies

$$\varphi_{(n)+}(s) = g(s).$$

Noting Theorem 5.1, the theorem follows.

Theorem 9.2.

$$\Omega_T = \pi/\theta \qquad \text{where}$$

$$\theta = \max_{s \in [0,S]} |\vartheta(s+) - \vartheta(s-)|.$$

Furthermore, given $\epsilon > 0$ there exists a compact degenerate operator of the form

$$(V\varphi)(s) = \frac{1}{\pi} \sum_{k=1}^{n} c_k(s) \int_C \varphi(\sigma)\, d_\sigma \bar{\psi}_k(\sigma), \tag{9.2}$$

where c_k, $\bar{\psi}_k \in \mathcal{C}$, and $\bar{\psi}_k$ is of bounded variation, such that

$$\|T - V\| \le (\theta + \epsilon)/\pi. \tag{9.3}$$

Proof: As in Lemma 3.5 let

$$\bar{\psi}(s,\sigma) = \begin{cases} \psi(s,\sigma), & \sigma < s, \\ \psi(s,s-), & \sigma = s, \\ \psi(s,\sigma) - [\psi(s,s+) - \psi(s,s-)], & \sigma > s. \end{cases}$$

Then $\bar{\psi}$ is continuous and of bounded variation as a function of σ. Let $\bar{\Pi}_s$ be the measure corresponding to $\bar{\psi}$.

Given $\epsilon > 0$ it follows from Lemma 4.4 that there is an $\eta(s) > 0$ such that

$$\int_C |\Pi_t - \bar{\Pi}_s|(d\sigma) \leq \theta + \epsilon ,$$

for all t satisfying $\|s - t\| < \eta(s)$. Set

$$I_s = \{t \in [0,S] ; \ s - \eta(s) < t < s + \eta(s)\} .$$

The intervals I_s cover $[0,S]$. Therefore, by the theorem of Heine-Borel, there is a finite set of intervals, $\{I_{s_k}\}$, $k = 1,2,\ldots,n$, which covers $[0,S]$. Clearly, it may be assumed that $s_k < s_{k+1}$ and that $I_{s_k} \subset I_{s_j}$ iff $k=j$.

Now let $c_k(s)$, $1 \leq k \leq n$, be the continuous functions such that

$$c_k(s) = \begin{cases} 0, & \text{if } s \notin I_{s_k} , \\ 1, & \text{if } s \in I_{s_k} - \bigcup\limits_{\substack{i=1 \\ i \neq k}}^{n} I_{s_i} , \\ \text{linear, otherwise} . \end{cases}$$

That is, the functions $c_k(s)$ are a partition of unity on $[0,S]$ subordinate to the cover $\{I_{s_k}\}$.

Let V be the mapping

$$(V\varphi)(s) = \frac{1}{\pi} \sum_{k=1}^{n} c_k(s) \int_C \varphi(\sigma) \bar{\Pi}_{s_k}(d\sigma) .$$

Then V is a degenerate compact operator of the form (9.2) (with $\bar{\psi}_k(\sigma) = \bar{\psi}(s_k, \sigma)$) and V satisfies (9.3).

Since ϵ is arbitrary, it follows from (9.3) that

$$\Omega_T \geq \pi / \theta ,$$

so that it remains only to show that

$$\Omega_T \leq \pi / \theta .$$

Suppose that, on the contrary, $\Omega_T > \pi / \theta$. Then there exists an $\eta > 0$ and a compact operator V such that if $B = T - V$ then

$$\|B\| \leq (\theta - \eta) / \pi .$$

Let s_0 be such that

$$\theta = | \vartheta(s_0+) - \vartheta(s_0-)| .$$

Set

$$s_n = s_0 + 1/n ,$$

$$\varphi_n(\sigma) = \begin{cases} 1, & s_0 \leq \sigma \leq s_n , \\ 0, & \text{otherwise} , \end{cases}$$

$$E_n = [s_0, s_0 + 1/n] ,$$

for $n = 1, 2, \ldots$. Since V is compact, the sequence $\{V \varphi_n\}$ must contain a convergent subsequence $\{V \varphi_{n_k}\}$. In particular, there must exist N such that

$$\|V \varphi_{n_k} - V\varphi_{n_\ell}\| \leq \eta/(4\pi) , \quad \text{if} \quad k, \ell \geq N .$$

We shall obtain a contradiction by showing that this is impossible.

Choose $n_k \geq N$ such that

$$|\psi(s_0, s_{n_k}) - \psi(s_0, s_0+)| \leq \eta / 10 .$$

Choose $\bar{s} < s_0$ such that

$$|\psi(s_0, \bar{s}) - \psi(s_0, s_0-)| \leq \eta / 10 ,$$

and

$$|\psi(\bar{s}, s_{n_k}) - \psi(s_0, s_{n_k})| \leq \eta / 10 .$$

Finally, choose $n_\ell \geq N$ such that

$$|\psi(s_{n_\ell}, \bar{s}) - \psi(s_0, \bar{s})| \leq \eta / 10 .$$

Next, observe that

$$\|B[\varphi_{n_k} - \varphi_{n_\ell}]\| \le \|B\| \, \|\varphi_{n_k} - \varphi_{n_\ell}\| \le (\theta - \eta)/\pi .$$

Now,

$$(T\varphi_n)(\bar{s}) = \frac{1}{\pi}\int_C \varphi_n(\sigma)\Pi_{\bar{s}}(d\sigma),$$

$$= \frac{1}{\pi}\,\Pi_{\bar{s}}(E_n),$$

$$= \frac{1}{\pi}[\psi(\bar{s}, s_n) - \psi(\bar{s}, s_0)] .$$

Therefore,

$$\|T\varphi_{n_k} - T\varphi_{n_\ell}\|$$

$$\ge \left|(T\varphi_{n_k})(\bar{s}) - (T\varphi_{n_\ell})(\bar{s})\right| ,$$

$$= \frac{1}{\pi}\left|\psi(\bar{s}, s_{n_k}) - \psi(\bar{s}, s_{n_\ell})\right| ,$$

$$\ge \frac{1}{\pi}\{ \left|\psi(s_0, s_0) - \psi(s_0, s_0-)\right| -$$

$$- \left|\psi(s_0, s_0) - \psi(s_0, s_{n_k})\right| - \left|\psi(s_0, s_{n_k}) - \psi(\bar{s}, s_{n_k})\right| -$$

$$- \left|\psi(s_0, s_0-) - \psi(s_0, \bar{s})\right| - \left|\psi(s_0, \bar{s}) - \psi(s_{n_\ell}, \bar{s})\right|\} ,$$

$$\ge [\theta - \eta/2]/\pi .$$

Hence ,

$$\| V\varphi_{n_k} - V\varphi_{n_\ell} \|$$

$$\geq \| T\varphi_{n_k} - T\varphi_{n_\ell} \| - \| B\varphi_{n_k} - B\varphi_{n_\ell} \| ,$$

$$\geq [\theta - \eta/2]/\pi - (\theta - \eta)/\pi$$

$$= \eta/(2\pi) ,$$

which is the desired contradiction.

Remarks

1. Let $T_{\mathcal{C}}$ denote the restriction of T to $\mathcal{C}$. Then by Theorem 8.1 , $T_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Theorem 9.2 , with T replaced by $T_{\mathcal{C}}$, was stated by Radon [110, p. 1149 ; 109 , p. 1110 and p. 1121] , but I am unable to follow the arguments of Radon [110, p. 1149] which purport to show that $\Omega_{T_{\mathcal{C}}} \leq \pi/\theta$. The same case was considered by Arbenz [7, p. 15], but Arbenz asserts only that $\Omega_{T_{\mathcal{C}}} \geq \pi/\theta$. Our proof uses ideas from both Arbenz and Radon.

2. It might appear that the theorem would follow immediately from the results for $T_{\mathcal{C}}$. This does not appear to be the case. For example, the operator V_s ,

$$(V_s \varphi)(\sigma) = \begin{cases} \varphi(s) , & \sigma = s , \\ 0 , & \text{otherwise} , \end{cases}$$

is a compact mapping of M into itself, but does not map $\mathcal{C}$ into $\mathcal{C}$.

3. The construction of the splitting $T = V + B$ in the proof is the same as that used by Arbenz. Radon used the following splitting . If

$$C_{t,\delta} = \{P(\sigma) : \|P(t) - P(\sigma)\| \leq \delta\} ,$$

let V_δ and B_δ be the operators ,

$$(V_\delta \varphi)(s) = \int_{C-C_{s,\delta}} \varphi(\sigma) \ \Pi_s (d\sigma) \ ,$$

$$(B_\delta \varphi)(s) = \int_{C_{s,\delta}} \varphi(\sigma) \ \Pi_s (d\sigma) \ .$$

Then $T = V_\delta + B_\delta$. It can be shown that V_δ is compact and that, given $\epsilon > 0$ there exists $\delta_0 > 0$ such that $\|B_\delta\| \leq (\theta + \epsilon)/\pi$ if $\delta \leq \delta_0$. The splitting of Radon is more elegant than that used in our proof, but has the disadvantage that it does not lead directly to the degenerate operators V of (9.2). It is interesting that Carleman [24, p. 12] used a splitting of T similar to that used by Radon.

Theorem 9.3.

Assume that $g \in M$ and that C has no cusps. Then there exists a unique $\varphi \in M$ such that (9.1) holds.

Proof: Consider the equation

$$(I + \lambda T) \varphi = g/\pi, \tag{9.4}$$

where λ is a complex-valued parameter. It is known (Radon [109, p. 1113 and p. 1119], Riesz and Sz.-Nagy [113, p. 217]) that the Fredholm alternative holds for (9.4) provided that $|\lambda| < \Omega_T$. From Theorem 9.2, $\Omega_T = \pi/\theta > 1$, so that the Fredholm alternative certainly holds when $\lambda = 1$. To prove the theorem it therefore suffices to show that if $\varphi_1 \in M$ satisfies

$$(I + T) \varphi_1 = 0 \tag{9.5}$$

then $\varphi_1 = 0$.

Assume that φ_1 satisfies (9.5). By virtue of Theorem 9.2 , there is a splitting ,

$$T = V + B ,$$

where V is of the form (9.2) and $\|B\| < 1$. Thus,

$$(I + B)\,\varphi_1 = \varphi_2 , \tag{9.6}$$

where $\varphi_2 = - V \varphi_1$. Since $\|B\| < 1$, (9.6) is solved by a Neumann series,

$$\varphi_1 = \sum_{k=0}^{\infty} (-1)^k \; B^k \varphi_2 . \tag{9.7}$$

Now it follows from (9.2) that $V : M \to \mathcal{C}$. In particular, $\varphi_2 \in \mathcal{C}$. Moreover, by Theorem 8.1, $T : \mathcal{C} \to \mathcal{C}$. Consequently, $B : \mathcal{C} \to \mathcal{C}$. Therefore, the series (9.7) is a uniformly convergent series of continuous functions. It follows that (9.7) converges to a continuous function.

It has therefore been shown that if $\varphi_1 \in M$ satisfies (9.5) then $\varphi_1 \in \mathcal{C}$. But, it is known (Radon [110, p. 1167], Arbenz [7, p. 20]) that if $\varphi_1 \in \mathcal{C}$ satisfies (9.5) then $\varphi_1 = 0$. The proof of the theorem is therefore complete.

10. The integral equation when C has cusps.

The results of the preceding section leave open the question of what happens in the case when C has cusps. It is this case which is studied in the present section.

It is easy to show that if C has an intruding cusp then (9.1) no longer has a unique solution. For (Radon [110, p. 1149]) let C have an intruding cusp at $P(s_1)$ and set

$$\varphi(\sigma) = \begin{cases} 1, & \sigma = s_1, \\ 0, & \text{otherwise}. \end{cases}$$

Then it follows from (2.8) that

$$(I + T)\varphi = 0 .$$

That is, $\lambda = -1$ is an eigenvalue of T.

The case when C has an extruding cusp is less easy to handle, and for a time the author believed that in this case $(I + T)^{-1}$ exists and is bounded. However, the following theorem shows that this is not the case.

Theorem 10.1.

Let C have an extruding cusp. Then there exists a sequence $\{\varphi_n\}$, $n = 1, 2, \ldots$, such that $\varphi_n \in M$, $\|\varphi_n\| = 1$, but $\|(I + T)\varphi_n\| \to 0$ as $n \to \infty$.

Proof: Let C have a cusp at $P(s)$. Set

$$E_{n+} = \{P(\sigma);\ s < \sigma \leq s + 1/n\},$$

$$E_{n-} = \{P(\sigma);\ s - 1/n \leq \sigma < s\},$$

$$E_n = E_{n+} \cup E_{n-},$$

$$\varphi_n(\sigma) = \begin{cases} n(\sigma - s) + 1, & \text{if } P(\sigma) \in E_{n-}, \\ n(\sigma - s) - 1, & \text{if } P(\sigma) \in E_{n+}, \\ 0, & \text{otherwise}, \end{cases}$$

for $n = 1, 2, \ldots$.

The notation of Theorem 2.1 and Lemma 3.7 will be used, and it will be assumed that

$$\frac{2}{n} \leq \frac{1}{\sqrt{n}} \leq \epsilon_1(s) .$$

It will be proved that

$$\|(I + T)\varphi_n\|$$

$$= \sup_{t \in C} |\varphi_n(t) + (T\varphi_n)(t)| ,$$

$$\leq \frac{1}{\pi} \int_{E_n} |\Theta|(d\sigma) + \frac{8}{\pi\sqrt{n}} + p_n , \qquad (10.1)$$

where p_n (which will be defined later) depends only upon n and is such that $p_n \downarrow 0$ as $n \to \infty$. Since $\varphi_n \in M$ and $\|\varphi_n\| = 1$, the theorem follows from (10.1) with the aid of Lemma 4.3.

Since $\psi(t, \sigma)$ is continuous on the right and φ_n is continuous on E_{n-}, it follows that (Dunford and Schwartz [26, p. 154])

$$\frac{1}{\pi} \int_{E_{n-}} \varphi_n(\sigma)\, d_\sigma \psi(t, \sigma)$$

$$= \frac{1}{\pi} [\varphi_n(\sigma)\, \psi(t, \sigma)]_{\sigma = (s - 1/n)+}^{\sigma = s-} - \frac{n}{\pi} \int_{E_{n-}} \psi(t, \sigma)\, d\sigma ,$$

$$= \frac{1}{\pi}\, \psi(t, s-) - \frac{n}{\pi} \int_{E_{n-}} \psi(t, \sigma)\, d\sigma ,$$

$$= I_-(n, t) ,$$

where

$$I_-(n,t) = \frac{n}{\pi} \int_{E_{n-}} [\psi(t, s-) - \psi(t, \sigma)]\, d\sigma .$$

Similarly,

$$\frac{1}{\pi} \int_{E_{n+}} \varphi_n(\sigma)\ d_\sigma\ \psi(t,\sigma) = I_+(n,\ t)\ ,$$

where

$$I_+(n,t) = \frac{n}{\pi} \int_{E_{n+}} [\psi(t, s+) - \psi(t,\sigma)]\, d\sigma\ .$$

Consequently,

$$(T\ \varphi_n)(t)$$

$$= \frac{1}{\pi} \int_{E_n} \varphi_n(\sigma)\ \Pi_t(d\ \sigma)\ ,$$

$$= \frac{1}{\pi} \int_{E_n} \varphi_n(\sigma)\, d_\sigma\, \psi(t,\ \sigma)\ ,$$

$$= I_-(n,t) + I_+(n,t)\ .$$

To establish (10.1) several cases must be considered:

<u>Case 1 : $P(t) \epsilon C_1 (s\ ;\ 1 / \sqrt{n}\ )$</u>

Applying Lemma 3.2, if $P(\sigma) \epsilon E_n$ then

$$|\psi(t, s) - \psi(t,\sigma)|$$

$$\leq \int_{E_n} |d_\tau\ \psi(t\ ,\ \tau)|\ ,$$

$$\leq [2 / n] / [1 / \sqrt{n} - \frac{1}{n}],$$

$$\leq 4 / \sqrt{n}\ .$$

Therefore,

$$
\begin{aligned}
&|(T\,\varphi_n)(t)| \\
&\leq \quad |I_+(n,t)| + |I_-(n,t)|\,, \\
&\leq \quad \frac{n}{\pi} \int_{E_n} |\,\psi(t,s) - \psi(t,\sigma)|\, d\sigma\,, \\
&\leq \quad \frac{n}{\pi} \int_{E_n} [4 / \sqrt{n}\,]\, d\sigma\,, \\
&= \quad 8 / (\pi\sqrt{n})\,.
\end{aligned}
$$

Since $\varphi_n(t) = 0$, (10.1) holds .

<u>Case 2 : $P(t) = P(s)$</u>

Then, using Lemma 4.2 ,

$$
\begin{aligned}
&|(T\,\varphi_n)(t)| \\
&= \left|\, \frac{1}{\pi} \int_{E_n} \varphi_n(\sigma)\, \Pi_s\,(d\,\sigma) \,\right| \\
&\leq \frac{1}{\pi} \int_{E_n} |\Pi_s|(d\,\sigma)\,, \\
&\leq \frac{1}{\pi} \int_{E_n} \left| \bigcirc\!\!\!\!\!\sim \right| (d\sigma)\,.
\end{aligned}
$$

Since $\varphi_n(t) = 0$, (10.1) holds.

Case 3 : $P(t) \in C_+(s;\ 1/\sqrt{n})$

Using Lemma 4.2 ,

$$|I_+(n,t)|$$

$$\le \frac{1}{\pi} \int_{E_{n+}} |\Pi_t|(d\sigma) \quad ,$$

$$\le \frac{1}{\pi} \int_{E_{n+}} |\tilde{\bigcirc}|\ (d\sigma)\ ,$$

$$\le \frac{1}{\pi} \int_{E_n} |\tilde{\bigcirc}|\ (d\sigma)\ . \tag{10.2}$$

To estimate $I_-(n,t)$ we begin by setting

$$a_n = \min\left(a\,(s;\ 2/\sqrt{n}\,),\ \ 1/4\ \right),$$

$$b_n = a_n^{1/2}\ .$$

Then $a_n \downarrow 0$ and $b_n \downarrow 0$ as $n \to \infty$. Next, set

$$\rho = |P(s) - P(t)|,$$

$$\rho_1 = \rho\,(1 - b_n)\ ,$$

$$\rho_2 = \rho\,(1 + b_n)\ ,$$

and let the circles $B(s;\rho_1)$ and $B(s;\rho_2)$ intersect $C_-(s;\ 2/\sqrt{n}\,)$ in $P(s_1)$ and $P(s_2)$, respectively (see Figure 10.1).

Set

$$C^{(1)} = \{\ P(\sigma) \in C_-(s\ ;\ 2/\sqrt{n}\,)\ ;\ \ \sigma > s_1\ \}\quad ,$$

$$C^{(2)} = \{\ P(\sigma) \in C_-(s;\ 2/\sqrt{n}\,)\ ;\ \ s_1 \ge \sigma \ge s_2\ \},$$

$$C^{(3)} = \{\ P(\sigma) \in C_-(s\ ;\ 2/\sqrt{n})\ ;\ \ \sigma < s_2\ \}\ .$$

Then

$$I_-(n,t) = \sum_{i=1}^{3} I^{(i)}(n,\ t)\ ,$$

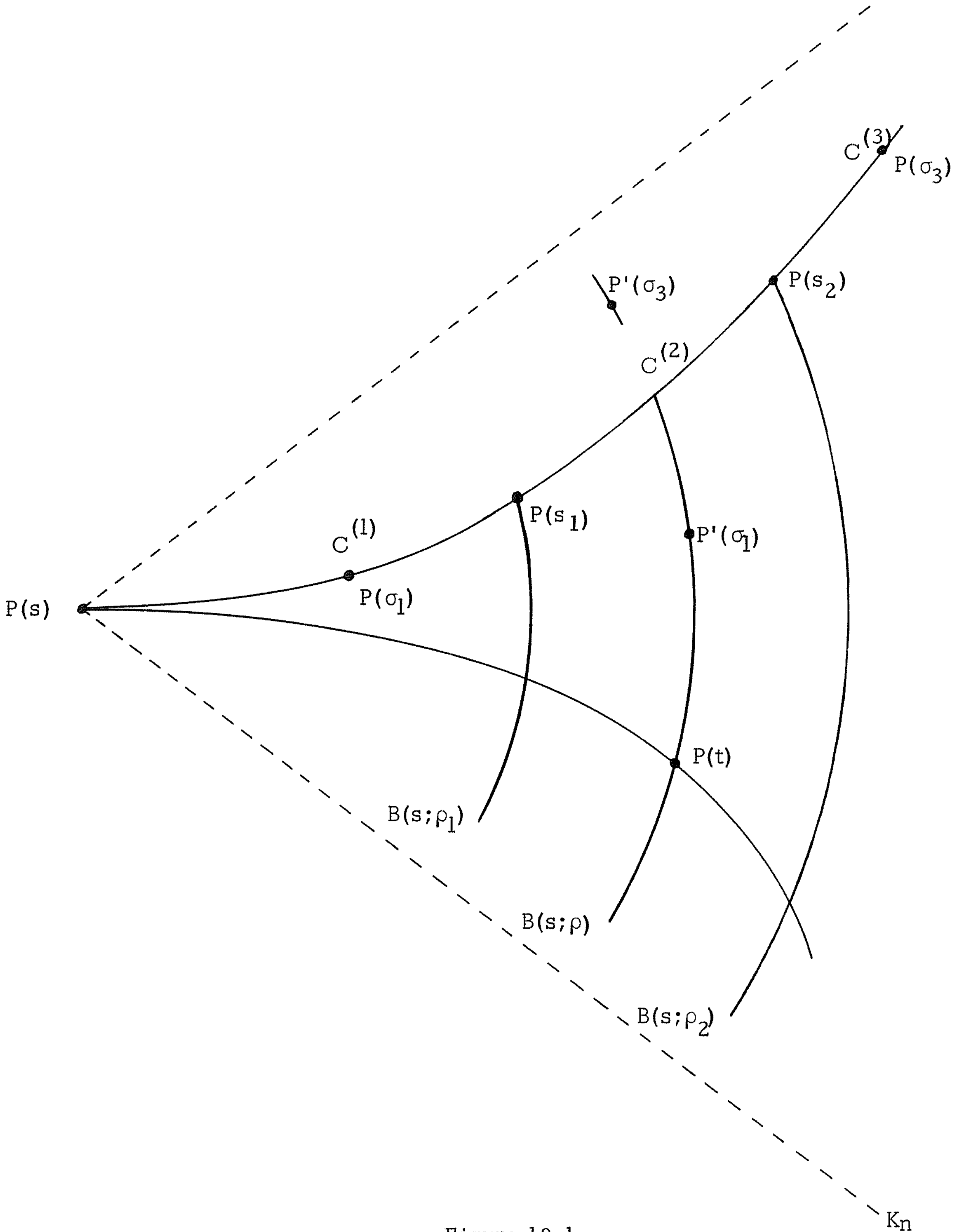

Figure 10.1

Untitled.

where

$$I^{(i)}(n,t) = \frac{n}{\pi} \int_{C^{(i)} \cap E_{n-}} [\psi(t,s-) - \psi(t,\sigma)]\, d\sigma .$$

Without loss of generality, it may be assumed that $\vartheta(s-) = -\pi$, $\vartheta(s+) = 0$. Then it follows from Lemma 3.7 that $C_0(s\,;\,2/\sqrt{n})$ lies in the cone K_n with vertex $P(s)$, axis the x-axis, and vertex angle c_n, where

$$c_n = 2 \text{ arc tan} \left[\frac{a_n}{1-a_n}\right] .$$

Clearly, $c_n \downarrow 0$ as $n \to \infty$.

Let $P(\sigma) \in C_-(s;\, 2/\sqrt{n})$. We denote by $P'(\sigma)$ the point of intersection of the line joining $P(s)$ and $P(\sigma)$ with $B(s;\,\rho)$ (see Figure 10.1).

$$\big|\,|P(s) - P(t)| - |P(s) - P(\sigma)|\,\big| = |P(\sigma) - P'(\sigma)| .$$

Since, for sufficiently large n, if $P(\tau) \in C_+(s\,;\,2/\sqrt{n})$ then $|\vartheta(\tau)| \leq \pi/4$, since $\psi(t,\sigma)$ is a continuous function of σ, and since, from Theorem 2.1, $P(\sigma)$ lies "above" $C_+(s\,;\,2/\sqrt{n})$, it may be assumed that

$$-\pi/4 \leq \psi(t,s) - \psi(t,\sigma) \leq 5\pi/4 ,$$

and

$$\cos[\psi(t,s) - \psi(t,\sigma)] = \cos \sphericalangle P(\sigma)\, P(t)\, P(s) .$$

Let $P(\sigma_1) \in C^{(1)}$, and consider the triangle $P(s)\, P(\sigma_1)\, P(t)$. Taking projections on $P(s)\, P(t)$

$$|P(s) - P(t)| = |P(s) - P(\sigma_1)| \cos \sphericalangle P(\sigma_1)\, P(s)\, P(t) + $$
$$+ |P(\sigma_1) - P(t)| \cos \sphericalangle P(\sigma_1)\, P(t)\, P(s) ,$$

so that

$$\cos \measuredangle\ P(\sigma_1)\, P(t)\, P(s)$$

$$\geq \frac{|P(s) - P(t)| - |P(s) - P(\sigma_1)|}{|P(\sigma_1) - P(t)|},$$

$$= \frac{|P(\sigma_1) - P'(\sigma_1)|}{|P(\sigma_1) - P(t)|},$$

$$= 1 - \frac{|P(\sigma_1) - P(t)| - |P(\sigma_1) - P'(\sigma_1)|}{|P(\sigma_1) - P(t)|},$$

$$\geq 1 - \frac{|P'(\sigma_1) - P(t)|}{|P(\sigma_1) - P(t)|},$$

$$\geq 1 - \frac{\rho\, c_n}{\rho\, b_n},$$

$$= 1 - d_n,$$

where $d_n = c_n / b_n$. Clearly, $d_n \downarrow 0$ as $n \to \infty$. Consequently,

$$|\psi(t, s) - \psi(t, \sigma_1)| \leq e_n, \tag{10.3}$$

where

$$e_n < \operatorname{arc\,cos}(1 - d_n),$$

and $e_n \downarrow 0$ as $n \to \infty$.

Let $P(\sigma_3) \in C^{(3)}$, and consider the triangle $P(s)\, P(\sigma_3)\, P(t)$. Taking projections on $P(s)\, P(\sigma_3)$,

$$\begin{aligned} |P(s) - P(\sigma_3)| &= |P(s) - P(t)| \cos \measuredangle\ P(\sigma_3)\, P(s)\, P(t) + \\ &\quad + |P(\sigma_3) - P(t)| \cos \measuredangle\ P(s)\, P(\sigma_3)\, P(t), \end{aligned}$$

so that

$$\cos \measuredangle P(s)\, P(\sigma_3)\, P(t)$$

$$\geq \frac{|P(s) - P(\sigma_3)| - |P(s) - P(t)|}{|P(\sigma_3) - P(t)|},$$

$$= \frac{|P(\sigma_3) - P'(\sigma_3)|}{|P(\sigma_3) - P(t)|},$$

$$= 1 - \frac{|P(\sigma_3) - P(t)| - |P(\sigma_3) - P'(\sigma_3)|}{|P(\sigma_3) - P(t)|},$$

$$\geq 1 - \frac{|P(t) - P'(\sigma_3)|}{|P(\sigma_3) - P(t)|},$$

$$\geq 1 - \frac{\rho\; c_n}{\rho\; b_n},$$

$$= 1 - d_n .$$

Consequently ,

$$|\measuredangle P(s)\, P(\sigma_3)\, P(t)| \leq e_n ,$$

so that

$$|[\psi(t,s) - \psi(t,\sigma_3)] - \pi| \leq c_n + e_n . \qquad (10.4)$$

Finally, using Lemma 3.7 ,

$$\rho(1 - b_n) \leq s - s_1 \leq \rho(1 - b_n)/(1 - a_n) , \qquad (10.5)$$

$$\rho(1 + b_n) \leq s - s_2 \leq \rho(1 + b_n)/(1 - a_n) , \qquad (10.6)$$

so that

$$s_1 - s_2 \leq \rho f_n , \qquad (10.7)$$

where

$$f_n < \frac{a_n + 2b_n - a_n b_n}{1 - a_n} .$$

Clearly, $f_n \downarrow 0$ as $n \to \infty$.

We can now bound the integrals $I^{(i)}(n,t)$. First, consider $I^{(1)}(n,t)$. Using (10.3)

$$|I^{(1)}(n,t)| \leq \frac{n}{\pi} \int_{E_{n-}} e_n \, d\sigma \leq \frac{e_n}{\pi} . \tag{10.8}$$

Next consider $I^{(2)}(n,t)$. Two cases arise:

<u>Case A: $s - s_1 \geq \frac{1}{n}$</u>

$$\text{Then } I^{(2)}(n, t) = 0 . \tag{10.9}$$

<u>Case B: $s - s_1 < 1/n$</u>

Then using (10.5),

$$\rho \leq \frac{1}{n} \frac{1}{1 - b_n} ,$$

so that, using(10.7),

$$|I^{(2)}(n,t)| \leq \frac{2}{\pi} n \, \|\psi\| \int_{C^{(2)}} d\sigma ,$$

$$= \frac{2}{\pi} n \, \|\psi\| \, |s_1 - s_2| ,$$

$$\leq g_n , \tag{10.10}$$

where

$$g_n = \frac{2f_n \|\psi\|}{\pi(1 - b_n)} .$$

Clearly, $g_n \downarrow 0$ as $n \to \infty$.

Finally, consider $I^{(3)}(n,t)$. Two cases arise:

<u>Case A: $|s_2 - s| \geq 1/n$</u>

Using (10.6),

$$\rho \geq \frac{1}{n} \frac{1 - a_n}{1 + b_n} ,$$

so that

$$t - s - \frac{1}{n} \geq \rho - \frac{1}{n},$$

$$\geq \frac{1}{n}\left[\frac{1 - a_n}{1 + b_n} - 1\right],$$

$$= \frac{1}{n} (-b_n - a_n) / (1 + b_n).$$

Therefore,

$$|\varphi_n(t)| \leq (a_n + b_n) / (1 + b_n).$$

Since $I^{(3)}(n, t) = 0$,

$$|I^{(3)}(n, t) + \varphi_n(t) | \leq (a_n + b_n) / (1 + b_n). \tag{10.11}$$

<u>Case B: $|s_2 - s| < 1/n$</u>

Then, from (10.6),

$$\rho \leq \frac{1}{n} \frac{1}{1 + b_n}.$$

Hence,

$$|s - t| \leq \frac{\rho}{1-a_n},$$

$$\leq \frac{1}{n} \frac{1}{(1-a_n)(1+b_n)},$$

$$= \frac{1}{n} \left[1 - \frac{b_n - a_n - a_n b_n}{(1-a_n)(1 + b_n)}\right],$$

$$= \frac{1}{n} \left[1 - \frac{b_n (1 - b_n - b_n^2)}{(1 - a_n)(1 + b_n)}\right],$$

$$< \frac{1}{n},$$

since $b_n < \frac{1}{2}$. Consequently,

$$\varphi_n(t) = n(t - s) - 1.$$

Also, using (10.4),

$$I^{(3)}(n,t) = \frac{n}{\pi} \int_{C^{(3)} \cap E_{n-}} [\psi(t,s) - \psi(t,\sigma_3)]\, d\sigma_3 ,$$

$$= \frac{n}{\pi} \int_{s-s_2}^{1/n^2} [\psi(t,s) - \psi(t,\sigma_3)]\, d\sigma_3 ,$$

$$= n[\frac{1}{n} - (s - s_2)] + H(n,t) ,$$

where

$$|H(n,t)| \leq \frac{n}{\pi} \int_{E_{n-}} [c_n + e_n]\, d\sigma_3$$

$$= (c_n + e_n)/\pi .$$

Therefore, using (10.6),

$$|I^{(3)}(n,t) + \varphi_n(t)|$$

$$\leq |[1 - n(s - s_2)] + H(n,t) + [n(t - s) - 1]| ,$$

$$\leq |H(n,t)| + n|(t-s) - \rho| + n|(s-s_2) - \rho| ,$$

$$\leq |H(n,t)| + n\, a_n |s-t| + n \frac{(a_n + b_n)\rho}{1 - a_n} ,$$

$$\leq h_n , \qquad (10.12)$$

where

$$h_n = \frac{c_n + e_n}{\pi} + \frac{a_n}{1 - a_n} + \frac{a_n + b_n}{1 - a_n} .$$

Clearly $h_n \downarrow 0$ as $n \to \infty$.

Setting

$$p_n = \frac{e_n}{\pi} + g_n + \frac{a_n + b_n}{1 + b_n} + h_n ,$$

inequality (10.1) follows from inequalities (10.2), (10.8), (10.9) and (10.10), (10.11) and (10.12).

APPENDIX A

SURVEY OF THE LITERATURE ON DOUBLE-LAYER POTENTIALS

A.1. Historical background

According to tradition, potential theory began in 1666 when Isaac Newton, while sitting in a garden at Woolsthorpe, was hit on the head by an apple, thereby discovering the laws of gravitation. The subsequent development of potential theory is an interesting example of the interaction between mathematics and physics.

At first mathematicians only studied the gravitational potential induced by non-negative distributions of mass in three dimensions, since this was the only physically meaningful problem known. Two-dimensional potential problems were not studied until about 1845, when physically meaningful two-dimensional problems arose. (In the present survey we will, as is customary in potential theory, distinguish between two-dimensional and three-dimensional potentials by calling the former logarithmic potentials and the latter Newtonian potentials).

In 1785 Couloumb discovered the similarity between the laws of gravitation and the laws of electrostatics and magnetostatics. This led to the study of potentials induced by negative distributions and thus to the study of potentials induced by double-layers or double-distributions. A double-layer may be visualized as a curve or surface one side of which is covered with a certain distribution of charge, and the other side of which is covered with charges of equal magnitude but opposite sign. The theory of Newtonian double-layer potentials was first considered in detail by Helmholtz in 1853, while the theory of logarithmic double-layer potentials was initiated by Beer in 1856. (see Burkhardt and Meyer [22]).

In 1870 Neumann published his famous paper [93] in which he solved the Dirichlet problem by means of double-layer potentials. We briefly summarize Neumann's results for the logarithmic case.

Let C be a smooth curve in the xy-plane with parametric representation

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq S, \tag{A.1.1}$$

where s denotes arc length along C in the positive (anti-clockwise) direction (see Figure A.1.1). C divides the plane into a simply-connected bounded domain $\mathcal{R} = \mathcal{R}_+$ and a simply-connected unbounded domain $\mathcal{R}_-$.

Let $g = g(s)$ be a smooth function on C and let $u = u(x, y)$ be the solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathcal{R}, \tag{A.1.2}$$

$$u = g, \quad (x, y) \in C. \tag{A.1.3}$$

To solve this problem Neumann assumed that u was the logarithmic potential corresponding to a double-layer of density $\varphi = \varphi(s)$ on C. That is, for $Q \in \mathcal{R}$,

$$u(Q) = \int_C \varphi(\sigma) \frac{\partial}{\partial \underline{n}(\sigma)} \log |Q - P(\sigma)| d\sigma \, , \tag{A.1.4}$$

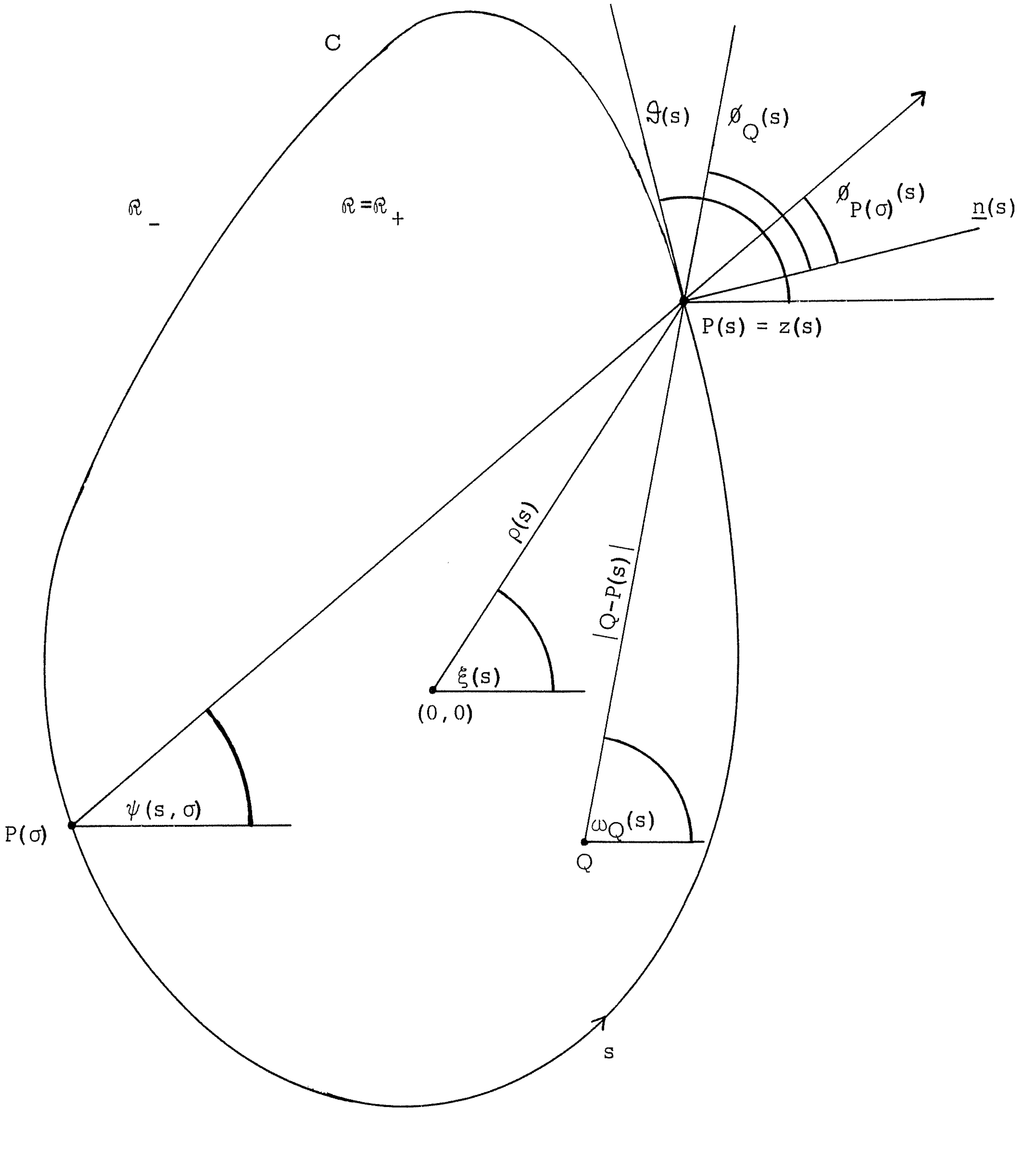

Figure A.1.1

The curve C .

where $\underline{n}(\sigma)$ is the unit outward normal to C at $P(\sigma)$ and $|Q - P(\sigma)|$ is the distance from Q to $P(\sigma)$ (see Figure A.1.1). If φ is smooth then it is easily verified that the (A.1.2) is satisfied. Neumann showed that

$$\left.\begin{aligned} &\lim_{Q\to P_+(s)} \int_C \varphi(\sigma) \frac{\partial}{\partial \underline{n}(\sigma)} \log|Q - P(\sigma)|d\sigma \\ &\qquad = \pi\varphi(s) + \pi\int_C \varphi(\sigma)K(s, \sigma)d\sigma , \end{aligned}\right\} \tag{A.1.5}$$

where

$$K(s, \sigma) = \frac{1}{\pi}\frac{d}{d\sigma} \arctan\left[\frac{y(\sigma) - y(s)}{x(\sigma) - x(s)}\right], \tag{A.1.6}$$

and where in (A.1.5) is to be understood in the sense that $Q \in \mathcal{R}_+$ tends to $P(s)$. Hence (A.1.3) is satisfied if φ satisfies the integral equation,

$$\varphi(s) + \int_0^S \varphi(\sigma)K(s, \sigma)d\sigma = g(s)/\pi, \quad 0 \le s \le S . \tag{A.1.7}$$

In his paper [93] Neumann proved, in outline, that (A.1.7) has a smooth solution φ when C is convex. In a subsequent paper [94] he generalized these results, in particular by allowing C to have corners. Detailed proofs appeared in his book of 1877 [95].

The work of Neumann was remarkable both in itself and in its consequences:

1. The connection between potential theory and Laplace's equation had been discovered by Laplace in 1782 and several attempts had been made to prove the existence of the solution to the Dirichlet problem (see Kellogg [56, p. 277] and Burkhardt and Meyer [22, p. 486]). However, Neumann's proof was the first correct proof to be published.

2. Although integral equations had arisen earlier (Hellinger and Toeplitz [45, p. 1345]) equation (A.1.7) was the first integral equation to arouse considerable interest. In 1896, while trying to generalize the results of Neumann to non-convex curves C, Poincare introduced the parameterized equation

$$\varphi(s) + \lambda \int_0^S \varphi(\sigma)K(s, \sigma)d\sigma = g(s)/\pi ,$$

and investigated the dependence of φ upon λ. Then, in his classic paper of 1900, Fredholm [32] created his theory of integral equations and used it to prove that (A.1.7) can be solved for every smooth C.

3. To solve (A.1.7) Neumann used a method which he called the "Methode des arithmetischen Mittels". This method was essentially the same as, and led to the development of, the Neumann series in functional analysis. (The name is historically incorrect since Liouville used such a series in 1837 (see Hellinger and Toeplitz [45, p. 1348])).

4. The reason why Neumann restricted himself to convex curves C was that for such curves the kernel $K(s, \sigma)$ of (A.1.7) is positive. In his use of the positivity of K, Neumann foreshadowed the theory of positive operators.

In the present survey an attempt is made to describe the development of the theory of double-layer potentials since the work of Neumann. It is therefore very appropriate that this is the centenary of Neumann's 1870 papers.

A.2. The classical theory.

In this section, unless explicitly stated to the contrary, we will be concerned with the logarithmic case. It will be assumed that C, φ, and g are "sufficiently smooth"; in particular it will be assumed that $x(s)$ and $y(s)$ are at least twice continuously differentiable, and that φ and g are at least once continuously differentiable.

Some useful formulas

First, we note that

$$\frac{\partial}{\partial \underline{n}(\sigma)} \log |Q - P(\sigma)| =$$

$$= \left[n_x(\sigma) \frac{\partial |Q-P(\sigma)|}{\partial x(\sigma)} + n_y(\sigma) \frac{\partial |Q-P(\sigma)|}{\partial y(\sigma)}\right] / |Q-P(\sigma)| ,$$

$$= [n_x(\sigma)[x(\sigma) - x_Q] + n_y(\sigma)[y(\sigma) - y_Q]] / |Q-P(\sigma)|^2 ,$$

$$= [\underline{n}(\sigma) \cdot (\underline{P(\sigma) - Q})] / |Q-P(\sigma)| ,$$

$$= [\cos(\phi_Q(\sigma))] / |Q-P(\sigma)| , \tag{A.2.1}$$

where $\phi_Q(\sigma)$ is as in Figure A.1.1. Hence, (A.1.4) is equivalent to

$$u(Q) = \int_C \varphi(\sigma) \frac{\cos[\phi_Q(\sigma)]}{|Q-P(\sigma)|} \, d\sigma . \tag{A.2.2}$$

Next, let $\omega_Q(s)$ be as in Figure A.1.1 (see also (2.9)). Then it is geometrically obvious, and can be proved directly from (2.9), that

$$\int_C \frac{d}{d\sigma}[\omega_Q(\sigma)]d\sigma = \begin{cases} 2\pi, & \text{if } Q \in \mathcal{R}_+ , \\ 0, & \text{if } Q \in \mathcal{R}_- . \end{cases} \tag{A.2.3}$$

Furthermore,

$$\lim_{\epsilon \to 0} \{ \lim_{Q \to P(s)} \int_{\substack{C \\ |P(\sigma)-P(s)| \geq \epsilon}} \frac{d}{d\sigma}[\omega_Q(\sigma)]d\sigma \} = \pi. \tag{A.2.4}$$

Since

$$\begin{aligned} &\frac{d}{d\sigma}[\omega_Q(\sigma)] \\ &= \frac{d}{d\sigma} \arctan [\frac{y(\sigma) - y_Q}{x(\sigma) - x_Q}] , \\ &= [[x(\sigma) - x_Q] \frac{dy}{d\sigma} - [y(\sigma) - y_Q] \frac{dx}{d\sigma}]/ |Q - P(\sigma)|^2 , \\ &= [[x(\sigma) - x_Q]n_x(\sigma) + [y(\sigma) - y_Q]n_y(\sigma)]/ |Q - P(\sigma)|^2 , \\ &= [\cos[\phi_Q(\sigma)]]/ |Q - P(\sigma)| , \end{aligned} \tag{A.2.5}$$

it follows from (A.2.2) that (A.1.4) is also equivalent to

$$u(Q) = \int_C \varphi(\sigma) \frac{d}{d\sigma} [\omega_Q(\sigma)]d\sigma . \tag{A.2.6}$$

Next, we consider $K(s, \sigma)$. The defining equation (A.1.6) is only meaningful for $P(s) \neq P(\sigma)$. However, using Taylor series,

$\pi K(s, \sigma)$

$$= \frac{\dot{y}(\sigma)[x(\sigma) - x(s)] - \dot{x}(\sigma)[y(\sigma) - y(s)]}{[x(\sigma) - x(s)]^2 + [y(\sigma) - y(s)]^2},$$

$$= \frac{1}{2} \frac{-\dot{y}(\sigma)\ddot{x}(\sigma') + \dot{x}(\sigma)\ddot{y}(\sigma')}{[\dot{x}(\sigma'')]^2 + \ddot{x}(\sigma'')[x(\sigma'') - x(\sigma)] + [\dot{y}(\sigma'')]^2 + \ddot{y}(\sigma'')[y(\sigma'') - y(\sigma)]}$$

where σ' and σ'' lie between s and σ. Hence

$$\pi K(s, \sigma) = \frac{1}{2}\kappa(\sigma) + \delta(|s-\sigma|),$$

where $\delta(|t|) \to 0$ as $|t| \to 0$ and $\kappa(\sigma)$ is the curvature of C at $P(\sigma)$,

$$\kappa(\sigma) = \ddot{y}(\sigma)\dot{x}(\sigma) - \ddot{x}(\sigma)\dot{y}(\sigma).$$

Therefore, if the domain of definition of K is extended by setting

$$K(s, \sigma) = \begin{cases} \dfrac{1}{\pi}\dfrac{d}{d\sigma} \arctan\left[\dfrac{y(\sigma) - y(s)}{x(\sigma) - x(s)}\right], & P(s) \neq P(\sigma), \\[2ex] \dfrac{1}{2\pi}\kappa(\sigma), & P(s) = P(\sigma), \end{cases} \tag{A.2.7}$$

K is continuous for $0 \leq s, \sigma \leq S$.

Finally, it follows from (2.9), (A.2.3), (A.2.4), and (A.2.7) that

$$\int_C K(s, \sigma) d\sigma = 1. \tag{A.2.8}$$

Other notation

If $\underline{n}(\sigma)$ is defined to be the unit inward normal instead of the unit outward normal then (A.1.4) takes the form

$$u(Q) = \int_C \varphi(\sigma) \frac{\partial}{\partial \underline{n}(\sigma)} \log[1/|Q-P(\sigma)|]d\sigma \ . \qquad (*)$$

Certain authors (including Kellogg [56]) use (*) and state that $\underline{n}$ is the "positive normal" but fail to define this concept.

Kantorowitsch and Krylow [55, p. 117] change the sign of $K(s, \sigma)$ and interchange s and σ so that

$$K^{K\&A}(s,\sigma) = -K(\sigma, s) \ .$$

Behavior near the boundary

Set

$$u_0(s) = \pi\int_C K(s, \sigma)\varphi(\sigma)d\sigma \ , \qquad (A.2.9)$$

and

$$\left.\begin{aligned} u_+(s) &= \lim_{Q \to P_+(s)} u(Q), \\ u_-(s) &= \lim_{Q \to P_-(s)} u(Q) \ , \end{aligned}\right. \qquad (A.2.10)$$

provided that the limits exist. Then (A.1.5) is contained in the following

Theorem A.2.1.

u_+ and u_- exist and satisfy:

$$\left.\begin{aligned} u_+(s) &= \pi\varphi(s) + u_0(s)\,, \\ u_-(s) &= -\pi\varphi(s) + u_0(s)\,, \\ 2u_0(s) &= u_+(s) + u_-(s)\,, \\ 2\pi\varphi(s) &= u_+(s) - u_-(s)\,. \end{aligned}\right\} \qquad \text{(A.2.11)}$$

Theorem A.2.1 is intuitively obvious from (A.2.6), and a proof is sketched by several authors (for example, Kantorowitsch and Krylow [55, p. 117]). Proofs using complex function theory are given by several authors (for example, Gakhov [34, p. 73]). So far as the author is aware, there is no elementary proof in the literature which does not make use of complex function theory.

The integral equation

It follows from (A.2.11) that the Dirichlet problem (A.1.2), (A.1.3), is solved by the double-layer potential (A.1.4) provided that φ satisfies (A.1.7). In operator notation,

$$(I + T)\varphi = g/\pi\,, \qquad \text{(A.2.12)}$$

where

$$(T\varphi)(s) = \int_C K(s, \sigma)\varphi(\sigma)d\sigma\ .$$

The most direct way of solving (A.2.12) is by means of the series solution

$$\varphi = [I - T + T^2 - T^3 \ldots]g/\pi\ . \qquad \text{(A.2.13)}$$

However, from (A.2.8),

$$T1 = 1, \tag{A.2.14}$$

so that T has the eigenvalue 1 and the convergence of the series (A.2.13) is by no means certain.

One way of establishing the convergence of (A.2.13) makes use of the following:

Lemma A.2.2.

Let C be any convex Jordan curve which is not the boundary of a quadrilateral or a triangle. Then there exists a constant k, $0 \leq k < 1$, which depends only upon C such that if $f = f(s)$ is any continuous function defined on C then

$$\operatorname{osc}(Tf) \leq k \operatorname{osc}(f), \tag{A.2.15}$$

where

$$\operatorname{osc}(f) = [\max_C f] - [\min_C f]. \tag{A.2.16}$$

The idea underlying (A.2.15), namely that Tf is smoother than f, was used by Beer in 1856 for Newtonian double-layer potentials (see Neumann [95, p. 221]). Lemma A.2.2 was first stated and proved by Neumann [95, p. 185], who called k the "configurations - constante" of C. Recently, Schober [118] has pointed out, and corrected, an error in Neumann's proof. It is known that $k = \frac{1}{2}$ if C is a circle (Neumann [95, p. 174]) and that

$k \leq 1 - S/(2\pi R)$ where $R \leq \infty$ is the supremum of the radii of all circles which intersect C in at least three points (Schober [118]).

If C and f are as in Lemma A.2.2 then $K(s, \sigma)$ is non-negative so that, using (A.2.8),

$$\min_C f \leq \min_C Tf \leq \max_C Tf \leq \max_C f . \qquad \text{(A.2.17)}$$

Combining this equation with (A.2.15) it follows that as n tends to infinity $T^n g$ tends to a constant, c say, and

$$\max_C |T^n g - c| \leq k^n \operatorname{osc.}(g) ,$$

$$\max_C |T^n(I - T)g| \leq k^n \operatorname{osc.}(g) .$$

Therefore, if

$$\varphi_n = \frac{c}{2} + (I - T)g/\pi + T^2(I - T)g/\pi + \ldots T^{2n}(I - T)g/\pi,$$

then as n tends to infinity, φ_n converges uniformly to a continuous function, φ say. Since

$$T\varphi = \lim_{n\to\infty} \left[c - \varphi_n + \frac{g}{\pi} - T^{2n+2} \frac{g}{\pi}\right] ,$$

$$= \frac{g}{\pi} - \varphi ,$$

φ satisfies (A.2.12). Furthermore, it follows easily from (A.2.15) and (A.2.17) that φ is the unique solution of (A.2.12).

To summarize, it has been shown that if C is convex then the series

$$\frac{c}{2} + (I - T)\frac{g}{\pi} + (I - T)T^2\frac{g}{\pi} + \ldots \tag{A.2.18}$$

is convergent and solves the equation (A.2.12).

We now turn to the general case when C is not necessarily convex. If $\lambda_1 = 1$, λ_2, λ_3, ..., are the eigenvalues of T, then it is known (Blumenfeld and Mayer [14], Gaier [33, p. 27]) that

(a) λ_1 is simple,

(b) For $i > 1$, $|\lambda_i| > 1$,

(c) For $i > 1$, λ_i is real and of finite multiplicity, (A.2.19)

(d) For $i > 1$, $(-\lambda_i)$ is an eigenvalue of T with the same multiplicity as λ_i.

The proof of (A.2.19) depends upon (i) the connection of T with the Dirichlet boundary value problem, and (ii) the fact that T is symmetrizable, that is if

$$H(s, t) = \int_C K(s, \sigma)\log[1/|P(\sigma) - P(t)|]d\sigma ,$$

then $H(s, t) = H(t, s)$.

It follows from (A.2.19a) and (A.2.19b) that the series (A.2.18) converges to the unique solution of (A.2.12).

<u>Connection with singular integrals</u>

There is a close connection between the theory of logarithmic double-layer potentials and the theory of singular integrals with Cauchy kernels. Some remarks on the historical background of this connection will be found in Muskhelishvili [91, p. 23] and Gakhov [34, p. 75]. A brief, but lucid, survey of the theory of singular integrals is given by Seeley [119].

Following Mikhlin [89, p.137], consider C as a curve in the complex plane and set $z = Q \in \mathcal{R}$, $\zeta = P(\sigma) \in C$ and

$$\zeta - z = re^{i\omega} . \tag{A.2.20}$$

Then, using the Cauchy-Riemann equations,

$$\begin{aligned} \text{Imag.}[\frac{d\zeta}{\zeta - z}] &= \text{Imag.}[d(\log[\zeta - z])] , \\ &= d\omega, \\ &= \frac{\partial \omega}{\partial \sigma} d\sigma , \\ &= \frac{\partial \log r}{\partial \underline{n}(\sigma)} d\sigma . \end{aligned} \tag{A.2.21}$$

Consequently, (A.1.4) can be written in the equivalent form,

$$u(z) = \text{Imag.}[\int_C \varphi(\zeta) \frac{d\zeta}{\zeta - z}] . \tag{A.2.22}$$

Now, it is well-known (Mikhlin [89, p. 117]) that if C is smooth, $t = P(s) \in C$, and

$$F(z) = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)}{\zeta - z} d\zeta , \tag{A.2.23}$$

then

$$F_+(t) = \lim_{z \to t_+} F(z) = \frac{1}{2} \varphi(t) + \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)}{\zeta - t} d\zeta , \tag{A.2.24}$$

the integral in (A.2.24) being a singular integral with Cauchy kernel. Formula (A.2.24) is sometimes called Morera's formula (Muskhelishvili [91, p. 41]) or Sokhotski's formula (Gakhov [34, p. 25]).

Remembering that $\zeta = P(\sigma)$, $t = P(s)$, and that φ is real-valued,

$$\text{Imag.} \int_C \frac{\varphi(\zeta)}{\zeta - t}\, d\zeta = \pi \int_C K(s,\sigma)\varphi(\sigma)d\sigma. \tag{A.2.25}$$

Equation (A.1.7) follows from (A.2.22) and (A.2.25) by taking the real part of (A.2.24).

In view of the above, it is natural to ask to what extent the theory of double-layer potentials can be developed via the theory of singular integrals. So far as the author is aware (and his knowledge is very limited) the answer is that the theory of singular integrals is of little use in this context for the following reasons.

First, the theory of singular integrals has been developed mainly for the case when C and φ are smooth. This is a sweeping generalization which requires clarification. There are some results for the case when C is not smooth. If C has a corner at t with interior angle α, $0 \leq \alpha \leq 2\pi$, then the Sokhotski formula (A.2.24) becomes (Gakhov [34, p. 31]),

$$F_+(t) = \left(1 - \frac{\alpha}{2\pi}\right)\varphi(t) + \frac{1}{2\pi i} \int \frac{\varphi(\zeta)}{\zeta - t}\, d\zeta, \tag{A.2.26}$$

which leads to a corresponding modification of (A.1.7). Kveseleva [73] has considered the case of intersecting contours, and Alekseev [2] has considered the case when C is "of class R" (this class is defined in section A.3). Quite general Cauchy integrals are considered by Goluzin [36] and Priwalow [107]. However, in all cases the conditions imposed upon φ and C seem to be stronger than those needed when singular integrals are not used.

Secondly, it appears that it is always necessary to appeal to the Fredholm theory: although Gakhov [34], Mikhlin [89], and Muskhelishvili [91], derive (A.1.7) via the Sokhotski formula (A.2.24), they establish the existence and uniqueness of the solution of (A.1.7) by appealing to the Fredholm theory for integral equations.

Connection with conformal mapping

Suppose that it is required to find the mapping

$$w = f(z) = f(x + iy)$$

which maps $\mathfrak{R}$ conformally into the unit circle in the w-plane, in such a way that $f(0) = 0$. Set

$$z(s) = x(s) + iy(s) = \rho(s)e^{i\xi(s)},$$

and denote by $\vartheta(s)$ the angle between the x-axis and the positive tangent to C at $z(s)$ (see Figure A.1.1). Finally, let

$$f(z(\sigma)) = e^{i\theta(\sigma)}.$$

Then it can be shown (Gaier [33, p. 7]) that θ satisfies the integral equation

$$[\theta(s) - \xi(s)] = \int_C [\theta(\sigma) - \xi(\sigma)]K(s, \sigma)d\sigma + \Phi(s), \tag{A.2.27}$$

where

$$\Phi(s) = \frac{1}{\pi} \int_C \log \rho(\sigma) \frac{\cos[\vartheta(\sigma)] - \psi(s, \sigma)}{|z(s) - z(\sigma)|} d\sigma .$$

It can also be shown (Gaier [33, p. 8]) that θ satisfies the integral equation,

$$\theta(s) = \int_C K(s, \sigma)\theta(\sigma)d\sigma - 2\beta(s), \qquad (A.2.28)$$

where

$$\beta(s) = \arg\left[\frac{z(0) - z(s)}{0 - z(s)}\right].$$

Equation (A.2.27) is due to Lichtenstein while (A.2.28) is due to Gershgorin. Several other equations with the Neumann kernel $K(s, \sigma)$ arise in the theory of conformal mapping; for details see Gaier [33, Chapter I].

The Lichtenstein and Gershgorin integral equations are clearly related to (A.1.7). There is, however, an even closer relationship. For if the exterior Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathcal{R}_-,$$

$$u = g, \quad (x, y) \in C,$$

is to be solved by the double-layer potential (A.1.4) then it can be shown that φ must satisfy the equation

$$-\varphi(s) + \int_C K(s, \sigma)\varphi(\sigma)d(\sigma) = g(s)/\pi, \qquad (A.2.29)$$

which includes the Lichenstein and Gershgorin equations as special cases.

Newtonian double-layer potentials

In the Newtonian case, C is a surface in E_3.

There is a close analogy between the theory of logarithmic and Newtonian double-layer potentials. The main differences are that

(i) The angle subtended by a unit sphere is 4π while the angle subtended by a unit circle is 2π,

(ii) $\log r$ must be replaced by $1/r$.

Thus, (A.1.4) becomes

$$u(Q) = \int_C \varphi(\sigma) \frac{\partial}{\partial \underline{n}(\sigma)} \frac{1}{|Q - P(\sigma)|} d\sigma \quad , \tag{A.2.30}$$

(A.1.6) becomes

$$K_3(s, \sigma) = \frac{1}{2\pi} \frac{\partial}{\partial \underline{n}(\sigma)} \frac{1}{|P(s) - P(\sigma)|} \quad , \tag{A.2.31}$$

and (A.1.7) becomes

$$\varphi(s) + \int_C K_3(s, \sigma)\varphi(\sigma)d\sigma = g(s)/2\pi \ . \tag{A.2.32}$$

Exhaustive expositions of the theory of Newtonian double-layer potentials are given by Kellogg [56] and Günter [38].

A.3. Generalizations of the classical theory

In the present section we consider various generalizations of the classical theory in which the smoothness restrictions upon C and φ are relaxed.

In his second paper of 1870 and his book of 1877 Neumann [94, 95] considered the case when C is a convex curve with corners and proved that the series (A.2.18) converges provided that C is not a triangle or rectangle. Neumann also obtained similar results for the Newtonian case. This work was subsequently forgotten.

The next work in the area is due to Carleman who, in his mammoth 200-page thesis of 1916 [24], considered the case when C is a piecewise smooth curve with corners $s = s_i$, $1 \leq i \leq n$, cusps not being allowed. Carleman split the kernel K into two parts,

$$K(s, t) = G(s, t) + H(s, t), \tag{A.3.1}$$

such that [24, p. 12]

(i) $H(s, t)$ is bounded for $0 \leq s,t \leq S$, and continuous except on a finite number of lines parallel to the s and t axes,

(ii) $G(s, t)$ is zero except when

$$|s - s_i|, \quad |t - s_i| \leq \delta$$

for some i, where δ is a small positive constant.

The splitting (A.3.1) induces a corresponding splitting

$$T = T_G + T_H$$

and Carleman showed that (A.1.7) has a solution by constructing $(I + \lambda T_G)^{-1}$ and $[I + (I + \lambda T_G)^{-1}\lambda T_H]^{-1}$. Carleman also briefly considered the third boundary value problem and Newtonian problems. Perhaps the most interesting part of the thesis is a detailed analysis of the case when C has only one corner. It is shown that [24, p. 118]

(i) There is a denumerable sequence of real numbers λ_i and continuous functions φ_i such that

$$[I - \lambda_i T]\varphi_i = 0$$

(ii) If the internal angle at the corner is $\pi - \theta$, and if $|\lambda| > |\pi/\theta|$, then there is a continuous function $\varphi(s)$, with a logarithmic singularity at the corner, such that

$$[I - \lambda T]\varphi = 0.$$

Finally, we note that, according to Gaier [33, p. 60] the techniques of Suharevskii [123] are similar to those used by Carleman.

In 1919 Radon [110] generalized the classical theory to the case when C is of bounded rotation; these results are described elsewhere in this report. In a series of papers written about 1937, Magnaradze [82, 83, 84] used Radon's techniques to solve problems in the theory of elasticity for domains whose boundaries are of bounded rotation. As we shall see below, Radon's work became important in the 1960's.

A somewhat different approach to double-layer potentials was taken by Lyapunov and Günter who studied Newtonian double-layer potentials on Lyapunov surfaces. A Lyapunov surface is a surface which satisfies the three Lyapunov conditions (Günter [38, p. 1]):

(i) At each point of the surface there exists a well-defined normal,

(ii) The normal satisfies a Hölder condition,

(iii) There is a positive constant d such that if M is a point of the surface and ℓ is a line parallel to the normal at M, then the portion of the surface within distance d of M intersects ℓ in at most one point.

Because of the restrictions upon the surface, Lyapunov and Günter were able to prove certain smoothness properties of the double-layer potential. The culmination of this approach was the work of Fichera [30] who required only that the density φ be Lebesgue integrable. A detailed exposition of all these results is given in the book of Günter [38].

Schauder [114, 115] also considered the behavior of double-layer potentials on surfaces satisfying conditions similar to those of Lyapunov.

Evans [27, 28] considered a general class of logarithmic double-layer potentials. Let C be a rectifiable Jordan curve of length S and let s denote length on C. For $P, P' \in C$ let $\theta(P, P')$ denote the angle between the x-axis and the line joining P to P'. For $P(s) \in C$ set

$$\overline{\theta}_+(s) = \limsup_{\epsilon \to 0+} \theta(P(s), P(s+\epsilon)),$$

$$\underline{\theta}_+(s) = \liminf_{\epsilon \to 0+} \theta(P(s), P(s+\epsilon)).$$

If either $\overline{\theta}_+(s)$ or $\underline{\theta}_+(s)$ is not finite, set

$$\theta_+(s) = 0;$$

otherwise, set

$$\theta_+(s) = [\overline{\theta}_+(s) + \underline{\theta}_+(s)]/2 + 2n\pi,$$

the integer n being chosen so that

$$0 \leq \theta_+(s) < 2\pi .$$

Define $\theta_-(s)$ similarly and set

$$\theta(s) = [\theta_+(s) + \theta_-(s)]/2 .$$

Then the unit outward normal $\underline{n}(s) = (n_1(s), n_2(s))$ is defined by

$$n_1(s) = \sin \theta(s),$$

$$n_2(s) = -\cos \theta(s) .$$

The curve C is said to satisfy conditions (γ) at a fixed point $P(s) \epsilon C$ if there exists a function $\gamma(t)$ such that

(i) $\gamma(t)$ is a positive, monotonic, non-decreasing function of t,

(ii) The integral $\int_0^s \frac{\gamma(t)}{t}\,dt$ is convergent,

(iii) $|\sphericalangle(\underline{n}(s), \underline{n}(\sigma))| < \gamma(|s - \sigma|)$.

Evans considered the logarithmic double-layer potential corresponding to a density φ which is a completely additive set function on C. That is, with the notation of Figure A.1.1,

$$u(Q) = \int_C \frac{\cos \phi_Q(\sigma)}{|Q - P(\sigma)|} \varphi(d\sigma) .$$

Evans showed that if (i) $P(s)$ is a point at which condition (γ) is satisfied, and (ii) φ has "a unique derivative $\varphi' = A$ at $P(s)$" then the double-layer potential has the usual boundary properties. For example [27, p. 217],

$$\lim_{Q \to P_+(s)} u(Q) = -2\pi A + \int_C \frac{\cos \phi_{P(\sigma)}(s)}{|P(s) - P(\sigma)|} \varphi(d\sigma) .$$

Evans and Miles [29] obtained similar results for Newtonian potentials of the form

$$u(Q) = \int_C \frac{\partial}{\partial \underline{n}(\sigma)} \left[\frac{1}{|Q-P(\sigma)|}\right] d\sigma ,$$

under the assumption that the surface C has a tangent plane at every point and satisfies certain additional smoothness conditions.

Recently there has been a resurgence of interest in the solution of boundary value problems for domains with corners:

In his thesis of 1958, Arbenz [7] extended the results of Radon. Arbenz obtained certain additional properties of the eigenvalues of the operator T, and also considered applications of the theory to conformal mapping and problems in elasticity. A very brief summary of these results is given in [6].

Leis [75, 76, 77] has considered the exterior boundary value problems in two and three dimensions for the Helmholtz wave equation

$$\nabla^2 u + k^2 u = 0,$$

for the case when the boundary C is piecewise smooth. He shows that the appropriate integral equations have solutions in the Lebesgue spaces L_1 or L_∞.

Wendland [135, 137] has extended the methods of Radon to the interior and exterior boundary value problems for Laplace's equation in three dimensions when the surface C is piecewise smooth.

The most important recent work is that of Kral [58,139], Burago et al [19, 20, 21], and Maz'ja and Sapoznikova [88] whose results overlap. Here, there is only space to indicate the central idea. Let C be rectifiable and φ continuous. Then the double-layer potential

$$u(Q) = \int_C \varphi(\sigma) \frac{\cos[\emptyset_Q(\sigma)]}{|Q - P(\sigma)|} \, d\sigma$$

is continuous for $Q \in \mathfrak{R}$. As Radon showed, if C is of bounded rotation, then u(Q) can be extended continuously to C. Kral, Burago, and Maz'ja turn this idea around, and study the class of curves C such that, for all continuous φ, $u(Q)$ can be extended continuously to C, thereby extending the techniques of Radon to their limit (for continuous densities φ). Both two-dimensional and n-dimensional problems are considered.

Two of the recurring themes in the theory of double-layer potentials are the need of impose certain smoothness conditions upon C and the need to establish the validity of the Fredholm alternative. Here we make some remarks on these topics:

<u>Conditions upon C</u>

We mention yet two more conditions which have been imposed upon curves C in the plane:

1. Let C be a rectifiable curve with the representation

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq S.$$

It is known (Riesz and Sz.-Nagy [113, p. 27]) that C has a tangent $\underline{t}$ at each point of a set $E \subset [0, S]$, where $([0, S] - E)$ is of measure zero. Then, according to Alekseev [2], C is of <u>class R</u> if, at each point $P(s)$ of C,

(i) Forward and backward tangents, $\underline{t}_+(s)$ and $\underline{t}_-(s)$, exist.

(ii) $$\underline{t}_{\pm}(s) = \lim_{\substack{\sigma \to s \pm 0 \\ \sigma \in E}} \underline{t}(\sigma) .$$

It is easily seen that piecewise smooth curves and curves of bounded rotation are of class R. Alekseev shows that curves of class R have the same geometric properties as curves of bounded rotation.

2. Paatero [100, 101] has generalized the concept of bounded rotation. Let G be a schlicht bounded simply-connected domain. Let $G_1, G_2, \ldots,$ be a sequence of open domains such that $G_i \subset G_{i+1}$ and $G = \bigcup_{i=1}^{\infty} G_i$. Now let ℓ be any smooth Jordan curve in G which contains G_i in its interior. Let s denote distance along ℓ and let $\vartheta(s)$ denote the angle between the x-axis and the tangent to ℓ at s. Set

$$a(\ell) = \int_{\ell} |d\vartheta(s)| ,$$

and

$$a_i = \lim\inf\ a(\ell) ,$$

the lim.inf. being taken over all possible curves ℓ. Then the <u>boundary rotation</u> of G, $a(G)$, is defined to be

$$a(G) = \lim_{i\to\infty} a_i ,$$

and G is said to be of <u>bounded boundary rotation</u> if $a(G) < \infty$. The above concepts can also be extended to domains G which are open, bounded, and simply-connected, but not schlicht. For recent work in this area see Lonka and Tammi [78].

<u>The Fredholm alternative</u>

Kantorovich and Akilov [54, p. 525] give interesting necessary and sufficient conditions for the Fredholm alternative to hold. See also Wendland [136].

In conclusion we draw the readers attention to some related work: see Agmon [1], Miranda [90, Chapter II], Gilbert [35], Lopatinskii [79], Sovin [120], Warschawski [134].

A.4. Numerical methods.

The method of double-layer potentials can be used to obtain numerical approximations to the solution of the Dirichlet problem (A.1.2), (A.1.3): one computes an approximate solution $\tilde{\varphi}$ of (A.1.7) and then uses (A.1.4) to compute the approximate solution $\tilde{u}$.

The first numerical solutions of (A.1.7) (or closely related equations) were obtained by Trefftz [131], Bairstow and Berry [9], and Lauck [74] all of whom used graphical methods to compute the integrals. The first fully numerical method was introduced in 1928 by Nyström [97, 98, 99]. Nyström's method, like all the methods to be discussed here, is a discretization method. That is, a system of linear algebraic equations is set up which, when solved, gives the approximate values of the density φ at a discrete number of points on C. In this connection it is amusing to recall that, writing in 1927 concerning the approximate solution of integral equations, Hellinger and Toeplitz [45, p. 1501] say "Alle Methoden, die das Problem auf die Aufgabe der Auflösung von n linearen Gleichungen mit n Unbekannten zurückfuhren . . . bedeuten numerisch keine wesentliche Forderung . . . ".

When discussing discretization methods it is necessary to distinguish between the case when C is smooth and C has corners:

Discretization when C is smooth

When C is smooth, φ satisfies (A.1.7). Hence φ can be obtained by any method for approximately solving Fredholm integral equations of the second kind. Discussions of methods for approximately solving Fredholm integral equations of the second kind are given by Bückner [17, 18], Gram [37], Kantorowitsch and Krylow [55], Kopal [57], and Walther and Dejon [133]. Recent papers on the subject include those of Anselone [4,5], Atkinson [8], and Noble and Tavernini [96].

As already mentioned, the first solutions of (A.1.7) by discretization techniques were obtained by Nyström [97, 98, 99]. We briefly summarize Nyström's method:

Let

$$\int_0^S f(\sigma)\, d\sigma \approx \sum_{j=1}^{n} w_j^{(n)} f(\sigma_j^{(n)}), \tag{A.4.1}$$

be a quadrature formula. Then (A.1.7) is approximated by

$$\tilde{\varphi}^{(n)}(s) + \sum_{j=1}^{n} w_j^{(n)} K(s, \sigma_j^{(n)}) \quad \tilde{\varphi}^{(n)}(\sigma_j^{(n)}) = g(s)/\pi . \tag{A.4.2}$$

The n-vector $\underline{\tilde{\varphi}}^{(n)} = \{\tilde{\varphi}^{(n)}(\sigma_j^{(n)})\}$ is obtained by solving the linear algebraic system

$$(I + \underline{A}^{(n)})\ \underline{\tilde{\varphi}}^{(n)} = \underline{b}^{(n)}, \tag{A.4.3}$$

where the n-vector $\underline{b}^{(n)}$ and the $n \times n$-matrix $\underline{A}^{(n)}$ are given by

$$\left.\begin{aligned} \underline{b}^{(n)} &= \{ g(\sigma_i^{(n)}) / \pi \}, \\ \underline{A}^{(n)} &= \{ w_j^{(n)} K(\sigma_i^{(n)}, \sigma_j^{(n)}) \} . \end{aligned}\right\} \tag{A.4.4}$$

The approximate solution $\tilde{\varphi}^{(n)}(s)$ is then obtained from (A.4.2):

$$\tilde{\varphi}^{(n)}(s) = g(s)/\pi - \sum_{j=1}^{n} w_j^{(n)} K(s, \sigma_j^{(n)}) \quad \tilde{\varphi}^{(n)}(\sigma_j^{(n)}) . \tag{A.4.5}$$

<u>Discretization when C has corners</u>.

When C has corners, equation (A.1.7) cannot be used. Instead, we must use equation (1.10) which may be written in the form

$$\varphi(s) + \frac{1}{\pi} \int_0^S \varphi(\sigma)\, d_\sigma \psi(s, \sigma) = g(s)/\pi . \tag{A.4.6}$$

The simplest discretization method for (A.4.6) is the following. Let n be an integer and let

$$0 = t_0^{(n)} < \sigma_1^{(n)} < t_1^{(n)} < \sigma_2^{(n)} < \ldots < t_{n-1}^{(n)} < \sigma_n^{(n)} < t_n^{(n)} = S . \tag{A.4.7}$$

Then the values of the approximate solution $\tilde{\varphi}^{(n)}$ at the points $\sigma_j^{(n)}$ are obtained by solving the n linear algebraic equations

$$(I + \underline{A}^{(n)}) \underline{\tilde{\varphi}}^{(n)} = \underline{b}^{(n)} , \qquad \text{(A. 4. 8)}$$

where

$$\left.\begin{aligned} \underline{\tilde{\varphi}}^{(n)} &= \{ \tilde{\varphi}^{(n)}(\sigma_j^{(n)}) \} , \\ \underline{b}^{(n)} &= \{ g(\sigma_i^{(n)})/\pi \} , \\ \underline{A}^{(n)} &= \{ \frac{1}{\pi} \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} d_t\, \psi(\sigma_i^{(n)}, t) \} \\ &= \{ [\psi(\sigma_i^{(n)}, t_j^{(n)}) - \psi(\sigma_i^{(n)}, t_{j-1}^{(n)})]/\pi \} . \end{aligned}\right\} \qquad \text{(A. 4. 9)}$$

The above method was used by Arbenz [7]. The case when the points $\sigma_j^{(n)}$ and $t_j^{(n)}$ are uniformly distributed over $[0, S]$ was considered in 1929 by Krylow and Bogoljubow [68]; see also Kantorowitsch and Krylow [55, p. 124] .

Equations (A. 4. 9) are based on the quadrature formula

$$\int_{t_1}^{t_2} \varphi(t)\, d_t\, \psi(s,t) \approx \varphi(t_3)[\psi(s,t_2) - \psi(s,t_1)], \qquad \text{(A. 4. 9)}$$

where $t_3 \in [t_1, t_2]$. Cryer [25, p. 9] has used the following quadrature formulas in place of (A. 4. 9):

$$\int_{t_1}^{t_2} \psi(t)\, d_t\, \psi(s,t) \approx \frac{[\varphi(t_1) + \varphi(t_2)]}{2} [\psi(s,t_2) - \psi(s,t_1)] , \qquad \text{(A. 4. 10)}$$

and

$$\int_{t_1}^{t_2} \varphi(t)\, d_t\, \psi(s,t) \approx A_1 \varphi(t_1) + A_2 \varphi(t_2) , \qquad \text{(A. 4. 11)}$$

Here, if $P(s) \neq P(t_1)$ and $P(s) \neq P(t_2)$,

$$A_1 = \frac{1}{2\pi r^2}\left[\frac{\Delta\psi}{2}(r^2 + r_2^2 - r_1^2) + r_1 r_2 \sin(\Delta\psi)\log\frac{r_1}{r_2}\right],$$

$$A_2 = \frac{1}{2\pi r^2}\left[\frac{\Delta\psi}{2}(r^2 + r_1^2 - r_2^2) + r_1 r_2 \sin(\Delta\psi)\log\frac{r_2}{r_1}\right],$$

where

$$\Delta\psi = \psi(s,t_2) - \psi(s,t_1),$$

$$r = |P(t_1) - P(t_2)|,$$

$$r_1 = |P(s) - P(t_1)|,$$

$$r_2 = |P(s) - P(t_2)|;$$

if $P(s) = P(t_1)$,

$$A_1 = [\pi + \psi(t_1, t_1+) - \psi(t_1, t_1-)]/4\pi,$$

$$A_2 = 0;$$

if $P(s) = P(t_2)$,

$$A_1 = 0,$$

$$A_2 = [\pi + \psi(t_2, t_2+) - \psi(t_2, t_2-)]/4\pi.$$

Formula (A.4.10) is obvious. Formula (A.4.11) is exact if $\varphi(t)$ is linear in t and the arc $P(t_1)P(t_2)$ is a line segment.

Numerical experiments.

Nyström's method has been used by many workers including Prager [106], Kandler [53], Richardson [112, p. 11], Kantorowitsch and Krylow [55, p. 129], Birkhoff et al [13], Todd and Warschawski [130], Andersen [3]. Several workers (including Nyström) have considered the case of an ellipse, and some comparisons are made by Gaier [33, p. 55].

Three-dimensional problems have been considered by Martenson [86, 87] and Kress [65, 66, 67].

Numerical results for the case when C has corners are given by Arbenz [7], Benveniste [12], Gaier [33, p. 57], Cryer [25, p. 11], Wendland [135, 137].

Finally, we remark that Mikhlin [89, p. 155] applies formulas of type (A.4.4) to a problem in which C is a square. The solution to two decimals of equations (6) on page 157 of Mikhlin is (correcting a misprint)

$$\mu_1 = 0.60\,, \quad \mu_2 = 0.80\,, \quad \mu_3 = 3.32\,.$$

Since μ_3 is the density at a corner of the square, the coefficient 1.1239 in equations (6) should be changed to 1.1239 + .5 = 1.6239 according to (A.4.9). The solution becomes

$$\mu_1 = 0.70\,, \quad \mu_2 = 1.07 \quad \mu_3 = 2.24\,.$$

The interpolating polynomial (8) changes from

$$\mu(x \pm i) = 2.56x^4 + 0.16x^2 + 0.60$$

to the far more reasonable

$$\mu(x \pm i) = 0.08x^4 + 1.46x^2 + 0.70\,.$$

Error Analysis

The error in Nystrom's method has been analyzed by Brakhage [15], Anselone [4, 5], Kantorowitsch and Krylow [55], and Gaier [33].

For the case when C has corners the error has been analysed by Benveniste [12], Bruhn and Wendland [16]; see also Petryshyn [102, 103], and Phillips [104], Haack and Wendland [39], Polsky [105].

For the purposes of error analysis it is important to note that φ is periodic with period S. It is well-known that, when integrating periodic functions, high accuracy can be obtained with simple quadrature formulas such as the trapezoidal rule. Provided that care is taken, this periodicity property can be used to substantially increase the accuracy. See, for example, Birkhoff et al [13, p. 123], Hãmmerlin [43], Kussmaul and Werner [72].

Solution of (A.4.3) and (A.4.8).

Equations (A.4.3) and (A.4.8) can be solved directly. However, many workers have used the iteration

$$\underset{\sim}{\tilde{\varphi}}^{(n,k+1)} = -\underset{\sim}{A}^{(n)} \underset{\sim}{\tilde{\varphi}}^{(n,k)} + \underset{\sim}{b}^{(n)} \quad , k=1,2,\ldots, \tag{A.4.12}$$

which is a discrete analogue of the iteration

$$\varphi^{(k+1)} = -T \varphi^{(k)} + g/\pi \, . \tag{A.4.13}$$

As is well-known, the rate of convergence of (A.4.12) and (A.4.13) depends upon the spectral radii of $A^{(n)}$ and T, respectively.

The iterations (A.4.12) and (A.4.13) have been extensively studied. See Todd [129], Todd and Warschawski [130], Schober [116, 117], Gaier [33, p. 21].

Comparison with finite difference methods for solving the Dirichlet problem.

The greatest advantage of the method of integral equations is its flexibility, particularly for two-dimensional problems. It is possible (Cryer [25, p. 8], Hayes [44]) to write programs which can handle very general boundaries. Moreover, the discretization points $\sigma_j^{(n)}$ can be chosen for convenience and so as to provide the greatest coverage in regions where the solution is expected to vary most rapidly.

However, it must be borne in mind that when $\tilde{\varphi}$ has been computed it still remains to compute $\tilde{u}$, which can take a surprising amount of time. For example, in one example Cryer [25, p. 12] found that $\underset{\sim}{\tilde{A}}$, $\underset{\sim}{\tilde{b}}$, and $\underset{\sim}{\tilde{\varphi}}^{(n)}$, could be computed in 20 minutes (on a computer with a multiplication time of 500μs) but that it took another 70 minutes to compute the values of $\tilde{u}$ on an appropriate grid.

Terry [128] has compared integral equation methods and finite difference methods for Neumann boundary value problems and finds that the finite difference methods are faster.

Related work.

See Hayes [44], Kupradse [69], Symm [125, 126, 127], Jaswon [50], Jeggle [51], Maiti [85], Kussmaul [70, 71], Kussmaul and Werner [72], Barnard et al [10, 11], Lynn and Timlake [80, 81], Ikebe et al [49], Seidel [121], Stiefel [122].

APPENDIX B

RADON'S PAPER: A REWORKING

In this appendix we rework certain parts of Radon [110] and correct some minor errors. Some of these errors, but by no means all, were connected in the translation [111]. This appendix has been written because several authors, including ourselves, have based their work on that of Radon.

We recall that C is of bounded rotation, if (Radon [110, p. 1126])

$$\left.\begin{aligned} x(s) &= x(0) + \int_0^s \cos[\vartheta(\sigma)]d\sigma\,, \quad 0 \le s \le S\,, \\ y(s) &= y(0) + \int_0^s \sin[\vartheta(\sigma)]d\sigma\,, \quad 0 \le s \le S\,, \end{aligned}\right\} \tag{B.1}$$

where $\vartheta = \vartheta(s)$ is of bounded variation on $[0, S]$, that is,

$$\int_0^S |d\vartheta(\sigma)| < \infty\,.$$

Hence, (Riesz and Sz.-Nagy [113, p. 9])

(a) ϑ is continuous except at a denumerable number of points,

(b) The derivative of $\vartheta(s)$, $\vartheta'(s)$, exists a.e. (almost everywhere in the sense of Lebesgue), (B.2)

(c) The following limits exist:

$$\vartheta(0+);\ \vartheta(S-);\ \text{and}\ \vartheta(s+),\ \vartheta(s-),\ 0 < s < S\,.$$

Properties (B.1) and (B.2) remain true if the initial point is changed, if multiples of 2π are added to ϑ, and if the values of ϑ are modified at the (countable) points of discontinuity of ϑ. Therefore, it may be assumed without loss of generality that

$$\left.\begin{array}{ll} \text{(a)} & \vartheta \text{ is continuous at } s = 0 \text{ and } s = S, \\ \text{(b)} & \vartheta(0) \equiv \vartheta(S)^{\dagger}, \\ \text{(c)} & \vartheta'(0) \text{ and } \vartheta'(S) \text{ exist and are equal}, \\ \text{(d)} & |\vartheta(s+) - \vartheta(s-)| \leq \pi, \quad 0 < s < S, \\ \text{(e)} & \vartheta(s) = \vartheta(s+), \quad 0 < s < S \, . \end{array}\right\} \tag{B.3}$$

Points of C at which ϑ is discontinuous are called <u>corners</u>, and points of C at which $|\vartheta(s+) - \vartheta(s-)| = \pi$ are called <u>cusps</u>. We have already seen that there are only countably many corners. Since ϑ is of bounded variation, there are only finitely many cusps.

When studying the properties of C, the main difficulty is that there may be points on C which are points of accumulation of corner points. Let $P(s) \in C$ be such a point of accumulation. Then there are corners $P(s_i) \in C$, $i = 1, 2, \ldots,$ such that $s_i \to s$ as $i \to \infty$. However, since ϑ is of bounded variation, $|\vartheta(s_i+) - \vartheta(s_i-)| \to 0$ as $i \to \infty$. In other words, although C has infinitely many kinks near $P(s)$, the kinks become smaller as they approach $P(s)$. In consequence, C has some of the properties of piecewise smooth curves, as can be seen from Theorem 2.1.

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†Throughout this appendix modulus equations are to be understood as being modulo 2π.

Radon's proof of Theorem 2.1 (Radon [110, pages 1128, 1130, and 1131]) contains three minor misprints:

(i) At the bottom of page 1128

$$\text{for } \int_s^\sigma \cos[\vartheta(\tau) - \vartheta(\sigma)]d\tau \text{ read } \int_s^\sigma \cos(\vartheta(\tau) - \vartheta(\sigma+)]d\tau ,$$

(ii) On line two of page 1129 for μ^2 read μ.

(iii) On line three of page 1129 for "alle s, für welche $|s - \sigma| < h$" read "alle σ, für welche $|s - \sigma| < h$ und $\sigma \neq s$" .

Remark.

At first sight it might appear that by using a compactness argument $\epsilon(s)$ could be taken to be independent of s in Theorem 2.1. This is not so as can be seen by considering the case when C is an equi-lateral triangle.

The function $\psi(s, \sigma)$ is defined by (2.1) for $(s, \sigma) \in \mathcal{I}$ and $(\sigma, s) \in \mathcal{I}$. Namely,

$$\left.\begin{aligned} \cos \psi(s, \sigma) &= [x(s) - x(\sigma)]/|P(s) - P(\sigma)|, \\ \sin \psi(s, \sigma) &= [y(s) - y(\sigma)]/|P(s) - P(\sigma)| , \\ &\text{for } (s, \sigma) \in \mathcal{I}, \end{aligned}\right\} \tag{B.4}$$

$$\psi(s, \sigma) = \psi(\sigma, s), \text{ for } (\sigma, s) \in \mathcal{I}. \tag{B.5}$$

We now consider the arguments whereby the domain of definition of ψ is extended to $[0, S] \times [0, S]$.

First, it follows from (B.1) and (B.4) that if s is a point of continuity of ϑ then†

$$\lim_{\substack{s'\to s\\ s''\to s}} \psi(s', s'') \equiv \vartheta(s) + \pi . \qquad (B.6)^*$$

Radon states [110, p. 1131] that

$$\lim_{\substack{s'\to s\\ s''\to s}} \psi(s', s'') \equiv \vartheta(s) ;$$

however, we believe that (B.6) is correct. Similar errors occur in most of Radon's equations connecting ψ and ϑ, but these errors do not propagate. We mark with an asterisk those equations where we differ from Radon. The equations of Radon would be correct if the roles of s and σ were interchanged in (B.4):

$$\cos \psi(s, \sigma) = [x(\sigma) - x(s)]/|P(s) - P(\sigma)| ,$$

$$\sin \psi(s, \sigma) = [y(\sigma) - y(s)]/|P(s) - P(\sigma)| ,$$
$$\text{for } (s, \sigma) \in \mathcal{P},$$

but this would require further changes. Benveniste [12] has used the above equations but it is not clear whether his use of them was intentional.

It follows from (B.6) that $\psi(s, \sigma)$ can be extended by continuity to the points $s = \sigma$ which are not corners of C. Since $P(0)$ is not a corner, ψ is uniquely defined if it is required that

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†Throughout this appendix equations such as (B.6) are to be understood as implying the existence of the limits involved.

$$0 \leq \psi(0,\ 0) < 2\pi\ . \tag{B.7}$$

Remembering that $P(0) = P(S)$ so that

$$\int_0^S \cos[\vartheta(\tau)]d\tau = \int_0^S \sin[\vartheta(\tau)]d\tau = 0,$$

it follows from (B.1), (B.3a), and (B.4), that

$$\lim_{\substack{s \to 0 \\ \sigma \to S}} \psi(s,\ \sigma) \equiv \vartheta(0), \tag{B.8}^*$$

so that ψ can be extended by continuity to the points $(0,\ S)$ and $(S,\ 0)$.

Finally, from (B.1), (B.4), and (B.5),

$$\left.\begin{aligned} \psi(s,\ s+) &= \lim_{\sigma \to s+} \psi(s,\ \sigma) \equiv \vartheta(s+) + \pi\ , \\ \psi(s,\ s-) &= \lim_{\sigma \to s-} \psi(s,\ \sigma) \equiv \vartheta(s-) + \pi\ , \\ &\qquad \text{for } 0 < s < S\ . \end{aligned}\right\} \tag{B.9}^*$$

Introducing

$$\psi_+(s) = \begin{cases} \psi(0,\ 0), & s = 0, \\ \psi(s,\ s+), & 0 < s < S, \\ \psi(S,\ S), & s = S, \end{cases} \tag{B.10}$$

$$\psi_-(s) = \begin{cases} \psi(0,\ 0), & s = 0, \\ \psi(s,\ s-), & 0 < s < S, \\ \psi(S,\ S), & s = S, \end{cases} \tag{B.11}$$

and setting

$$\psi(s, s) = \psi_+(s), \quad 0 \leq s \leq S, \tag{B.12}$$

the definition of ψ on $[0, S] \times [0, S]$ is complete.

By evaluating a certain contour integral on C, two useful (and geometrically obvious) formulas can be obtained:

$$\psi(s, S) - \psi(s, 0) = \pi, \quad 0 \leq s \leq S, \tag{B.13}$$

$$\psi_+(s) - \psi_-(s) = W_-(s) - \pi, \quad 0 \leq s \leq S, \tag{B.14}$$

where $W_-(s)$ is the exterior angle at $P(s)$ (see Theorem 2.1.).

The final arguments concern the connection between ϑ and ψ. It is asserted that (B.1), (B.2), and (B.3) remain valid if ϑ is redefined by means of

$$\vartheta(s) = \psi_+(s) + \pi . \tag{B.15}$$

To justify (B.15) it is necessary to obtain some further properties of ψ. ψ is continuous on $[0, S] \times [0, S]$ except at the points $s = \sigma$ corresponding to the corners of C. To analyse the behavior of ψ at these points one notes from (B.1), (B.4), and (B.5), that if $s \in (0, S)$ and $\epsilon > 0$ are given then there is a $\delta = \delta(s, \epsilon) > 0$ such that

$$\left.\begin{aligned} &|\cos[\psi(s', s'')] - \cos[\vartheta(s+) + \pi]| \leq \epsilon, \\ &|\sin[\psi(s', s'')] - \sin[\vartheta(s+) + \pi]| \leq \epsilon, \\ &\qquad \text{for } s \leq s' < s'' \leq s + \delta, \end{aligned}\right\} \tag{B.16}$$

and

$$\left.\begin{aligned} &|\cos[\psi(s', s'')] - \cos[\vartheta(s-) + \pi]| \leq \epsilon, \\ &|\sin[\psi(s', s'')] - \sin[\vartheta(s-) + \pi]| \leq \epsilon, \\ &\text{for } s - \delta \leq s' < s'' \leq s. \end{aligned}\right\} \quad (B.17)$$

It follows that

$$\left.\begin{aligned} &\psi_+(s+) = \psi_+(s), \quad \psi_+(s-) = \psi_-(s), \\ &\psi_-(s+) = \psi_+(s), \quad \psi_-(s-) = \psi_-(s). \end{aligned}\right\} \quad (B.18)$$

For example, it follows from (B.16) that

$$\lim_{\substack{s', s'' \to s- \\ s' < s''}} \psi(s', s'')$$

exists. Hence

$$\begin{aligned} \psi_+(s-) &= \lim_{s' \to s-} \left[\lim_{s'' \to s'+} \psi(s', s'') \right], \\ &= \lim_{\substack{s', s'' \to s- \\ s' < s''}} \psi(s', s''), \\ &= \lim_{s' \to s-} \psi(s', s), \\ &= \psi(s-, s), \\ &= \psi_-(s). \end{aligned}$$

From (B.3e), (B.6), (B.9), (B.12), and (B.18),

$$\vartheta(s) \equiv \psi_+(s) + \pi, \quad 0 \leq s \leq S, \quad (B.19)$$

while it follows from (B.14) and (B.18) that

$$|\psi_+(s+) - \psi_+(s-)| \leq \pi. \quad (B.20)$$

Hence, since $\psi_+(s)$ is continuous when ϑ is continuous, ψ_+ is of bounded variation. It is now easy to justify (B.15).

APPENDIX C

PROPERTIES OF THE SPACE $\mathcal{B}$

The space $\mathcal{B}$ was apparently first introduced by Hahn [41, p. 53] who proved most of the results given here.

We recall that $\mathcal{B}$ consists of functions f for which the limits

$$\left.\begin{aligned} f(s+0) &= \lim_{\epsilon \to 0} f(s+|\epsilon|), \quad 0 \le s < S, \\ f(s-0) &= \lim_{\epsilon \to 0} f(s-|\epsilon|), \quad 0 < s \le S, \end{aligned}\right\} \tag{C.1}$$

exist and satisfy

$$f(s) = [f(s+0) + f(s-0)]/2, \quad 0 < s < S\ . \tag{C.2}$$

$\mathcal{B}$ is equipped with the maximum norm. We also recall that $\mathcal{S}$ denotes the subspace of stepfunctions.

Theorem C.1.

$\mathcal{B}$ is a Banach space.

Proof: See Hahn [41, p. 53].

Since $\mathcal{B}$ is clearly a normed linear space, it suffices to prove that $\mathcal{B}$ is complete.

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{B}$. Then

$$f(s) = \lim_{n \to \infty} f_n(s) \tag{C.3}$$

is well-defined. It remains to show that (C.1) and (C.2) hold.

Let $0 \leq s < S$. Choose $\epsilon_1 > 0$. Since $\{f_n\}$ is a Cauchy sequence there is an $N_1 = N_1(\epsilon_1)$ such that

$$|f_n(t) - f_m(t)| \leq \epsilon_1/3, \quad \text{for} \quad m, n \geq N_1, \quad \text{and } t \in [0, S]. \tag{C.4}$$

Let $m, n > N_1$. Then there is an $s' > s$ such that

$$|f_n(s') - f_n(s+0)|, \quad |f_m(s') - f_m(s+0)| \leq \epsilon_1/3 . \tag{C.5}$$

Combining (C.4) and (C.5),

$$|f_n(s+0) - f_m(s+0)| \leq \epsilon_1, \quad \text{if} \quad n, m \geq N_1 .$$

Consequently, $\{f_n(s+0)\}$ is a Cauchy sequence. Set

$$p(s) = \lim_{n \to \infty} f_n(s+0), \quad 0 \leq s < S. \tag{C.6}$$

Now choose $\epsilon > 0$. Then there is an N such that

$$\left.\begin{aligned} &|f(t) - f_N(t)| \leq \epsilon/3, \quad \text{for} \quad 0 \leq t < S , \\ &|p(s) - f_N(s+0)| \leq \epsilon/3 . \end{aligned}\right\} \tag{C.7}$$

Given N there is a $\delta > 0$ such that

$$|f_N(s+0) - f_N(t)| \leq \epsilon/3 , \quad \text{for} \quad s < t \leq s + \delta . \tag{C.8}$$

Combining (C.7) and (C.8),

$$|p(s) - f(t)| \leq \epsilon, \quad \text{for} \quad s < t \leq s + \delta . \tag{C.9}$$

Consequently, $f(s + 0)$ exists for $0 \leq s < S$ and satisfies

$$f(s+0) = p(s) = \lim_{n\to\infty} f_n(s + 0), \quad 0 \leq s < S . \tag{C.10}$$

In similar fashion it can be shown that $f_n(s - 0)$ exists for $0 < s \leq S$ and satisfies

$$f(s + 0) = p(s) = \lim_{n\to\infty} f_n(s - 0), \quad 0 < s \leq S. \tag{C.11}$$

The proof of (C.1) is therefore complete.

To prove (C.2) it sufficies to note that

$$f_n(s) = [f_n(s + 0) + f_n(s - 0)]/2 ,$$

and use (C.3), (C.10), and (C.11).

<u>Theorem C.2.</u>

If $f \in \mathcal{B}$ then f has at most a denumerable number of points of discontinuity.

Let $\eta > 0$ and set

$$J_\eta = \{s \in (0, S); \ |f(s + 0) - f(s - 0)| \geq \eta\} . \tag{C.12}$$

Then J_η is a finite set.

<u>Proof</u>: See Hahn [40, p. 216] or Hobson [48, p. 304].

Suppose that, for some η, J_η is not a finite set. Then J_η has at least one limit point, s say, in $[0, S]$. Set $I = (s, S)$ if s is a limit point of $(s,S) \cap J_\eta$; otherwise set $I = (0,s)$. For $\epsilon > 0$ set

$$I_\epsilon = I \cap (s-\epsilon\ ,\ s+\epsilon\ ) .$$

Since $I_\epsilon \cap J_\eta$ is not empty, there is an $s' \in I_\epsilon$ such that

$$|f(s'+0) - f(s'-0)| \geq \eta$$

Consequently, there are points $s_1, s_2 \in I_\epsilon$ such that

$$|f(s_1) - f(s_2)| \geq \eta/2 .$$

But this contradicts (C.1) (since I_ϵ lies on one side of s). Consequently, as asserted in the theorem, J_η is a finite set for all η.

Now let $\eta_1, \eta_2, \ldots$ be a decreasing sequence of numbers which tend to zero. Set

$$J = [\ \bigcup_{i=1}^{\infty} J_{\eta_i}\ ] \cup \{0,S\} .$$

Then J is denumerable. But J contains all the points of discontinuity of f. The theorem follows.

We recall that f is said to have <u>discontinuities of at most the first kind</u> in $[0,S]$ if the limits

$$f(s+0)\ , \qquad 0 \leq s < S\ ,$$

$$f(s-0), \qquad 0 < s \leq S\ ,$$

exist (Hobson [48, p. 301]).

Theorem C.3.

Let f be defined on $[0,S]$. In order that f have discontinuities of at most the first kind, it is necessary and sufficient that for every $\epsilon > 0$ there exist a_i,

$$0 = a_0 < a_1 < \ldots < a_n = S\ , \tag{C.13}$$

such that

$$\omega\,[f:(a_{i-1},a_i)] \leq \epsilon, \quad 1 \leq i \leq n, \tag{C.14}$$

where

$$\omega\,[f:(a,b)] = \sup_{a < t_1, t_2 < b} |f(t_1) - f(t_2)|. \tag{C.15}$$

Proof: See Hahn [40, p. 217].

Necessity: Assume that f has discontinuities of at most the first kind. Let $\epsilon > 0$ be given. Then to each $s \in [0,S]$ there corresponds a $\delta = \delta(s,\epsilon) > 0$ such that

$$\omega\,[f:(s, s+\delta) \cap [0,S]] \leq \epsilon,$$

$$\omega\,[f:(s-\delta, s) \cap [0,S]] \leq \epsilon.$$

Set

$$I_s = (s-\delta, s+\delta) \cap [0,S].$$

Then the open intervals I_s cover $[0,S]$. Hence $[0,S]$ is covered by a finite number of the I_s, say by

$$I_{s_i} = (s_i - \delta_i, s_i + \delta_i) \cap [0,S], \quad 1 \leq i \leq m.$$

Clearly, it may be assumed that $I_{s_i} \subset I_{s_j}$ iff $i = j$, and that $s_i < s_{i+1}$. Then I_{s_i} and $I_{s_{i+1}}$ overlap, and there exist s_i' such that

$$s_i' \in (s_i, s_i + \delta_i) \cap (s_{i+1} - \delta_{i+1}, s_{i+1}), \quad 1 \leq i \leq m-1.$$

Setting $n = 2m$, $a_0 = 0$, $a_n = S$, $a_{2i-1} = s_i$, $1 \leq i \leq m$, and $a_{2i} = s_i'$, $i \leq i \leq m-1$, it follows that that (C.13) and (C.14) hold.

Sufficiency: Assume that at the point $s \in [0,S]$ f has a discontinuity which is not of the first kind, that is, either.

Case (i): $s \in [0,S)$ and $f(s+0)$ does not exist,

or

Case (ii): $s \in (0,S]$ and $f(s-0)$ does not exist .

Consider case (i). Then there is an $\epsilon_1 > 0$ and a $\delta_1 > 0$ such that

$$\omega\,[f: (s,\ s+\delta)] \geq \epsilon_1 \quad \text{for all} \quad \delta \in (0,\ \delta_1). \tag{C.16}$$

Now suppose that (C.13) holds. Then there is a $\delta \in (0,\ \delta_1)$ such that $(s, s+\delta) \subset (a_{i-1}, a_i)$ for some i . But then, noting (C.16), (C.14) does not hold for $\epsilon = \epsilon_1$. That is, in case (i), (C.13) and (C.14) cannot hold for all ϵ .

In similar fashion it can be shown that, in case (ii), (C.13) and (C.14) also cannot hold for all ϵ . Sufficiency has therefore been proved.

Theorem C.4.

$\mathcal{D}$ is not separable.

Proof: Assume that $\mathcal{D}$ is separable. Then (Dunford and Schwartz [26 , p. 21]) there exists a denumerable sequence $\{f_i\}$, $f_i \in \mathcal{D}$, such that for any $f \in \mathcal{D}$ and any $\epsilon > 0$ there is an f_n for which

$$\|f - f_n\| \leq \epsilon\ . \tag{C.17}$$

Let J_i be the set of points of discontinuity of f_i and $J = \cup J_i$. From Theorem C.2 we know that J_i is denumerable so that J is denumerable. Since $(0,S)$ is not denumerable, there is an $s \in (0,S)$ such that $s \notin J$. Set

$$f(t) = \begin{cases} 0\ , & 0 \leq t < s, \\ \frac{1}{2}\ , & t = s, \\ 1\ , & s < t \leq S\ . \end{cases}$$

Since s is a point of continuity of f_i for all i , it follows that

$$\|f - f_i\| \geq \frac{1}{2}, \quad \text{for all } i\ . \tag{C.18}$$

Comparing (C.17) and (C.18) we see that we have reached a contradiction.

Theorem C.5

$\mathcal{S}$ is dense in $\mathcal{D}$.

Proof: See Hahn [41, p.55].

Let $f \in \mathcal{D}$. Choose $\epsilon > 0$. It follows from Theorem C.3 that there exist a_i such that (C.13) and (C.14) hold. Hence, there exist c_i such that

$$|f(s) - c_i| \leq \epsilon \,, \quad s \in (a_{i-1}, a_i) \,, \quad 1 \leq i \leq n \,.$$

Set

$$f^*(s) = \begin{cases} c_i \,, \quad s \in (a_{i-1}, a_i) \,, & 1 \leq i \leq n \,, \\ (c_i + c_{i+1})/2 \,, \quad s = a_i \,, & 1 \leq i < n-1 \,, \\ f(0) \,, \quad s = a_0 \,, & \\ f(S) \,, \quad s = a_n \,. & \end{cases}$$

Then $f^* \in \mathcal{S}$. Clearly,

$$|f(s) - f^*(s)| \leq \epsilon \,, \qquad s \in (a_{i-1}\, a_i) \,, \quad 1 \leq i \leq n \,,$$

while it follows from (C.2) that

$$|f(s) - f^*(s)| \leq \epsilon \,, \quad s = a_i \,, \quad 0 \leq i \leq n \,.$$

Thus, $\|f - f^*\| \leq \epsilon$. Consequently, $\mathcal{S}$ is dense in $\mathcal{D}$.

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