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BOUNDARY VALUE CONTROL OF THE HIGHER DIMENSIONAL WAVE EQUATION: PART II*

by

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1. INTRODUCTION

This paper is a sequel to [13], where we began our study of the approximate controllability of the higher dimensional wave equation with boundary value controls. There, and here, we let Ω be a bounded, open, connected domain in R^n whose boundary, Γ, is an analytic (or C^∞ and piecewise analytic) (n - 1)-dimensional surface in R^n. We parametrize Γ with an (n - 1)-dimensional vector variable s and indicate points on Γ by $x(s)$. Integrals over Ω are written as $\int_\Omega (\quad)dx$ while integrals over Γ are written $\int_\Gamma (\quad)ds$. Taking $\tilde{\Gamma}$ to be a relatively open subset of Γ and T a positive number, we define an admissible control to be a function $f: \Gamma \otimes [0,T] \to R^1$ such that $f \in C^\infty(\Gamma \otimes [0,T])$ and f vanishes identically outside a compact subset of $\tilde{\Gamma} \otimes (0,T)$.

For all such admissible controls f we let $w^f(x,t)$ solve the linear hyperbolic mixed initial-boundary value problem

$$(1.1) \qquad \rho(x)\, w^f_{tt} - \sum_{i,j=1}^{n} (\alpha_{ij}(x) w^f_i)_j = 0 \quad \text{in } \Omega \otimes [0,T]\,,$$

$$(1.2) \qquad w^f_x(x(s),t)\, A(x(s))\eta(x(s)) = f(s,t), \text{ on } \Gamma \otimes [0,T]\,,$$

$$(1.3) \qquad w^f(x,0) \equiv w^f_t(x,0) \equiv 0,\; x \in \Omega\,.$$

The subscripts t and i denote partial differentiation with respect to t and x^i (the i-th component of $x \in R^n$) respectively. The subscript x indicates the gradient vector of the vector function to which it is applied. The vector $\eta(x(s))$ is the outward unit normal to Γ at $x(s) \in \Gamma$. The real analytic functions $\rho(x)$, $\alpha_{ij}(x)$, $i, j = 1,2,\ldots,n$, are such that

$$\alpha_{ij}(x) = \alpha_{ji}(x),$$

$$\rho(x) \geq \rho_0 > 0 ,$$

$$v'A(x)v \geq \delta_0 \|v\|^2, \quad \delta_0 > 0,$$

in some open set which includes $\Omega \cup \Gamma$. Here $A(x)$ is the $n \times n$ symmetric matrix whose entries are $\alpha_{ij}(x)$.

From [3] and [8] we learn that (1.1), (1,2), (1.3) has a unique C^∞ solution in $\Omega \otimes [0,T]$. Thus we may let R_T denote the set of all terminal states $(w^f(\cdot,T), w_t^f(\cdot,T))$. The set R_T is a subspace of the Hilbert space $H_E(\Omega)$ of finite energy states with inner product

$$<(u,u_t); (v,v_t)>_E = \int_\Omega [\rho(x)u(x)v(x) + u_x(x)A(x)v_x(x)'] \, dx$$

(here ' denotes the transpose of a vector) and norm

$$\|(v,v_t)\|_E = (<(v,v_t); (v,v_t)>_E)^{\frac{1}{2}} .$$

The gradients u_x, v_x are defined in the sense of the theory of distributions. To avoid an indefinite innner product, two states which differ by (c,o), where c is a constant function on Ω, are identified. However, we will continue to speak of elements of $H_E(\Omega)$ as "states" rather "equivalence classes of states".

The control system (1.1), (1.2) is said to be approximately controllable in time T if R_T is dense in $H_E(\Omega)$, i.e., if the validity of the equation

$$<(w^f(\cdot,T),\, w_t^f(\cdot,T));\, (\hat{v},\hat{v}_t)>_E \;=\; 0$$

for all $f \in R_T$ implies that $(\hat{v},\hat{v}_t) = (c,o)$, a zero energy state in $H_E(\Omega)$.

In [13] we showed that Ω, Γ, ρ and A determine a positive number T_0 such that:

(i) if $T < 2T_0$ the system (1.1), (1.2), (1.3) is not approximately controllable in time T;

(ii) if $T > 2T_0$ and $n \leq 3$ then the system is approximately controllable in time T.

We will refer to $2T_0$ as the <u>critical time</u>. When $n = 1$ it is known (see [5], [14], [15], e.g.) that approximate controllability continues to hold for $T = 2T_0$.

The purpose of the present paper is two-fold. First, we show in Section 2 that if $T > 2T_0$ approximate controllability holds without any restriction on the dimension n. Second, we show in the remaining sections that if $n \geq 2$ approximate controllability may or may not hold for $T = 2T_0$, the critical time, depending on certain relationships between Γ, ρ and A. Because the proofs for $T = 2T_0$ are very detailed, they are given only for special examples. In the concluding remarks we describe the form which a general theory of critical time approximate controllability would take.

2. A NEW PROOF OF APPROXIMATE CONTROLLABILITY FOR $T > 2T_0$.

The theorem which we will prove in this section replaces Theorem 4 in [13]. The new result has the advantage of being valid for all positive integers n . Many of the details of the proof are the same as in the earlier result. Therefore we will concentrate on the essential differences, referring the reader to [13] for complete treatment of parts common to both proofs.

Let $(\hat{v}, \hat{v}_t)$ be a finite energy state, i.e., $\|(\hat{v}, \hat{v}_t)\|_E < \infty$, and assume that $(\hat{v}, \hat{v}_t)$ is orthogonal to all states $(w^f(\cdot, T), w_t^f(\cdot, T))$ in R_T relative to the energy inner product. Thus

$$(2.1) \quad \langle (w^f(\cdot,T),\, w_t^f(\cdot,T));\, (\hat{v}, \hat{v}_t) \rangle_E = \int_\Omega [\rho(x) w_t^f(x,T) \hat{v}_t(x) + w_x^f(x,T) A(x) \hat{v}'_x(x)]\, dx = 0$$

for all admissible controls f . We let $v(x,t)$ be the generalized solution of the mixed problem

$$(2.2) \quad \rho(x)\, v_{tt} - \sum_{i,j=1}^{n} (\alpha_{ij}(x) v_i)_j = 0 \quad \text{in } \Omega \otimes [0,T],$$

$$(2.3) \quad v_x(x(s),t)\, A(x(s))\, \eta(x(s)) = 0, \quad (x(s),t) \in \Gamma \otimes [0,T],$$

$$(2.4) \quad v(x,T) = v(x), \quad v_t(x,T) \quad v_t(x) .$$

The existence of such a solution is proved, e.g., in [8] and [10], where it is likewise shown that $v(\cdot,t)$ and $v_t(\cdot,t)$ define continuous functions from $[0,T]$ into $H^1(\Omega)$ and $H^0(\Omega) = L(\Omega)$, respectively. (Recall that if m is a non-negative integer, then $H^m(\Omega)$ consists of functions $u(x)$ whose derivatives of order $\leq m$, taken in the sense of the theory of distributions, lie in $L^2(\Omega)$. $H^m(\Omega)$ is a Hilbert space with inner product

$$(u,\hat{u})_{H^m(\Omega)} = \sum_{o \leq \|p\| \leq m} \int_\Omega [D^p u(x)\, D^p \hat{u}(x)]\, dx.$$

Here p is an n-vector with non-negative integer components $p_1, p_2, \ldots, p_n$, $\|p\| = p_1 + p_2 + \cdots + p_n$, and D^p denotes $\dfrac{\partial^{\|p\|}}{(\partial x^1)^{p_1} (\partial x^2)^{p_2} \ldots (\partial x^n)^{p_n}}$).

As in [13] we smooth the solution $v(x,t)$ by a process of antidifferentiation and formation of finite differences. The innovation lies in the way in which the antiderivatives are defined. We consider the elliptic operator

$$Bu = \frac{1}{\rho(x)} \sum_{i,j=1}^{n} (\alpha_{ij}(x) u_i)_j$$

which is defined on functions $u \in C^2(\bar{\Omega})$ $(\bar{\Omega} = \Omega \cup \Gamma)$ satisfying the boundary conditions

$$u(x(s))\, A(x(s))\, \eta(x(s)) = 0, \quad x(s) \in \Gamma .$$

This unbounded symmetric operator has an unbounded self-adjoint extension, which we will still call B, defined on a domain D dense in $L^2(\Omega)$. (See e.g., [4], [6]). Moreover, if $(u,1)_{L^2(\Omega)} = 0$, then there is a positive number λ_0, the smallest eigenvalue of B except 0, such that

$$\|Bu\| \geq \lambda_0 \|u\| .$$

From this it follows that if we let $\hat{B}$ denote the restriction of B to $D \cap \{u \in L^2(\Omega) | (u, 1)_{L^2(\Omega)} = 0\}$ then $\hat{B}^{-1}$ is defined, bounded and self adjoint on $\{u \in L^2(\Omega) | (u,1)_{L^2(\Omega)} = 0\}$, which we will call $\hat{D}$.

From the work of Lions-Magenes ([9], p. 165 ff.) it is known that if $g \in \hat{D} \cap H^m(\Omega)$, $m \geq 0$, then $\hat{B}^{-1} g \in \hat{D} \cap H^{m+2}(\Omega)$ and the mapping $g \in \hat{D} \cap H^m(\Omega) \rightarrow \hat{B}^{-1} g \in \hat{D} \cap H^{m+2}(\Omega)$ is continuous with respect to the norms $\| \ \|_{H^m(\Omega)}$, $\| \ \|_{H^{m+2}(\Omega)}$.

We return to $(\hat{v}, \hat{v}_t)$ and let c_1 and c_2 be real constants such that

$$(2.5) \quad \int_\Omega (\hat{v}(x)-c_1)\, dx = \int_\Omega (\hat{v}_t(x) - c_2)\, dx = 0 .$$

Then

$$\tilde{v}(x,t) = (v(x,t) - c_1 - c_2(t - T)$$

satisfies (2.2) and (2.3) and $\tilde{v}(\cdot,t) \in \hat{D} \cap H^1(\Omega)$, $t \in [0,T]$. Likewise

$$\tilde{v}_t(x,t) = v_t(x,t) - c_2$$

is such that $\tilde{v}_t(\cdot,t) \in \hat{D} \cap H^0(\Omega)$, $t \in [0,T]$. We define, for each non-negative integer k,

$$D^{-2k} \tilde{v} = \hat{B}^{-k} \tilde{v}, \quad D^{-2k+1} = \hat{B}^{-k} \tilde{v}_t$$

and conclude from the above cited work in [9] that for a non-negative integer m,

$$D^{-m} \tilde{v}(\cdot,t) \in \hat{D} \cap H^{m+1}(\Omega), \quad t \in [0,T] ,$$

and that $D^{-m}\tilde{v}(\cdot,t)$ is a continuous function of t relative to the norm $\| \; \|_{H^{m+1}(\Omega)}$. Since $\tilde{v}(\cdot,t)$ is a generalized solution of $\tilde{v}_{tt} = B\tilde{v}$ (i.e. (2.2)) one can verify without difficulty that $D^{-m}\tilde{v}$ satisfies the same equation (in the strict sense if $m > 0$) and that

$$\frac{d^m}{dt^m}\left(D^{-m}\tilde{v}(\cdot,t)\right) = \tilde{v}(\cdot,t) .$$

Next we define

$$D^{-m} v(\cdot,t) = D^{-m}\tilde{v}(\cdot,t) + c_1\frac{(t-T)^m}{m!} + c_2\frac{(t-T)^{m+1}}{(m+1)!} \tag{2.6}$$

and verify that

$$\frac{d^m}{dt^m}\left(D^{-m} v(\cdot,t)\right) = v(\cdot,t) .$$

It is not in general true that $D^{-m} v(\cdot,t)$ is a solution of $v_{tt} = Bv$. But since c_1 and c_2 are constants it is clear that we still have

$$D^{-m} v(\cdot,t) \in H^{m+1}(\Omega), \quad t \in [0,T], \quad m \geq 0 . \tag{2.7}$$

We now refer to the theorem of Sobolev (see, e.g., [1], p. 32) which states that if $v \in H^m(\Omega)$ and if ℓ is a positive integer strictly less than $m - n/2$ then $v \in C^\ell(\Omega)$. Moreover, there is a constant K, independent of v, such that

$$\|v\|_{C^\ell(\bar{\Omega})} \leq K\|v\|_{H^m(\Omega)} . \tag{2.8}$$

We choose $m = 2k$ to be a positive integer such that $m - n/2 > 1$. Then from (2.7) and the Sobolev theorem we have

$$D^{-m} v(\cdot,t) \in C^2(\bar{\Omega}), \quad D^{-m+1} v(\cdot,t) = D^{-m} v_t(\cdot,t) \in C^1(\bar{\Omega}) .$$

The continuity of $D^{-m} v(\cdot,t)$, $D^{-m+1} v(\cdot,t)$, as functions of t, with respect to $\| \ \|_{H^{m+1}(\Omega)}$, $\| \ \|_{H^m(\Omega)}$, respectively, combined with (2.8), then shows that $D^{-m} v(\cdot,t) \in C^2(\bar{\Omega} \otimes [0,T])$.

Now, for $\delta > 0$, we define

$$\Delta(D^{-m} v(\cdot,t)) = D^{-m} v(\cdot,t+\delta) - D^{-m} v(\cdot,t), \quad t \in [0,T-\delta] ,$$

$$\Delta^k(D^{-m} v(\cdot,t)) = \Delta(\Delta^{k-1}(D^{-m} v(\cdot,t))), \quad t \in [0,T-k\delta] .$$

Noting (2.6), the fact that $D^{-m} \tilde{v}$ solves $\tilde{v}_{tt} = B\tilde{v}$, and the fact that $v \in C^2(\bar{\Omega} \otimes [0,T])$, we see that $\Delta^m(D^{-m} v(\cdot,t))$, which we will call $v^\delta(\cdot,t)$, is such that $v^\delta(x,t)$ is a C^2 solution of (2.2), (2.3) in $\bar{\Omega} \otimes [0,T]$.

The rest of the proof proceeds much as in [13] and we will give an outline only. The interested reader should consult the earlier paper for details, noting that there $\tilde{\Gamma}$ of this paper was called Γ_1.

Using the divergence theorem one shows that (2.1) implies (with D denoting $\frac{\partial}{\partial t}$)

$$(2.9) \qquad \int_{\tilde{\Gamma} \otimes [0,T]} [D^{-m+1} v(x(s),t) \, D^m f(s,t)] dx \, dt = 0$$

for all admissible controls f. This implies that $D^{-m+1} v(x(s),t) = (D^{-m} v(x(s),t))_t$, is a polynomial in t of degree at most $m - 1$ whose coefficients are C^1 functions of $x(s)$, for $(x(s),t) \in \tilde{\Gamma} \otimes [0,T]$. Then

$$(\Delta^m(D^{-m} v(x(s),t)))_t = \Delta^m((D^{-m} v(x(s),t))_t) \equiv 0, \ (x(s),t) \in \tilde{\Gamma} \otimes [0,T-m\delta].$$

This, combined with the fact that $\Delta^m(D^{-m} v)$ satisfies the boundary condition (2.3), enables us to use the Holmgren-Fritz John uniqueness theorem [7] to show that $(\Delta^m(D^{-m} v))_t$ must vanish identically for $(x,t) \in K(\tilde{\Gamma},0,T-m\delta)$, the intersection of the forward cone of influence of $\tilde{\Gamma}$ at time 0 with the backward cone of influence of $\tilde{\Gamma}$ at time $T - m\delta$. If $T > 2T_0$ the set $K(\tilde{\Gamma},0,T-m\delta)$ includes a set $\bar{\Omega} \otimes [(T/2)-\varepsilon,(T/2)+\varepsilon]$ for some $\varepsilon > 0$, provided $\delta > 0$ is sufficiently small. (See figures in [13].) Thus,

$$(\Delta^m(D^{-m} v(x,t)))_t \equiv 0, \quad (x,t) \in \bar{\Omega} \otimes [(T/2)-\varepsilon,(T/2)+\varepsilon]$$

which clearly implies

$$(\Delta^m(D^{-m} v(x,t)))_{tt} \equiv 0, \quad (x,t) \in \bar{\Omega} \otimes [(T/2)-\varepsilon,(T/2)+\varepsilon].$$

Since $\Delta^m(D^{-m} v)$ is a C^2 solution of (2.2), (2.3) we conclude that

$$v(x,t) \equiv v(x), \quad (x,t) \in \bar{\Omega} \otimes [(T/2)-\varepsilon, (T/2)+\varepsilon]$$

where $v(x)$ is a C^2 solution of the elliptic boundary value problem

(2.10) $$\sum_{i,j=1}^{n} (\alpha_{ij}(x)u_i)_j = 0, \quad x \in \Omega$$

(2.11) $$u_x(x(s))\, A(x(s))\, \eta(x(s)) = 0, \quad x(s) \in \Gamma .$$

But the only solutions of (2.10), (2.11) have the form

$$u(x) = c, \text{ a constant}, \quad x \in \Omega .$$

Thus

$$\Delta^m (D^{-m} v(x,t)) = c, \quad (x,t) \in \Omega \otimes [(T/2) - \varepsilon, (T/2) + \varepsilon]$$

so that $D^{-m} v(x,t)$ is a polynomial in t of degree at most m whose coefficients are C^2 functions of x for $x \in \Omega$. Then $v(x,t) = D^m(D^{-m} v(x,t))$ is a constant in $\Omega \otimes [(T/2) - \varepsilon, (T/2) + \varepsilon]$. In particular,

$$(v(\cdot, T/2), v_t(\cdot, T/2)) = (c, 0) ,$$

a zero energy state. Applying to conservation of energy principle, which is valid for generalized solutions of (2.2), (2.3), we infer that

$$(v(\cdot, T), v_t(\cdot, T)) = (v, v_t) = (c, 0).$$

We see therefore that if (2.1) holds for all admissible controls f, so that (v, v_t) is orthogonal, relative to the energy inner product $\langle ; \rangle_E$, to every state in R_T, then $\| (v, v_t) \|_E = 0$ and (v, v_t) is the null element in $H_E(\Omega)$. We have proved this without making any special assumptions on n, the dimension of the space in which Ω lies. Thus Theorem 4 of [13] can be replaced by the stronger.

<u>Theorem 4(a)</u> <u>The system</u> (1.1), (1.2) <u>is approximately controllable in time</u> T <u>if</u> $T > 2T_0$.

Combined with Theorem 2 of [13], which states that the system (1.1), (1.2) is not approximately controllable in time T if $T < 2T_0$, we see that we are justified in referring to $2T_0$ as the critical time. We will see in the sequel that, if $n \geq 2$, critical time approximate controllability is a rather delicate question.

3. THE CRITICAL TIME CONTROL PROBLEM

We are going to study the problem for a particular partial differential equation in certain special domains. In Section 6 we will indicate a more general theory.

In R^n, $n \geq 2$, we consider "rectangles" Σ_r, $r = 1,2,\ldots,n$, of dimension $n - r$, defined by

$$\Sigma_r = \{x = (x^1,x^2,\ldots,x^n) \in R^n \,|\, x^i = 0,\ i = 1,\ldots,r,\ 0 \leq x^j \leq 1,\ j = r+1,\ldots,n\}.$$

Of course, Σ_n is just the origin in R^n. For all real ξ we define

$$\tilde{\rho}(\xi) = \exp\left(1 - \frac{1}{\xi^2}\right)$$

and for all $x = (x^1,x^2,\ldots,x^n) \in R^n$ we put

$$\rho(x) = \tilde{\rho}(x^1) + \tilde{\rho}(x^2) + \ldots + \tilde{\rho}(x^n) .$$

We define domains $\Omega_r \subseteq R^n$ as follows:

$$\Omega_r = \{x \in R^n \,|\, \inf_{y \in \Sigma_r} \rho(x-y) < 1\}.$$

Then Ω_r is an open, bounded, simply connected region in R^n whose boundary

$$\Gamma_r = \{x \in R^n \,|\, \inf_{y \in \Sigma_r} \rho(x-y) = 1\}$$

is an n-dimensional surface of class C^∞ which is piecewise analytic.

In Ω_r we consider a boundary value control problem for the ordinary wave equation:

$$(3.1)\qquad w^f_{tt} - \sum_{i=1}^{n} w^f_{ii} = 0 \text{ in } \Omega_r \otimes [0,T],$$

$$(3.2)\qquad w^f_x(x(s),t)\eta(x(s)0 = f(s,t),\ (x(s),t) \in \Gamma_r \otimes [0,T],$$

$$(3.3)\qquad w^f(x,0) \equiv w^f_t(x,0) \equiv 0,\quad x \in \Omega_r.$$

We take $\tilde{\Gamma} = \Gamma_r$, i.e., admissible controls are C^∞ functions whose supports are compact subsets of the interior of $\Gamma_r \otimes [0,T]$. Thus control forces operate over the whole boundary of Ω_r.

For (3.1) there is a universal wave propagation speed, 1. Thus, given an instant t_0, the forward cone of influence of Γ_r at time t_0 is given by

$$(3.4)\qquad K^+(\Gamma_r, t_0) = \{(x,t) \in \Omega_r \otimes [t_0, +\infty] \mid \inf_{y \in \Gamma_r} \|x-y\| \le t - t_0\}$$

and the backward cone of influence of Γ_r at time t_0 is

$$K^-(\Gamma_r, t_0) = \{(x,t) \in \Omega_r \otimes (-\infty, t_0] \mid (x, 2t_0-t) \in K^+(\Gamma_r, t_0)\}.$$

(In (3.4) $\|\ \ \|$ denotes the Euclidean norm in R^n.) We define, for $T > 0$,

$$K(\Gamma_r, 0, T) = K^+(\Gamma_r, 0) \cap K^-(\Gamma_r, T).$$

As shown in [13], Section 3, the critical time T_0 has the property that

$$\Omega_r \otimes \{T_0\} \subseteq K(\Gamma_r, 0, 2T_0)$$

but $\Omega_r \otimes \{T/2\}$ is not a subset of $K(\Gamma_r, 0, T)$ if $T < 2T_0$. In the present case it follows that $T_0 = 1$, and hence the critical time is $T = 2$, because

$$\sup_{x \in \Omega_r} \{ \inf_{y \in \Gamma_r} (\|x-y\|)\} = 1.$$

We will prove two theorems regarding approximate controllability of (3.1), (3.2) in the critical time $T = 2$. We give these theorems the numbers 5 and 6 since they complement the four theorems proved in [13] and Section 2 of the present paper.

Theorem 5 *If* $r = 1$, *the system* (3.1), (3.2) *is not approximately controllable in the critical time* $T = 2$.

Theorem 6 *If* $2 \leq r \leq n$, *the system* (3.1), (3.2) *is approximately controllable in the critical time* $T = 2$.

The reader should be aware that these theorems apply for $n \geq 2$ only. When $n = 1$ the analog of Theorem 5 is not true, for it has already been shown in [5], [14], [15] that we do have critical time approximate controllability in this case.

In order to prove Theorems 5 and 6 we need certain results from the theory of distributions.

4. DISTRIBUTIONS IN $H^{-1}(\Omega_r)$ WITH SUPPORT IN Σ_r.

As in Section 2, we denote by $H^1(\Omega_r)$ real valued functions $v(x)$ defined on Ω_r which lie in $H^0(\Omega_r) = L^2(\Omega_r)$ and have first order partial derivatives, defined in the sense of the theory of distributions, which also lie in $H^0(\Omega_r)$. With the inner product

$$(u,v)_{H^1(\Omega_r)} = \int_{\Omega_r} [u(x)v(x) + \sum_{i=1}^{n} u_i(x)\, v_i(x)]\, dx$$

(again the subscript i refers to partial differentiation with respect to x^i) $H^1(\Omega_r)$ is a Hilbert space. We have

$$H^1(\Omega_r) \subseteq H^0(\Omega_r)$$

and for each $v \in H^1(\Omega_r)$

$$\|v\|_{H^1(\Omega_r)} \geq \|v\|_{H^0(\Omega_r)} ,$$

which shows that the injection mapping of $H^1(\Omega_r)$ into $H^0(\Omega_r)$ is continuous.

We will now indicate the construction of a third Hilbert space $H^{-1}(\Omega_r)$ with

$$H^1(\Omega_r) \subseteq H^0(\Omega_r) \subseteq H^{-1}(\Omega_r),$$

and the injection of $H^0(\Omega_r)$ into $H^{-1}(\Omega_r)$ is likewise continuous. To begin, let $u \in H^0(\Omega_r)$. We define a continuous linear functional on $H^0(\Omega_r)$:

(4.1) $$\ell_u(v) = (u,v)_{H^0(\Omega_r)}, \quad v \in H^0(\Omega_r).$$

Now if $v \in H^1(\Omega_r)$

$$|\ell_u(v)| \leq \|u\|_{H^0(\Omega_r)} \|v\|_{H^0(\Omega_r)} \leq \|u\|_{H^0(\Omega_r)} \|v\|_{H^1(\Omega_r)}$$

and we conclude that (4.1) also defined ℓ_u as a continuous linear functional on $H^1(\Omega_r)$. It follows that there is a unique element $\hat{u} \in H^1(\Omega_r)$ such that

(4.2) $$\ell_u(v) = (\hat{u},v)_{H^1(\Omega_r)} .$$

We define

(4.3) $$\|u\|_{H^{-1}(\Omega_r)} = \|\hat{u}\|_{H^1(\Omega_r)} .$$

Now for all $u \in H^0(\Omega_r)$

$$\|u\|_{H^{-1}(\Omega_r)} = \sup_{\substack{v \in H^1(\Omega_r) \\ v \neq 0}} \frac{|(\hat{u},v)_{H^1(\Omega_r)}|}{\|v\|_{H^1(\Omega_r)}}$$

$$= \sup_{\substack{v \in H^1(\Omega_r) \\ v \neq 0}} \frac{|(u,v)_{H^0(\Omega_r)}|}{\|v\|_{H^1(\Omega_r)}} \leq \sup_{\substack{v \in H^1(\Omega_r) \\ v \neq 0}} \frac{|(u,v)_{H^0(\Omega_r)}|}{\|v\|_{H^0(\Omega_r)}}$$

$$= \sup_{\substack{v \in H^0(\Omega_r) \\ v \neq 0}} \frac{|(u,v)_{H^0(\Omega_r)}|}{\|v\|_{H^0(\Omega_r)}} = \|u\|_{H^0(\Omega_r)} ,$$

the second last equality being true because $H^1(\Omega_r)$ is dense in $H^0(\Omega_r)$ relative to the topology induced by the norm $\| \ \|_{H^0(\Omega_r)}$.

We define $H^{-1}(\Omega_r)$ to be the completion of $H^0(\Omega_r)$ relative to the norm $\| \ \|_{H^{-1}(\Omega_r)}$. Now $\|u\|_{H^{-1}(\Omega_r)} = \|\hat{u}\|_{H^1(\Omega_r)}$ holds for $u \in H^0(\Omega_r)$, which is clearly dense in $H^{-1}(\Omega_r)$, and this relationship extends (see [12]) to an isometry $u \longleftrightarrow \hat{u}$ between $H^{-1}(\Omega_r)$ and $H^1(\Omega_r)$. The space $H^{-1}(\Omega_r)$ is a Hilbert space with

(4.4) $$(u,v)_{H^{-1}(\Omega_r)} = (\hat{u},\hat{v})_{H^1(\Omega_r)} .$$

The elements ϕ of $H^{-1}(\Omega_r)$ correspond to distributions ℓ_ϕ of order at most 1 (see [16]) on Ω_r.

We are now ready to prove two lemmas which will be of great importance in the proofs of Theorems 5 and 6.

Lemma 1. If $n \geq 2$ there exists a non-trivial element $\phi \in H^{-1}(\Omega_1)$ such that:

(i) the support of ℓ_ϕ is a subset of Σ_1

(ii) if c is a constant function on Ω_1 then $\ell_\phi(c) \equiv (\hat{\phi},c)_{H^1(\Omega_1)} = 0$.

Lemma 2. If $2 \leq r \leq n$ there is no non-trivial distribution in $H^{-1}(\Omega_r)$ with support in Σ_r.

The reader uninterested in the proofs of these lemmas may proceed to Section 5 without any loss of continuity.

Proof of Lemma 1. Let ψ denote a real valued function of $n-1$ variables $x^2, x^3, \ldots, x^n$ such that, with $\tilde{\Sigma}_1$ defined by

$$\tilde{\Sigma}_1 = \{\tilde{x} = (x^2, \ldots, x^n) \in R^{n-1} \mid (0, \tilde{x}) \in \Sigma_1\} ,$$

$\psi \in C^2(\tilde{\Sigma}_1)$, vanishes outside a compact subset of the interior of $\tilde{\Sigma}_1$, and

$$\text{(4.5)} \qquad \int_{\tilde{\Sigma}_1} \psi(\tilde{x})\, d\tilde{x} = 0 , \quad \int_{\tilde{\Sigma}_1} (\psi(\tilde{x}))^2\, d\tilde{x} \neq 0 .$$

For positive integers $k = 4, 5, 6 \ldots$ define

$$\text{(4.6)} \qquad \tilde{\theta}_k(\xi) = \begin{cases} 0 , & -1 \le \xi \le -\frac{3}{4} \\ -\frac{1}{2}(\xi + \frac{3}{4})^2 , & -\frac{3}{4} \le \xi \le -\frac{1}{4} , \\ -\frac{1}{2}\xi - \frac{1}{4} , & -\frac{1}{4} \le \xi \le \frac{1}{k} , \\ \frac{k}{4}\xi^2 + \frac{1}{4k} - \frac{1}{4} , & -\frac{1}{k} \le \xi \le \frac{1}{k} , \\ \tilde{\theta}_k(-\xi) , & \frac{1}{k} \le \xi \le 1 . \end{cases}$$

Then, for $x \in \Omega_1$, put

$$\text{(4.7)} \qquad \theta_k(x) = \theta_k(x^1, \tilde{x}) = \begin{cases} 0 \quad \text{if } \tilde{x} \notin \tilde{\Sigma}_1 , \\ \tilde{\theta}_k(x^1)\psi(\tilde{x}) , \quad x \in \tilde{\Sigma}_1 . \end{cases}$$

Then θ_k is defined as a function of class C^2 in Ω_1 for $k = 4, 5, 6, \ldots$.

Now compute, for any $v \in H^1(\Omega_1)$,

$$\int_{\Omega_1} \frac{\partial \theta_k(x)}{\partial x^1} \frac{\partial v(x)}{\partial x^1} \, dx$$

$$= -\int_{\Omega_1} \frac{\partial^2 \theta_k(x)}{(\partial x^1)^2} \, v(x) \, dx$$

$$= -\left[\int_{[-\frac{3}{4}, -\frac{1}{4}] \otimes \tilde{\Sigma}} -\psi(\tilde{x}) \, v(x) \, dx + \int_{[-\frac{1}{k}, \frac{1}{k}] \otimes \tilde{\Sigma}} \frac{k}{2} \, \psi(\tilde{x}) \, v(x) \, dx \right.$$

$$\left. + \int_{[\frac{1}{4}, \frac{3}{4}] \otimes \tilde{\Sigma}} -\psi(\tilde{x}) \, v(x) \, dx \right] .$$

Thus

$$\int_{[-\frac{1}{k}, \frac{1}{k}] \otimes \tilde{\Sigma}} \frac{k}{2} \, \psi(\tilde{x}) \, v(x) \, dx = \int_{[-\frac{3}{4}, -\frac{1}{4}] \otimes \tilde{\Sigma}} \psi(\tilde{x}) \, v(x) \, dx + \int_{[\frac{1}{4}, \frac{3}{4}] \otimes \tilde{\Sigma}} \psi(\tilde{x}) \, v(x) \, dx$$

$$- \int_{\Omega_1} \frac{\partial \theta_k(x)}{\partial x^1} \frac{\partial v(x)}{\partial x^1} \, dx$$

and, for $k = 4,5,6,\ldots$, $j = 4,5,6,\ldots$,

$$\int_{[-\frac{1}{k},\frac{1}{k}]\otimes\tilde{\Sigma}} \frac{k}{2}\,\psi(\tilde{x})\,v(x)\,dx - \int_{[-\frac{1}{j},\frac{1}{j}]\otimes\tilde{\Sigma}} \frac{j}{2}\,\psi(\tilde{x})\,v(x)\,dx$$

$$= \int_{\Omega_1} \left(\frac{\partial\theta_j(x)}{\partial x^1} - \frac{\partial\theta_k(x)}{\partial x^1} \right) \frac{\partial v(x)}{\partial x^1}\,dx\,.$$

Applying the Schwartz inequality

$$\left| \int_{[-\frac{1}{k},\frac{1}{k}]\otimes\tilde{\Sigma}} \frac{k}{2}\,\psi(\tilde{x})\,v(x)\,dx - \int_{[-\frac{1}{j},\frac{1}{j}]\otimes\tilde{\Sigma}} \frac{j}{2}\,\psi(\tilde{x})\,v(x)\,dx \right| \tag{4.8}$$

$$\leq \left\| \frac{\partial\theta_j}{\partial x^1} - \frac{\partial\theta_k}{\partial x^1} \right\|_{H^0(\Omega_1)} \left\| \frac{\partial v}{\partial x^1} \right\|_{H^0(\Omega_1)}$$

$$\leq \| \theta_j - \theta_k \|_{H^1(\Omega_1)} \| v \|_{H^1(\Omega_1)}\,.$$

An inspection of (4.6), (4.7) readily shows that

$$\| \theta_j - \theta_k \|_{H^1(\Omega)} = \varepsilon_{jk}$$

where

$$\lim_{\substack{j\to\infty \\ k\to\infty}} \varepsilon_{jk} = 0\,.$$

Let us put

$$(4.9) \qquad \phi_k(x) = \phi_k(x^1, \tilde{x}) = \begin{cases} \frac{k}{2}\ \psi(\tilde{x}),\ (x^1, \tilde{x}) \in [-\frac{1}{k}, \frac{1}{k}] \otimes \tilde{\Sigma}, \\ 0 \quad \text{otherwise} . \end{cases}$$

Then (4.8) shows that the continuous linear functionals ℓ_{ϕ_k}, defined on $H^1(\Omega_1)$ as in (4.1), (4.2) have the property that

$$|\ell_{\phi_k}(v) - \ell_{\phi_j}(v)| \leq \varepsilon_{jk} \|v\|_{H^1(\Omega_1)}$$

which implies (c.f. (4.2)) that

$$\|\hat{\phi}_k - \hat{\phi}_j\|_{H^1(\Omega_1)} \leq \varepsilon_{jk}$$

and therefore, from (4.3),

$$\|\phi_k - \phi_j\|_{H^{-1}(\Omega_1)} \leq \varepsilon_{jk} \quad .$$

Thus, in $H^{-1}(\Omega_1)$, $\{\phi_k\}$ is a Cauchy sequence and has a limit $\phi \in H^{-1}(\Omega_1)$. It remains only to show that ϕ has properties (i) and (ii) stated in Lemma 1.

Let $v \in C^\infty(\Omega_1)$ have support K which is a compact subset of $\Omega_1 - \Sigma_1$. Then, for all sufficiently large k, $K \cap ([-\frac{1}{k}, \frac{1}{k}] \otimes \tilde{\Sigma})$ is empty and

$$\ell_{\phi_k}(v) = (\phi_k, v)_{H^0(\Omega_1)} = (\hat{\phi}_k, v)_{H^1(\Omega_1)} = 0 .$$

Since ϕ_k converges to ϕ in $H^{-1}(\Omega_1)$, $\hat{\phi}_k$ converges to $\hat{\phi}$ in $H^1(\Omega_1)$, by virtue of the isometry discussed just prior to (4.4). Therefore

$$\ell_\phi(v) = (\hat{\phi}, v)_{H^1(\Omega_1)} = \lim_{k\to\infty} (\hat{\phi}_k, v) = 0 .$$

Thus ℓ_ϕ vanishes when applied to $v \in C^\infty(\Omega_1)$ with support K not meeting Σ_1 and we have shown that the support of ℓ_ϕ must be a subset of Σ_1.

Similarly, for $k = 4,5,6,\ldots,$ c constant,

$$\ell_{\phi_k}(c) = (\phi_k, c)_{H^0(\Omega_1)} = \int_{[-\frac{1}{k},\frac{1}{k}]\otimes\tilde{\Sigma}_1} \frac{k}{2}\,\psi(\tilde{x})c\,dx^1\,d\tilde{x}$$

$$= c\int_{\tilde{\Sigma}_1} \psi(\tilde{x})\,d\tilde{x} = 0,$$

from (4.5). Thus part (ii) of Lemma 1 is proved. The second part of (4.5) readily shows that ϕ is non-trivial and the proof of Lemma 1 is complete.

Proof of Lemma 2.

For $p > 0$ we define

(4.10) $$h_p(x^1,x^2,\ldots,x^r) = 1 - [(x^1)^2 + (x^2)^2 + \ldots + (x^r)^2]^{\frac{1}{p}} .$$

We compute

(4.11) $$\sum_{i=1}^{r} \left(\frac{\partial h_p}{\partial x^i}\right)^2 = \frac{4}{p^2}\,[(x^1)^2 + (x^2)^2 + \ldots + (x^r)^2]^{\frac{2}{p}-1} .$$

Integrating (4.11) over the unit ball in R^r we obtain the integral $4\omega_{r-1}/(4p + (r-2)p^2)$, where ω_{r-1} is the integral of 1 over the (r-1) dimensional sphere of radius 1. Thus we see that if B is any bounded open set in R^r then $h_p \in H^1(B)$ for $p > 0$ and

$$(4.12) \qquad \lim_{p \to +\infty} \| h_p \|_{H^1(B)} = 0 .$$

(Note that $r \geq 2$ is necessary for these conclusions.)

Given $x = (x^1, \ldots, x^r, x^{r+1}, \ldots, x^n) \in R^n$, let us set $y = (x^1, \ldots, x^r)$, $z = (x^{r+1}, \ldots, x^n)$. Each distribution ℓ defined on R^{n-r} has a natural extension to a distribution $\hat{\ell}$ defined on R^n. If $\hat{v}$ $(= \hat{v}(y,z)) \in C^\infty(R^n)$ we let $\hat{v}$ be defined on R^{n-r} by $v(z) = v(0,z)$. Then

$$\hat{\ell}(\hat{v}) = \ell(v)$$

defines the extension $\hat{\ell}$ of ℓ.

A result in [9] (p. 78) shows that if $\phi \in H^{-1}(\Omega_r)$ then the distribution ℓ_ϕ associated with ϕ can be expressed in the form

$$\ell_\phi(u) = (g_0, u)_{H^0(\Omega_r)} + \sum_{i=1}^{n} \left(g_i, \frac{\partial u}{\partial x^i}\right)_{H^0(\Omega_r)}, \quad u \in C^\infty(\Omega_r),$$

where $g^i \in L^2(\Omega_r)$, $i = 0, 1, \ldots, n$. This shows that ℓ_ϕ is a distribution of order at most 1 (i.e. if v_k are C^∞ functions converging to zero in $C^1(\Omega_r)$ as $k \to \infty$, then $\lim_{k \to \infty} \ell_\phi(v_k) = 0$.)

Thus if we take $\phi \in H^{-1}(\Omega)$, ℓ_ϕ has order at most 1. A theorem in [16] (p. 99, ff.) then shows that if the support of ℓ_ϕ is a subset of Σ_r, there are distributions $\ell_0, \ell_1, \ell_2, \ldots, \ell_r$ defined on R^{n-r} with support in $\tilde{\Sigma}_r = \{z \mid (0,z) \in \Sigma_r\}$ such that

$$\ell_\phi = \hat{\ell}_0 + \sum_{i=1}^{r} \frac{\partial \hat{\ell}_i}{\partial x^i} .$$

Let $\psi(z) \in C^\infty(R^{n-r})$ have support K contained in some small neighborhood of $\tilde{\Sigma}_r$ in R^{n-r}. Define $v(= v(y,z))$ in Ω_r by

(4.13) $$v(x) = v(y,z) = h_4(y)\psi(z)$$

where h_4 is given by (4.10). Then $v \in H^1(\Omega_r)$. Since $\phi \in H^{-1}(\Omega_r)$, the linear functional ℓ_ϕ can be defined on all of $H^1(\Omega_r)$ and we have

$$\ell_\phi(v) = \hat{\ell}_0(v) + \sum_{i=1}^{r} \left(\frac{\partial \hat{\ell}_i}{\partial x^i}\right)(v)$$

$$= \hat{\ell}_0(v) - \sum_{i=1}^{r} \hat{\ell}_i\left(\frac{\partial v}{\partial x^i}\right) .$$

Let $e_j = (0,\ldots 0,1,0,\ldots,0)$ in R^r, the 1 being in the jth position. Define

$$v_\varepsilon(x) = v_\varepsilon(y,z) = v(y + \varepsilon e_j, z) .$$

One verifies without difficulty that

(4.14) $$\lim_{\varepsilon \to 0} \|v - v_\varepsilon\|_{H^1(\Omega_r)} = 0 .$$

On the other hand

$$\ell_\phi(v_\varepsilon) = \hat{\ell}_0(v_\varepsilon) - \sum_{i=1}^{r} \hat{\ell}_i\left(\frac{\partial v_\varepsilon}{\partial x^i}\right)$$

$$= h_4(0 + \varepsilon e_j)\,\ell_0(\psi) - \sum_{i=1}^{r} \frac{\partial h_4}{\partial x^i}(0 + \varepsilon e_j)\ell_i(\psi)$$

$$= h_4(\varepsilon e_j)\,\ell_0(\psi) - \frac{\partial h_4}{\partial x^i}(\varepsilon e_j)\ell_j(\psi)$$

$$= (1 - \varepsilon^{\frac{1}{2}})\,\ell_0(\psi) + \frac{1}{2}\varepsilon^{-\frac{1}{2}}\,\ell_j(\psi) .$$

Thus if $\ell_j(\psi)$ is different from zero

$$\lim_{\varepsilon \to 0} \ell_\phi(v_\varepsilon) = +\infty$$

and then (4.14) shows that ℓ_ϕ cannot be a continuous linear functional on $H^1(\Omega_r)$, contrary to our assumption that $\phi \in H^{-1}(\Omega_r)$. We conclude that $\ell_j(\psi) = 0$. Since this is true for all such ψ and the support of ℓ_j is a subset of $\tilde{\Sigma}_r$, we conclude that $\ell_j = 0$. We can do this for $j = 1, 2, \ldots, r$ and conclude that

$$\ell_\phi = \hat{\ell}_0 .$$

Now, for $p > 0$, define $v_p(x)$ as in (4.13), replacing 4 by p. Compute

$$\ell_\phi(v_p) = h_p(0)\ell_0(\psi) = \ell_0(\psi)$$

for all $p > 0$, since $h_p(0) = 1$. But (4.12) is easily seen to imply that

$$\lim_{p \to \infty} \|v_p\|_{H^1(\Omega_r)} = 0$$

and thus ℓ_ϕ cannot be a continuous linear functional on $H^1(\Omega_r)$ unless $\ell_0(\psi) = 0$. Therefore, since ℓ_ϕ is a continuous linear functional on $H^1(\Omega_r)$, $\ell_0(\psi) = 0$ for all ψ of the form prescribed above. Then the fact that ℓ_0 has support in $\tilde{\Sigma}_r$ shows that $\ell_0 = 0$. We have now shown that

$$\ell_\phi = 0$$

and Lemma 2 has been proved.

Remarks Some readers may find the dual role of ℓ_ϕ, as a distribution of order ≤ 1 and as a linear functional on $H^1(\Omega_r)$, slightly confusing. Given $\phi \in H^{-1}(\Omega_r)$ there is associated with it a unique element $\hat{\phi} \in H^1(\Omega_r)$ and for all $v \in H^1(\Omega_r)$

$$\ell_\phi(v) = (\hat{\phi}, v)_{H^1(\Omega_r)} .$$

This also defines ℓ_ϕ as a continuous linear functional on $C^\infty(\Omega_r)$, since convergence in $C^\infty(\Omega_r)$ implies convergence in $H^1(\Omega_r)$. Thus ℓ_ϕ is also a distribution in the sense of Schwartz [16].

One can easily see that Lemma 1, part (i) continues to hold for $n = 1$. (Just put $\phi = \delta$, the Dirac distribution.) But (ii) cannot hold for $n = 1$. The function ψ cannot be constructed as in (4.5). This explains why Theorem 5 is true for $n \geq 2$ but not for $n = 1$.

5. PROOF OF THEOREM 5.

A result in Lions-Magenes [9] (p. 202) states that if $\tilde{\phi} \in H^{-1}(\Omega)$ satisfies (ii) of Lemma 1, then there is a unique function $\tilde{v} \in H^1(\Omega)$ with $\int_{\Omega_1} \tilde{v}(x)\, dx = 0$ such that, in the sense of the theory of distributions,

$$(5.1) \qquad \sum_{i=1}^{n} \tilde{v}_{ii} = \tilde{\phi} \text{ in } \Omega_1 ,$$

$$(5.2) \qquad \tilde{v}_x(x(s))\, \eta(x(s)) = 0, \quad x(s) \in \Gamma_1 .$$

(The sense in which (5.2) holds is also explained in [9]. In our applications $\tilde{v}$ is harmonic outside a compact subset of Ω_1 and (5.2) holds in the classical sence. Moreover, there is a constant $M > 0$ such that

$$(5.3) \qquad \|\tilde{v}\|_{H^1(\Omega_1)} \le M \|\phi\|_{H^{-1}(\Omega_1)} .$$

Let the functions ϕ_k be defined on Ω_1 as in (4.9) and let $\tilde{v}^k$ be the corresponding solutions of (5.1), (5.2) with $\tilde{\phi}$ replaced by ϕ_k. Also, let $\tilde{v}$ satisfy (5.1), (5.2) with $\tilde{\phi}$ replaced by the element $\phi \in H^{-1}(\Omega_1)$ constructed in Lemma 1. Since $\lim_{k\to\infty} \|\phi - \phi_k\|_{H^{-1}(\Omega)} = 0$, (5.3) implies that

$$\lim_{k\to\infty} \|\tilde{v} - \tilde{v}^k\|_{H^1(\Omega_1)} = 0 .$$

It is clear that $\tilde{v}$ cannot be a constant on Ω_1, therefore $(\tilde{v}, 0)$ is a non-zero energy state. We let $v(x,t)$, $v^k(x,t)$, $k = 4,5,6, \ldots$ be generalized solutions in $\Omega_1 \otimes [0,2]$ of

(5.4) $$v_{tt} - \sum_{i=1}^{n} v_{ii} = 0 ,$$

(5.5) $$v_x(x(s),t)\eta(x(s)) = 0, \quad (x(s)t) \in \Gamma_1 \otimes [0,2]$$

satisfying

$$v(x,1) \equiv \tilde{v}(x),\ v_t(x,1) \equiv 0,\ v^k(x,1) \equiv \tilde{v}^k(x),\ v_t^K(x,1) \equiv 0 .$$

By the principle of conservation of energy, $(v(\cdot,2), v_t(\cdot,2))$ is also a non-zero energy state.

Let f be an admissible control. Then the support of f lies in a set $\Gamma_1 \otimes [\delta, 2-\delta]$ for some $\delta > 0$. Since the support of ϕ_k is $[-\frac{1}{k}, \frac{1}{k}] \otimes \tilde{\Sigma}_1$, $\tilde{v}^k$ is harmonic in $\Omega_1 - ([-\frac{1}{k}, \frac{1}{k}] \otimes \tilde{\Sigma}_1)$. Then, by a familiar uniqueness result in the theory of hyperbolic partial differential equations (see e.g. [2])

$$\left.\begin{aligned} v^k(x,t) &\equiv \tilde{v}^k(x) \\ v_t^k(x,t) &\equiv 0 \end{aligned}\right\} \qquad |t-1| \le \inf_{y \in [-\frac{1}{k},\frac{1}{k}] \otimes \tilde{\Sigma}_1} \{\|x-y\|\} .$$

Thus, for sufficiently large k, $v_t^k(x(s),t) \equiv 0$, $(x(s),t) \in \Gamma_1 \otimes [\delta, 2-\delta]$ and an application of the divergence theorem (c.f. Theorem 1 in [13]) shows that

(5.6) $$\int_{\Omega_1} [w_t^f(x,2)\, v_f^k(x,2) + \sum_{i=1}^{n} w_i^f(x,2)\, v_i^k(x,2)]\,dx$$
$$= \int_{\Gamma_1 \otimes [0,2]} v_t^k(x(s),t)\, f(s,t)\,ds = 0 .$$

(The solution $w^f \in C^\infty(\Omega_1 \otimes [0,2])$ and it is proved in [10] that $v^k \in H^2(\Omega^1 \otimes [0,2])$. This enables one to use the divergence theorem without difficulty.)

Noting that

$$\lim_{k\to\infty} \| v_t(\cdot, 2) - v_t^k(\cdot, 2)\|_{H^0(\Omega_1)} = 0$$

$$\lim_{k\to\infty} \| v_i(\cdot, 2) - v_i^k(\cdot, 2)\|_{H^0(\Omega_1)} = 0, \; i = 1, 2, \ldots, n,$$

we conclude from (5.6) that

$$\int_{\Omega_1} [w_t^f(x, 2)\, v_t(x, 2) + \sum_{i=1}^{n} w_i^f(x, 2)\, v_i(x, 2)]\, dx = 0 .$$

Since f is an arbitrary admissible control we have shown that the non-zero energy state $(v(\cdot, 2), v_t(\cdot, 2))$ lies in $R_2^{\perp}$ and thus that R_2 is not dense in $H_E(\Omega_1)$ relative to the norm $\| \quad \|_E$. Thus Theorem 5 is proved.

6. PROOF OF THEOREM 6.

Much of the work necessary to prove Theorem 6 has already been done in Section 2 in the proof of Theorem 4a. We again assume that (v, v_t) is a finite energy state which satisfies (2.1) (with $\rho \equiv 1$, $A \equiv I$ and $\Omega = \Omega_r$) and we let $v(x,t)$ be the generalized solution of (5.4), (5.5) satisfying the terminal conditions (2.4). The solution v is smoothed by the same process of forming antiderivatives and finite time differences as described in (2.5) - (2.8) ff. The divergence theorem can again be used to obtain (2.9) (with $\tilde{\Gamma}$ replaced by Γ_r), and thus, via the Holmgren–Fritz John uniqueness theorem [7] to prove that $(\Delta^m(D^{-m}v))_t$ must vanish identically for $(x,t) \in K(\Gamma_r, 0, T - m\delta)$, the intersection of the forward cone of influence of Γ_r at time 0 with the backward cone of influence of Γ_r at time $T - m\delta$.

The essential difference between the proof of Theorem 6 and that of Theorem 4a lies in the fact that when $T = 2$, the critical time, $K(\Gamma_r, 0, 2-m\delta)$ does not include any set $\overline{\Omega}_r \times [1-\varepsilon, 1+\varepsilon]$ for any $\varepsilon > 0$, no matter how small we take $\delta > 0$ to be.

If $\delta > 0$ is small, the functions $\Delta^m(D^{-m}v(x,1))$ are defined and twice continuously differentiable for $x \in \overline{\Omega}_r$. Now the operator Δ depends on δ, and we define

$$v^\delta(x) = \delta^{-m}\Delta^m(D^{-m}v(x,1)), \quad x \in \overline{\Omega}_r .$$

The continuity of $v(\cdot,t)$ as a mapping from R^1 into $H^1(\Omega_r)$ enables one to show by elementary means that

(6.1) $$\lim_{\delta \to 0} \| v^{\delta}(x) - v(x,1) \|_{H^1(\Omega_r)} = 0 .$$

Now $\delta^{-m}\Delta^{m}(D^{-m} v(x,t)) \equiv v^{\delta}(x,t)$ is twice continuously differentiable in $\overline{\Omega}_r \otimes [0, 2 - m\delta]$ and there satisfies

$$\sum_{i=1}^{n} v_{ii}^{\delta} = v_{tt}^{\delta}$$

and boundary conditions of the form (5.5). Thus the functions

$$g^{\delta}(x) = v_{tt}^{\delta}(x, T_0)$$

are, for $\delta > 0$, continuous in $\overline{\Omega}_r$ and we have

$$\sum_{i=1}^{n} v_{ii}^{\delta}(x) \equiv g^{\delta}(x), \quad x \in \overline{\Omega}_r .$$

Now $v_t^{\delta}(x,t)$ $(= \delta^{-m}\Delta^{m}(D^{-m}v)_t\,(x,t))$ has been shown to vanish in $K(\Gamma_r, 0, 2-\delta)$, which implies that $v_{tt}^{\delta}(x,t)$ vanishes there also. We conclude therefore that

(6.2) $$g^{\delta}(x) \equiv v_{tt}^{\delta}(x, T_0) \equiv 0, \quad x \in \Omega_r^{\delta}$$

where

(6.3) $$\Omega_r^{\delta} = \{x \in \Omega_r \,|\, (x,1) \in K(\Gamma_r, 0, 2-m\delta) \cap (\Omega_r \otimes \{1\})\} .$$

The sets Ω_r^{δ} are monotone increasing as $\delta \to 0$ with the property

(6.4) $$\bigcap_{\delta > 0} (\Omega_r - \Omega_r^{\delta}) = \Sigma_r .$$

Let $u \in H^1(\Omega_r) \subseteq H^0(\Omega_r)$. Since $g^\delta \in C^0(\overline{\Omega}_r) \subseteq H^0(\Omega_r)$ we can form the inner product $(g^\delta, u)_{H^0(\Omega_r)}$. Integrating by parts we find that

$$|(g^\delta, u)_{H^0(\Omega_r)}| \equiv |(\sum_{i=1}^{n} v^\delta_{ii}, u)_{H^0(\Omega_t)}|$$

$$= |-\sum_{i=1}^{n} (v^\delta_i, u_i)_{H^0(\Omega_r)}| \leq \|v^\delta\|_{H^1(\Omega_r)} \|u\|_{H^1(\Omega_r)} .$$

Thus g^δ is an element of $H^0(\Omega_r)$ which defines, via $(g^\delta, u)_{H^0(\Omega_r)}$, a continuous linear functional ℓ_{g^δ} on $H^1(\Omega_r)$ for which

$$\|\ell_{g^\delta}\| \leq \|v^\delta\|_{H^1(\Omega_r)} .$$

There is an element $\hat{g}^\delta \in H^1(\Omega_r)$ such that

$$\ell_{g^\delta}(u) = (\hat{g}^\delta, u)_{H^1(\Omega_r)}, \quad u \in H^1(\Omega_r) .$$

Then, reasoning as in Section 4,

$$\|g^\delta\|_{H^{-1}(\Omega_r)} = \|\hat{g}^\delta\|_{H^1(\Omega_r)} \leq \|v^\delta\|_{H^1(\Omega_r)} .$$

Similarly for $\delta_1 > 0$, $\delta_2 > 0$,

$$\|g^{\delta_1} - g^{\delta_2}\|_{H^{-1}(\Omega_r)} \leq \|v^{\delta_1} - v^{\delta_2}\|_{H^1(\Omega_r)} .$$

Now if we take a sequence $\{\delta_k\}$ of positive numbers with $\lim_{k\to\infty} \delta_k = 0$, we have

$$\lim_{k\to\infty} \| v^{\delta_k} - v(\cdot,1)\|_{H^1(\Omega_r)} = 0$$

from (6.1). Thus

$$\lim_{\substack{k\to\infty\\ j\to\infty}} \| g^{\delta_k} - g^{\delta_j}\|_{H^{-1}(\Omega_r)} = \lim_{\substack{k\to\infty\\ j\to\infty}} \| v^{\delta_k} - v^{\delta_j}\|_{H^1(\Omega_r)} = 0$$

and we see that $\{g^{\delta_k}\}$ is Cauchy in $H^{-1}(\Omega_r)$, converging to an element $g \in H^{-1}(\Omega_r)$.

Let ℓ_g be the distribution (also linear functional on $H^1(\Omega_r)$) associated with g. We claim that the support of g is contained in Σ_r. For, if $u \in C^\infty(\Omega_r)$ has support K which does not meet Σ_r, then (6.4) shows that K is a subset of Ω_r^δ for sufficiently small $\delta > 0$. Then

$$\ell_g(u) = \lim_{k\to\infty} \ell_{g_k^\delta}(u) = (g^{\delta_k}, u)_{H^0(\Omega_r)} = 0 ,$$

as we see from (6.2). Thus $\ell_g(u)$ vanishes whenever the support of $u \in C^\infty(\Omega_r)$ does not meet Σ_r and we conclude that the support of ℓ_g lies in Σ_r.

In Section 4 we showed that if $g \in H^{-1}(\Omega_r)$ and ℓ_g has support in Σ_r, $2 \leq r \leq n$, then $g = 0$. Thus

$$0 = \|g\|_{H^{-1}(\Omega_r)} = \lim_{k\to\infty} \|g^{\delta_k}\|_{H^{-1}(\Omega_r)}$$

and for every $u \in H^1(\Omega_r)$

(6.5) $$\lim_{k\to\infty} \ell_{g_k^{\delta}}(u) = (g^{\delta_k}, u)_{H^0(\Omega_r)} = 0 .$$

Set $u = -v(\cdot, 1)$ in (6.5) and we have

$$0 = \lim_{k\to\infty} (g^{\delta_k}, -v(\cdot, 1))_{H^0(\Omega_r)} = \lim_{k\to\infty} (\sum_{i=1}^{n} (v_i^{\delta_k}, v_i(\cdot, 1))_{H^0(\Omega_r)})$$

Since v^{δ_k} converges to $v(\cdot, 1)$ in $H^1(\Omega_r)$ we have

$$0 = \lim_{k\to\infty} (\sum_{i=1}^{n} (v_i^{\delta_k}, v_i(\cdot, 1))_{H^0(\Omega_r)}) = \sum_{i=1}^{n} (v_i(\cdot, 1), v_i(\cdot, 1))_{H^0(\Omega_r)}$$

and we conclude that there is a constant c such that

$$v(x, T_0) \equiv c, \quad x \in \Omega_r .$$

Since $v_t(\cdot, t)$ is a continuous mapping from $[0, 2]$ into $H^0(\Omega_r)$ one can show by elementary means that

(6.6) $$\lim_{\delta\to 0} \| v_t^{\delta}(\cdot, 1) - v_t(\cdot, 1) \|_{H^0(\Omega_r)} = 0 .$$

But $v_t^{\delta}(x, 1) \equiv 0$ for $x \in \Omega_r^{\delta}$ as we see from (6.3) and the fact that $v_t^{\delta}(x, t) \equiv 0$ $K(\Gamma_r, 0, 2-m\delta)$. Combined with (6.4) this shows that

$$\lim_{\delta\to 0} v_t^{\delta}(x, 1) = 0, \text{ a.e. in } \Omega_r$$

and then (6.6) shows that

$$v_t(\cdot, T_0) = 0$$

in $H_0(\Omega_r)$. Thus $(v(\cdot,T_0), v_t(\cdot,T_0)) = (c,0)$ is a zero energy state. The conservation of energy principle then shows that $(v(\cdot,2), v_t(\cdot,2))$ is likewise a zero energy state and, reasoning as in the proof of Theorem 4a, (3.1), (3.2) is approximately controllable in time $T = 2$. Thus Theorem 6 is proved.

Remark: The Holmgren-Fritz John uniqueness theorem [7] cited here and in Section 2, was originally proved under the assumption that the boundary Γ of Ω is analytic. The boundaries Γ_r of the sets Ω_r constructed in Section 3 do not have this property -- they are C^∞ and piecewise analytic. However, the results of [7] can be extended to such boundaries with very little difficulty. If $\Gamma_r = \Gamma_r^1 \cup \cdots \cup \Gamma_r^s$, where the Γ_r^k are relatively closed in Γ_r with disjoint relative interiors $\overset{o}{\Gamma}{}_r^k$ and if each $\overset{o}{\Gamma}{}_r^k$ is an analytic surface then $(\Delta^m(D^{-m}v))_t \equiv 0$ on $\overset{o}{\Gamma}{}_r^k$ implies, via [7], that this identity continues to hold in $K(\overset{o}{\Gamma}{}_r^k, 0, 2-m\delta)$. But the interior of $K(\overset{o}{\Gamma}{}_r^k, 0, 2-m\delta)$ is included in the set $\bigcup_k K(\overset{o}{\Gamma}{}_r^k, 0, 2-m\delta)$ and thus the continuous function $(\Delta^m(d^{-m}v))_t$ also vanishes in $K(\Gamma_r, 0, 2-m\delta)$, as we need for our proof.

7. CONCLUDING REMARKS.

While Theorems 5 and 6 are stated for special domains Ω_r and a special hyperbolic partial differential equation, it is not difficult to extrapolate these results to systems of the form (1.1), (1.2) in more general domains Ω with boundary Γ which includes a relatively open subset $\widetilde{\Gamma}$ whereon control is exercised.

Given the critical time $T = 2T_0$ one forms sets $K(\Gamma_1, 0, 2T_0 - m\delta)$ as in the proof of Theorem 6. (See [13] for complete description). Then we form the sets

$$\Omega^\delta = \{x \mid (x, T_0) \in K(\Gamma_1, 0, 2T_0 - m\delta) \cap [\Omega \otimes \{T_0\}]\}.$$

As δ tends to zero the sets Ω^δ increase. The complementary sets $\Omega - \Omega^\delta$ decrease and we put

$$\Sigma = \bigcap_{\delta > 0} (\Omega - \Omega^\delta).$$

The dimension of Σ is what is critical. If Σ contains a smooth manifold of dimension $n - 1$ the system will not be controllable in time $T = 2T_0$. For one can construct a distribution $\phi \in H^{-1}(\Omega)$ with properties (i) and (ii) of Lemma 1, solve

$$\sum_{i,j=1}^{n} (\alpha_i, (x) \widetilde{v}_i)_j = \phi ,$$

set

$$(7.1) \qquad v(x,T_0) \equiv \tilde{v}(x), \quad v_t(x,T_0) \equiv 0$$

and then let $v(x,t)$ be the generalized solution of (2.2), (2.3) satisfying (7.1). The state $(v(\cdot,2T_0), v_t(\cdot,2T_0))$ will then lie in $R^{\perp}_{2T_0}$ relative to the energy inner product in $H_E(\Omega)$. If Σ has dimension $n-2$ or less one can show, as in Lemma 2, that Σ cannot be the support of a non-trivial distribution in $H^{-1}(\Omega)$ and prove critical time controllability as in Theorem 6.

It is clear that in the "typical" case Σ will have dimension less than $n-1$. In fact Σ will be a single point in many instances. It seems reasonable to conjecture that Σ cannot have dimension greater than $n-2$ if Γ is an analytic surface. Thus critical time approximate controllability is the rule, not the exception.

The results of [13] and the present paper leave the theory of approximate boundary value controllability of systems (1.1), (1.2) in a fairly satisfactory state. However, much remains to be done. Perhaps the most important task is that of characterizing all finite energy states which can be reached (from a zero initial state) in a time $T \geq 2T_0$ using controls $f \in L^2(\Gamma_1 \otimes [0,T])$. A first step is to consider $f \in C^\infty(\Gamma_1 \otimes [0,T])$ as in the present paper and try to bound $\|f\|_{L^2(\Gamma_1 \otimes [0,T])}$ in terms of $w^f(\cdot,T)$ and its derivatives. This work has already been done in [5], [15] for the wave equation in one space dimension. Results in this direction would enable one to undertake a systematic study of the applicability of the quadratic criterion to hyperbolic boundary value control problems, as has been done, e.g., in [11] for the case of spatially distributed controls.

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