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NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS WITH MULTIPLE SOLUTIONS

by

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Nonlinear Two-Point Boundary Value Problems with Multiple Solutions.

Seymour V. Parter and Víctor Pereyra.

Abstract: In the first part of this paper we study the convergence of finite difference methods to approximate the maximal solution of problems of the form: $u'' + \lambda f(x,u) = 0$, with boundary conditions either $u(0) = u(b) = 0$ or $u(0) = u'(b) = 0$, $0 < b \leq 1$.
The function $f(x,u)$ satisfies several conditions that are explicitly given in § 1. This work extends earlier results of Parter (see references at the end).

Since this problem has in general more than one solution we develop in the second part two algorithms to approximate solutions characterized by the number of their zeros in $(0,1)$. We include in the last Section numerical results and some additional comments on the implementation of the algorithms on a digital computer.

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NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS WITH MULTIPLE SOLUTIONS

Seymour V. Parter* and Víctor Pereyra+

1. INTRODUCTION.

In several earlier papers Parter [9],[10] studied numerical methods for finding the "maximal" solution u(t) of boundary value problems of the form

$$u''+f(t,u)=0 \quad , \quad u(0)=u(1)=0. \tag{1.1}$$

In [10] Parter discussed certain pathological examples in which U(t;h), the maximal solution of some naturally associated finite difference equation, does not converge (when $h\downarrow 0$) to u(t).

In 1955, I. Kolodner [6] considered the special problem

$$u''+\frac{\lambda u}{(t^2+u^2)^{1/2}}=0 \quad (0<t<1) \quad , \quad u(0)=u'(1)=0. \tag{1.2}$$

He proved the following remarkable theorem.

Let $\quad 0<\lambda_0<\lambda_1 \ \cdots$

be the eigenvalues of the linear eigenvalue problem

$$v''+\frac{\lambda v}{t}=0 \quad 0<t<1 \quad , \quad v(0)=v'(1)=0. \tag{1.3}$$

Let

$$\lambda_n<\lambda\le\lambda_{n+1} \tag{1.4}$$

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Then there exist exactly (n+1) distinct solutions $u_0(t)$, $u_1(t),\ldots,u_n(t)$ *of equation* (1.2), *normalized so that* $u'(0)>0$. *These solutions are characterized by the fact that* $u_j(t)$ *has exactly* j *nodal zeros in* (0,1), *and no other interior zeros*.

Note that $-u(t)$ is a solution whenever $u(t)$ is a solution.

These results of Kolodner have been extended to more general problems by G. Pimbley [13], [14], C.V. Coffman [3], and Parter [11].

In this Report we consider the problems studied by Coffman [3] and their numerical solution. Our basic tool is the construction of maximal solutions as discussed in [9]*. As we shall see, these problems do not exhibit the pathologies mentioned earlier. This nice behavior is then the basis for obtaining numerical approximations to the solution with a specified number of zeros.

The nonlinear boundary value problems we consider are then of the form

(1.5 a) $\quad v''+\lambda f(x,v)=0 \quad 0<x<b\leq 1.$

The function $v(x)$ is required to satisfy either the boundary conditions

(1.5 b) $\quad v(0)=v(b)=0,$

or

(1.5 c) $\quad v(0)=v'(b)=0.$

The function $f(x,v)$ satisfies

$$|f(x,v)|<L\ ,$$

and is of the form

(1.6 a) $\quad f(x,v)=g(x,v)v\ ,$

where $g(x,v)$ is positive and of class C^2 in (x,v) on the strip

(1.6 b) $\quad R=\{(x,v):x>0\ ,\ \ |v|<\infty\}\ .$

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* A research announcement of Simpson [21] where a similar problem is discussed has just come to our attention.

Moreover, the following conditions hold in R:

(1.6 c) $f_v(x,v)=g(x,v)+vg_v(x,v)>0$,

(1.6 d) $vg_v(x,v)<0$, $v\neq 0$,

(1.6 e) for some $\delta>0$,

$xf_v(x,0)\leq M<\infty$, $0<x<\delta$.

Finally,

(1.6 f) $\lim_{|v|\to\infty} g(x,v)=0$.

Coffman [3] proves that equation (1.5 a) with boundary conditions (1.5 c), and with f satisfying (1.6 a) through (1.6 f) has at least one solution with exactly n nodal zeros in (0,1) if and only if

(1.7) $\lambda_n<\lambda$,

where λ_n is the nth eigenvalue of the linear problem

(1.8 a) $u''+\lambda g(x,0)u=0$,

(1.8 b) $u(0)=u'(1)=0$.

In Section 2 we introduce some notation and preliminary results.

In Section 3 we obtain maximal, positive solutions for problem (1.5), (1.6). A complete discussion is made in order to consider the new difficulties associated with the singular behavior at the origin. Also new proofs of old facts result in this case thanks to the stronger hypotheses (1.6).

In Section 4 we introduce, without proofs, the notation and results of Coffman which are relevant for the ensuing developments. We include also there a few other results of our own that will be used in the following sections.

In Sections 5 and 6 we study in detail two different procedures to approximate solutions of (1.5), (1.6) with a specified number of zeros in (0,1). Convergence and order of convergence are established. Finally, in Section 7, we present some numerical results which clearly show the viability of these procedures.

2. PRELIMINARIES.

We consider two cases corresponding to the two possible boundary conditions.

Case 1. Let N be a given integer and set

(2.1) $$h=h(N)=b/(N+1) \; .$$

Let

(2.2 a) $$G_0(h)\equiv\{jh;\; j=1,2,\ldots,N\} \; ,$$

(2.2 b) $$\bar{G}_0(h)\equiv\{jh;\; j=0,1,\ldots,N+1\} \; .$$

Let $P_0=P_0(h)$ be set of all piecewise linear functions $\phi(t,h)$ defined on $[0,b]$ which are determined by their values at the points of $\bar{G}_0(h)$ and which satisfy

(2.3) $$\phi(0;h)=\phi(b;h)=0$$

Case 2. Let N be a given integer and set

(2.4) $$h=h(N)=\frac{b}{N+1/2} \; .$$

Let

(2.5 a) $$G_1(h)\equiv\{jh;\; j=1,\ldots,N\} \; ,$$

(2.5 b) $$\bar{G}_1(h)\equiv\{jh;\; j=0,1,\ldots,N+1\} \; .$$

Let $P_1=P_1(h)$ be the set of all piecewise linear functions $\phi(t,h)$ defined on $[0,b+\frac{1}{2}h]$ which are determined by their values at the points of $\bar{G}_1(h)$ and which satisfy

(2.6) $$\begin{aligned}&\phi(0;h)=0\\ &\phi(b-\tfrac{1}{2}h;h)=\phi(b+\tfrac{1}{2}h;h) \; .\end{aligned}$$

Definition: Let Δ_h be the linear operator mapping P_k into P_k $(k=0,1)$ given by

(2.7) $$[\Delta_h\phi](jh)=\frac{\phi((j+1)h;h)-2\phi(jh;h)+\phi((j-1)h;h)}{h^2}\ ,\quad jh\varepsilon G_k(h).$$

For the sake of completeness we restate the fundamental Maximum Principle:

Lemma 2.1 Let $\phi(t;h)\varepsilon P_k$. If

(2.8a) $$[\Delta_h\phi](t)\geq 0 \qquad t\varepsilon G_k(h)\ ,$$

then

(2.8b) $$\phi(t;h)\leq 0$$

Proof: See [8], [10], [15].

As an immediate consequence of this lemma and the "consistency" of Δ_h with D^2, the second derivative operator, we obtain some basic estimates.

Lemma 2.2: Let $\phi(t;h)\varepsilon P_k$. Then

(2.9a) $$|\phi(t;h)|\leq\frac{t(b-t)}{2}\max_{G_0(h)}|\Delta_h\phi|,\quad \phi\varepsilon P_0$$

(2.9b) $$|\phi(t;h)|\leq\frac{t(2b-t)}{2}\max_{G_1(h)}|\Delta_h\phi|,\quad \phi\varepsilon P_1$$

Proof: See [9].

We will also make use of "Energy" estimates. Thus we introduce the standard notation and some basic facts.

(2.10a) $$u_x(x)=\frac{u(x+h)-u(x)}{h}$$

(210b) $$u_{\bar{x}}(x)=\frac{u(x)-u(x-h)}{h}$$

(2.10c) $$[\Delta_h u](x)=u_{x\bar{x}}(x)=u_{\bar{x}x}(x)\ .$$

A simple summation by parts shows that, if $\phi(t;h)\varepsilon P_k$, then

(2.11) $$h\sum_{j=1}^{N}[\Delta_h\phi](jh).\phi(jh)=-h\sum_{j=1}^{N+1}\phi^2_{\bar{x}}(jh)\quad .$$

Let M>0 be a given constant.

Lemma 2.3. Let $\phi(t;h)\varepsilon P_k$ and satisfy

(2.12a) $$|h\sum_{j=1}^{N+1}\phi_{\bar{x}}^2(jh)|\leq M^2 \quad .$$

Then

(2.12b) $$|\phi(t;h)-\phi(s;h)|\leq M|t-s|^{1/2} \quad ,$$

(2.12c) $$\|\phi\|_\infty\leq M \quad .$$

Proof: See [1], [8].

Lemma 2.4. Let $\phi(t;h)\varepsilon P_k$ and satisfy

(2.13) $$h\sum_{j=1}^{N}|[\Delta_h\phi](jh)|\leq M \quad .$$

Let $0<j<r\leq N+1$. Then

(2.14a) $$\phi_{\bar{x}}(rh)-\phi_{\bar{x}}(jh)=h\sum_{s=j}^{r-1}[\Delta_h\phi](sh) \quad .$$

Moreover

(2.14b) $$|\phi_{\bar{x}}(jh)|\leq M \quad ,$$

(2.14c) $$|\phi(jh)|\leq jhM \quad .$$

Proof: Equation (2.14a) follows from equation (2.10c). When k=1 we observe that $\phi_{\bar{x}}((N+1)h)=0$. Thus, in that case, inequality (2.14b) follows from (2.14a) and (2.13).

When k=0 we observe that it is impossible for $\phi_{\bar{x}}(jh)$ to be of constant sign unless it is identically zero. Hence, given an integer j there is an integer r so that

$$|\phi_{\bar{x}}(jh)-\phi_{\bar{x}}(rh)|=|\phi_{\bar{x}}(jh)|+|\phi_{\bar{x}}(rh)|\leq M \quad .$$

Thus, we have established (2.14b). In order to obtain inequality (2.14c) we merely observe that

$$|\phi(jh)|=|h\sum_{s=1}^{j}\phi_{\bar{x}}(sh)|\le jhM \ .$$

With these estimates we are able to prove a basic convergence theorem for the smallest eigenvalue of singular problems. The method is essentially the method used in [7]. We are not concerned with the "rate of convergence". However, once one has the basic convergence theorem, one can obtain rates as in [2], [4], or using some further estimates derive them from [20].

Lemma 2.5. Let $A(t)>0$ for $t\varepsilon(0,b)$. Consider the eigenvalue problem

$$(2.15) \qquad \Delta_h\phi+\mu A(t).\phi=0, \qquad \phi\varepsilon P_k \ .$$

Then, all the eigenvalues μ_j $(j=1,2,\dots,N)$ are positive, and the smallest eigenvalue, $\mu_1(A;h)$, is characterized by

$$(2.16) \qquad \mu_1(A;h)=\min_{\phi\not\equiv 0}\frac{h\sum_{j=1}^{N+1}\phi_{\bar{x}}^2(jh)}{h\sum_{j=1}^{N}A(jh)\phi^2(jh)}, \quad \phi\varepsilon P_k \ .$$

Moreover, the associated eigenfunction $\phi^1(t;h)$ is simple, of one sign, and may be chosen so that

$$(2.17) \qquad \phi^1(t;h)>0, \quad t\varepsilon G_k(h) \ .$$

Proof: These are standard results, see [4], [7], [15].

Corollary: If $A(t)\gneq B(t)$, then either

$$A(t)\equiv B(t), \quad t\varepsilon G_k(h)$$

or

$$\mu_1(A;h)<\mu_1(B;h) \ .$$

Let

$$(2.18) \qquad \mu_1(h)\equiv\mu_1(g(t,0);h),$$

where $g(x,v)$ is as in (1.6), and let $\phi^1(t;h)$ be the associated eigenfunction normalized so that (2.17) holds and

(2.19) $$\mu_1(h)\{h\sum_{j=1}^{N} g(jh,0)\left[\phi^1(jh;h)\right]^2\}=1 .$$

Let λ_0 denote the smallest eigenvalue of the differential equation

(2.20) $$V''+\lambda g(t,0)V=0 ,$$

subject to the appropriate boundary conditions.

<u>Lemma 2.6.</u> <u>There is a constant</u> $h_0>0$ <u>such that if</u> $h<h_0$ <u>then</u>,

(2.21) $$0<\mu_1(h)<2\lambda_0 .$$

<u>Proof</u>: Let $V(t)$ be the eigenfunction associated with λ_0. Let $\hat{V}(t;h)\varepsilon P_k(h)$ be determined by

$$\hat{V}(t;h)=V(t), \quad t\varepsilon\bar{G}_k(h) .$$

Then, an elementary argument shows that

$$\mu_1(h)\leq\frac{h\sum_{j=1}^{N+1}(\hat{V}_{\bar{x}}(jh))^2}{h\sum_{j=1}^{N} g(jh,0)\hat{V}^2(jh;h)}\to\lambda_0 \quad \text{as } h\to 0 .$$

<u>Lemma 2.7</u>. <u>There is a constant</u> $M_1<\infty$ <u>such that</u>, <u>for all</u> $h<h_0$

(2.22) $$|\phi^1(t;h)|<M_1 t , \quad 0\leq t\leq b$$

<u>Proof:</u> Since $\phi^1(t;h)$ satisfies $\Delta_h\phi^1+\mu_1(h)g(t,0)\phi^1=0$ and the normalization (2.19) it follows, from (2.11) and (2.12b) (with $s=0$) that

(2.23) $$|\phi^1(t;h)|<\sqrt{t} .$$

Using (1.6e), (2.21) and the basic equation (2.15) we see that

(2.24) $$|\Delta_h\phi^1(jh)|<\frac{2\lambda_0 M}{\sqrt{jh}} .$$

An elementary computation based on the "integral test" of advanced calculus shows that

$$h\sum_{j=1}^{N}|\Delta_h\phi^1(jh)|<2\lambda_0 M\sqrt{b+1}=M_1 \ .$$

The lemma follows from Lemma 2.4.

<u>Lemma 2.8</u>: <u>Let</u> $h<h_0$ <u>and set</u>

$$\psi(t;h)=\sqrt{g(t,0)}\,\phi^1(t;h) \ .$$

<u>Then the functions</u> $\phi^1(t;h)$ <u>and the functions</u> $\psi(t;h)$ <u>are uniformly bounded and equicontinuous</u>.

<u>Proof:</u> The uniformly boundedness and equicontinuity of $\phi^1(t;h)$ follow from the estimates of the preceding Lemma and Lemmas 2.3, 2.4. The equicontinuity of $\psi(t;h)$ follows from Lemma 2.7 and the equicontinuity of $\phi^1(t;h)$.

<u>Theorem 2.1</u>. <u>As</u> $h\to 0$, <u>we have</u>

$$\mu_1(h)\to\lambda_0$$

$$\phi^1(t;h)\to V(t),\ \underline{\text{uniformly for}}\ t\in[0,b] \ .$$

<u>Proof</u>: We may extract a subsequence h_k so that

$$\mu_1(h_k)\to\bar{\mu}$$

and the functions $\phi^1(t;h_k)$, $\psi(t;h_k)$ converge uniformly to functions $\Phi(t)$, $\Psi(t)$.

Using the conclusions of Lemma 2.4 (or merely (2.24) and (2.14a)) we may even assume that

$$h\sum_{j=1}^{N+1}\phi^1_{\bar{x}}(jh;h)^2\to\int_0^b|\Phi'(t)|^2dt \ .$$

Thus, we have

$$\int_0^b |\Phi'(t)|^2 dt = \bar{\mu} \int_0^b g(t,0)\Phi^2(t)dt = 1.$$

A glance at the proof of Lemma 2.6 shows that

$$0 \le \bar{\mu} \le \lambda_0 .$$

We claim that $\bar{\mu} > 0$. For if not,

$$\Phi(t) \equiv \Psi(t) \equiv 0 .$$

However

$$\bar{\mu} \int_0^b \Psi^2(t)dt = 1 .$$

But $\Phi(t)$ satisfies the boundary conditions of the limiting eigenvalue problem. Thus, the variational characterization of λ_0 shows that

$$\lambda_0 \le \bar{\mu} .$$

The theorem follows at once.

3. MAXIMAL SOLUTIONS

Consider the discrete analog of equation (1.5a). That is, we seek a function $U(t;h)\varepsilon P_k$ which satisfies

(3.1) $$\Delta_h U+\lambda f(t,U)=0 \ .$$

Lemma 3.1. Let

$$\lambda<\mu_1(h) \ .$$

Then there does not exist a solution of equation (3.1).

Proof: Suppose $U(t;h)\varepsilon P_k$ satisfies equation (3.1). Then, clearly λ is an eigenvalue and $U(t;h)$ is an eigenfunction of the linear eigenvlue problem

(3.2a) $$\Delta_h U+\lambda A(t)U=0, \quad U\varepsilon P_k \ ,$$

where

(3.2b) $$A(t)=g(t,U(t;h)) \ .$$

Let

$$\alpha=\mu_1(A;h) \ ,$$

and let $V(t;h)\varepsilon P_k$ be the associated eigenfunction. Applying Lemma 2.5 we see that

(3.3) $$\alpha\leq\lambda \ .$$

But since

(3.4a) $$A(t)\leq g(t;0) \ ,$$

we see that

$$\mu_1(h)\leq\alpha\leq\lambda \ .$$

Thus the lemma is proven.

Corollary: Suppose $\lambda<\lambda_0$. Then there is an $h_1>0$ such that there is no solution of equation 3.1 if $h\leq h_1$.

Proof: Apply Theorem 2.1 and Lemma 3.1.

Lemma 3.2. Let M be a bound for $f(t,u)$. That is,

$$0\leq f(t,u)\leq M, \quad \text{for } 0\leq u.$$

Let $U^0(t;h)\varepsilon P_k$ be the solution of

(3.5) $$\Delta_h U^0+\lambda M=0 \ .$$

Let $U^{n+1}(t;h)\varepsilon P_k$ be the solution of

(3.6) $$\Delta_h U^{n+1}+\lambda f(t,U^n)=0 \ .$$

Then the functions $U^n(t;h)$ satisfy

(3.7) $$0\leq U^{n+1}(t;h)\leq U^n(t;h) \ .$$

Proof: A direct induction argument establishes this lemma. See [9], [10] for similar arguments.

Lemma 3.3. Let $Z(t;h)\varepsilon P_k$ satisfy

(3.8) $$\Delta_h Z+\lambda f(t,Z)\geq 0 \ .$$

Let $\{U^n(t;h)\}_{n=0}^{\infty}$ be the sequence of functions generated in Lemma 3.2.

Then

(3.9) $$Z(t;h)\leq U^n(t;h) \ .$$

Proof: Let

$$E^n(t;h)= Z(t;h)-U^n(t;h)\varepsilon P_k \ .$$

Then

$$\Delta_h E^0\geq\lambda(M-f(t,Z))\geq 0 \ .$$

Hence, using Lemma 2.1,

$$E^0(t;h)<0, \quad t\varepsilon G_k(h) \ .$$

Assume

$$E^n(t;h)\leq 0 \ .$$

Then, using (1.6c),

$$\Delta_h E^{n+1} = \lambda\left[f(t,U^n) - f(t,Z)\right] \geq 0 \ .$$

Hence

$$E^{n+1} \leq 0 \ ,$$

and the lemma is proven.

Lemma 3.4. Let $\mu_1(h)<\lambda$. Let $V^0(t;h)$ be the eigenfunction associated with $\mu_1(h)$ normalized so that inequality (2.17) holds and

$$\left[\frac{\lambda}{\mu_1(h)} g(t,V^0) - g(t,0)\right] = R(t;h) \geq 0 \ . \tag{3.10}$$

For $n=1,2,\ldots$ let $V^n(t;h)\varepsilon P_k$ be the solution of

$$\Delta_h V^n + \lambda f(t,V^{n-1}) = 0 \ . \tag{3.11}$$

Then for all n and $r>0$ we have

$$0 < V^n(t;h) \leq V^{n+1}(t;h) \leq U^r(t;h) \ . \tag{3.12}$$

Proof: Using inequality (3.10) we have

$$\Delta_h V^0 + \lambda f(t,V^0) = \left[\Delta_h V^0 + \mu_1(h) g(t,0) V^0\right] + \mu_1(h) R(t;h) V^0(r;h) \geq 0 \ .$$

Assume that

$$\Delta_h V^n + \lambda f(t,V^n) \geq 0 \ . \tag{3.13a}$$

Then

$$\Delta_h V^{n+1} = -\lambda f(t,V^n) \leq \Delta_h V^n \ .$$

Thus, using Lemma 2.1,

$$V^{n+1}(t;h) \geq V^n(t;h) \ , \tag{3.13b}$$

and furthermore

$$\Delta_h V^{n+1} + \lambda f(t,V^{n+1}) = \lambda\left[f(t,V^{n+1}) - f(t,V^n)\right] \geq 0 \ . \tag{3.13c}$$

The Lemma follows from Lemma 3.3 and (3.13b). The construction of the sequence $\{V^n(t;h)\}$ based on inequality (3.10) has been inspired for a similar one that Picard [12] has used for the differential equation (1.5a).

<u>Theorem 3.1</u>. <u>Suppose</u>

$$\mu_1(h)<\lambda \;.$$

<u>Then there exists a positive maximal solution</u> $U(t;h)$ <u>of equation</u> (3.1).

<u>That is</u>

(3.14) $\quad U(t;h)>0, \quad t\varepsilon(0,b)$,

<u>and, if</u> $T(t;h)$ <u>is any other solution, then</u>

(3.15) $\quad T(t;h)\leq U(t;h)$.

<u>Moreover, if</u> $T(t;h)$ <u>is another solution and</u>

(3.16a) $\quad 0\leq T(t;h)$,

<u>then either</u>

(3.16b) $\quad T(t;h)\equiv 0$

<u>or</u>

(3.16c) $\quad T(t;h)\equiv U(t;h)$.

<u>Proof</u>: The functions $U^n(t;h)$ generated in Lemma 3.2 converge monotonically to a function $U(t;h)$ which satisfies (3.14). Inequality (3.15) follows from Lemma 3.3.

Suppose $T(t;h)$ is a nontrivial solution which satisfies (3.16a).

Then

(3.17) $\quad T(t;h)\leq U(t;h)$.

Moreover, if

$$A(t)=g(t,T(t;h)),\; B(t)=g(t,U(t;h)) \;.$$

we see that

$$A(t)\geq B(t) \;.$$

But, since

$$\mu_1(A;h)=\mu_1(B;b)=\lambda \quad ,$$

equality (3.16c) follows from the corollary to Lemma 2.5.

Theorem 3.2. Let $u(t)$ be the unique positive solution of equation (1.5) (see [3], [10], [13]). Let $U(t;h)$ be the unique positive solution of equation (3.1). Then,

$$\lim_{h\to 0+} |U(t;h)-u(t)|=0 \ , \quad 0\le t\le b \ . \tag{3.18}$$

Proof: We observe that the functions $V^n(t;h)$ also converge to $U(t;h)$. Moreover, the analogous iterations for the differential equation generate two sequences $u^n(t)$, $v^n(t)$, $n=0,1,\ldots$. It is an easy matter to show that, for every n

$$\begin{aligned} &\lim_{h\to 0}|U^n(t;h)-u^n(t)| =0 \ , \quad 0\le t\le b \ . \\ &\lim_{h\to 0}|V^n(t;h)-v^n(t)|=0 \ , \quad 0\le t\le b \ . \end{aligned} \tag{3.19}$$

The theorem now follows from Fatou's Lemma (see [9], [10]).

Lemma 3.5. Let $\lambda_0<\lambda$. Let $u(t)$ be the unique positive solution of equation (1.5). Let $\alpha_0(u)$ denote the smallest eigenvalue of the linear eigenvalue problem

$$\phi''+\alpha f_u(t,u)\phi=0 \tag{3.20}$$

subject to the appropriate boundary conditions. Then

$$\lambda<\alpha \ .$$

Proof: This result follows from the fact that

$$f_u(t,u)\le g(t,u) \ , \quad f_u(t,u)\not\equiv g(t,u) \ , \tag{3.21}$$

and the monotonicity of the smallest eigenvalue, i.e. the continuous analog of the Corollary to Lemma 2.5, (see [11]).

Lemma 3.6. Let $\xi(t;h)$ be any function which is between $u(t)$ and $U(t;h)$. Then

$$\lim_{h\to 0} h\ \Sigma|f_u(jh,u(t))-f_u(jh,\xi(t;h))|(jh)=0 \ .$$

Proof: Note that

$$|f_u(jh,u(t))-f_u(jh,\xi(t;h))|(jh) \le 2M_0 ,$$

and, for every $t\varepsilon(0,b]$ we have

$$\lim_{h\to 0+} |f_u(t,u(t))-f_u(t,\xi(t;h))|t=0 .$$

Thus, the lemma follows from the Dominated Convergence Theorem.

Theorem 3.3. Let $\lambda_0<\lambda$ and let $u(t)$ be the unique positive solution of equation (1.5). For h sufficiently small $\mu_1(h)<\lambda$. Let $U(t;h)$ be the unique positive solution of equation (3.1). Let

$$h\sum_{j=1}^{N} |[\Delta_h u](jh)+\lambda f(jh,u(jh))|=\tau(h) . \tag{3.22}$$

Then, there is a constant $h_1>0$ and a constant $L>0$ such that for all $h<h_1$

$$\mathrm{Max}|U(jh;h)-u(jh)|\le L\tau(h) . \tag{3.23}$$

Proof: There is an $h_0>0$ and a constant $\delta>0$ such that

$$\mu_1(f_u(t,u);h)\ge\lambda+\delta , \quad h<h_0 . \tag{3.24}$$

This fact follows immediately form Lemma 3.5 and the proof of Theorem 2.1. Let

$$W(t;h)=U(t;h)-u(t) .$$

Then

$$\Delta_h W+\lambda f_u(t,u)W=\lambda[f_u(t,u)-f_u(t,\xi(t;h))]W+r(t;h) \tag{3.25a}$$

where

$$r(t;h)=-[\Delta_h u+\lambda f(t,u)] . \tag{3.25b}$$

Multiplying equation (3.25a) by W and summing by parts we obtain

$$-h\sum_{j=1}^{N+1} W_{\bar{x}}^2+\lambda h\sum_{j=1}^{N} f_u(jh,u)W^2(jh;h)=A_0+A_1 , \tag{3.26}$$

where

$$(3.26a)\qquad A_0=h\sum_{j=1}^{N}\left[f_u(jh,u)-f_u(jh,\xi)\right]W^2(jh;h)\ ,$$

$$(3.26b)\qquad A_1=h\sum_{j=1}^{N} r(jh;h)W(jh;h)\ .$$

Thus

$$(3.27)\qquad h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}(jh;h)\le\lambda h\sum_{j=1}^{N} f_u(jh,u)W^2+|A_0|+|A_1|\ .$$

From (2.16) and (3.24) we obtain,

$$(3.28)\qquad \lambda\left[h\sum_{j=1}^{N} f_u(jh,u)W^2\right]\le\frac{\lambda}{\mu_1(f_u;h)}\left[h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}\right]<\frac{\lambda}{\lambda+\delta}\left[h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}\right]\ .$$

Hence, (3.27) is transformed into

$$(3.29)\qquad \frac{\delta}{\lambda+\delta}\left[h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}(jh;h)\right]<|A_0|+A_1|\ .$$

Applying Schwartz's inequality to $W(jh;h)=\sum_{s=1}^{j} W_{\bar{x}}(sh;h)$

we get

$$|W(jh;h)|<\sqrt{jh}\{h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}\}^{1/2}\ ,$$

and therefore,

$$(3.30)\qquad |A_0|<h\sum_{j=1}^{N}\left[f_u(jh;u)-f_u(jh;\xi)\right](jh).\{h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}\}\ .$$

From (3.29), (3.30) follows immediately

$$(3.31)\qquad h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}(jh;h)\le\frac{\lambda+\delta}{\delta}\{\tau(h)||W||_\infty+\varepsilon(h)\left[h\sum_{j=1}^{N+1} W_{\bar{x}}^{\,2}(jh;h)\right]\}$$

where

$$\varepsilon(h)=h\sum_{j=1}^{N}|f_u(jh,u(t))-f_u(jh,\xi(t;h))|(jh)\ .$$

Applying Lemma 3.6 we see that there is an $h_1>0$ such that

$$\frac{\lambda+\delta}{\delta}\varepsilon(h)\leq\frac{1}{2}\ ,\quad h\leq h_1\ .$$

Thus, using Lemma 2.3 we finally have

$$\|W\|_{\infty}\leq\frac{2(\lambda+\delta)}{\delta}\tau(h),\quad h\leq h_1\ .$$

4. PROPERTIES OF ZEROS OF SOLUTIONS OF PROBLEM (1.5a), (1.5c).

We rewrite the boundary value problem studied by Coffman [3]:

(4.1a) $v''+\lambda f(x,v)=0$, $0<x\leq 1$,

(4.1b) $v(0)=v'(1)=0$,

where f satisfies all the hypotheses (1.6a)-(1.6f).

Let $0<\lambda_0<\lambda_1<...<\lambda_n$ ... be the eigenvalues of the linearization at the origin (1.8), and let us only consider those solutions of (4.1) that satisfies the normalization condition $v'(0)>0$. Then, under an additional hypotheses that will be explicited later (see (g) below or [3,p.757 Th.3.1]) Coffman proves that there are exactly (n+1) nontrivial solutions to problem (4.1): $v_0(x)$, $v_1(x),...,v_n(x)$, provided that

(4.2) $\lambda_n<\lambda<\lambda_{n+1}$.

Furthermore, as in Kolodner's special case, the j:th solution has exactly j nodal zeros in (0,1).

Given any $k, 0\leq k\leq n$, we will construct algorithms capable of producing convergent approximations to $v_k(x)$.

The homotopy method is frequently used in the approximated solution of bifurcation problems of which (4.1) is an instance. The problem is immersed into a problem depending continuously upon a parameter γ. For $\gamma=0$ (say) the resulting problem is linear and can be readily solved. By moving γ continuously it is possible to recover the original problem and an iterative procedure can be employed to approximate its solutions. Unfortunately, and due to the character of the problem, it may be difficult to control to which of the many solutions of the problem the procedure will converge. In fact, the homotopy method is equivalent in this case to moving in function space on a branch of solutions parametrized by γ. Jumping from one branch to another in an unpredictable way and without any clear justification is a common happening (see [16], [17], [19]).

In the present case we have considerable information about the structure of solutions and it will be possible to avoid those difficulties by using a completely different approach. As an additional feature our methods will be more economical than the homotopy method.

We shall describe now, mostly without proofs, some properties which will be of use in the algorithms (for proofs see Coffman's paper and also Kolodner [6]).

(a) For any $a>0$ there exists a unique solution $v(x,a)$ on $0\leq x<\infty$, of the initial value problem:

$$v''+\lambda f(x,v)=0\ , \qquad v(0)=0,\ v'(0)=a\ . \tag{4.3}$$

Let $\Delta(x,a)$ denote the solution of the variational equation at $v(x,a)$ (which exists uniquely)

$$\Delta''+\lambda_v(x,v(x,a))\Delta=0\ , \qquad \Delta(0,a)=0,\ \ \Delta'(0,a)=1\ . \tag{4.4}$$

(b) The zeros of $v(x,a)$ separate those of $\Delta(x,a)$.

(c) $\Delta(x,a)=\frac{\partial}{\partial a}v(x,a)\ ,\quad \Delta'(x,a)=\frac{\partial}{\partial a}v'(x,a)\ .$

Let $u_0(x)$ be the solution of the initial value problem

$$u''+\lambda g(x,0)u=0\ , \qquad u(0)=0,\ \ u'(0)=1\ . \tag{4.5}$$

(d) The zeros of $u_0(x)$ separate those of $v(x,a)$.

(e) $\lim_{a\downarrow 0}\frac{1}{a}v(x,a)=\lim_{a\downarrow 0}\Delta(x,a)=u_0(x)\ ,$

$\lim_{a\downarrow 0}\frac{1}{a}v'(x,a)=\lim_{a\downarrow 0}\Delta'(x,a)=u_0'(x)\ .$

Let

$$w(x,a)=a+\lambda\int_0^x g_x(t,v(t,a))v(t,a)\Delta(t,a)dt\ . \tag{4.6}$$

For any given $a>0$ we shall denote by $y_k(a)$ the k^{th} positive zero of $v(x,a)$, and respectively by $z_k(a)$, $\alpha_k(a)$, $\beta_k(a)$, those of $v'(x,a)$, $\Delta(x,a)$, and $\Delta'(x,a)$.

If $v(x,a)$ has m positive zeros then it follows that

$$\begin{aligned}
&(-1)^k v(x,a) \geq 0 \quad \text{on} \quad [y_k \, , \, y_{k+1}] \, , \quad 0 \leq k < m \, , \\
&(-1)^k v'(x,a) \geq 0 \quad \text{on} \quad [z_k \, , \, z_{k+1}] \, , \quad 1 \leq k < m \, , \\
&(-1)^{k-1} \Delta(x,a) \geq 0 \quad \text{on} \quad [\alpha_{k-1}, \alpha_k] \, , \quad 1 \leq k \leq m \, , \\
&(-1)^{k-1} \Delta'(x,a) \geq 0 \quad \text{on} \quad [\beta_{k-1}, \beta_k] \, , \quad 1 < k \leq m \, .
\end{aligned} \tag{4.7}$$

Let n be a positive integer and assume that $v(x,a)$ has at least n positive zeros.

(f) If $w(x,a)>0$ on $(0,\alpha_{n-1}(a)]$ then the following ordering holds:

$$\begin{aligned}
&0=y_0(a)=\alpha_0(a)<z_1(a)<\beta_1(a)<y_1(a)<\alpha_1(a)<\ldots<y_{k-1}(a)< \\
&<\alpha_{k-1}(a)<z_k(a)<\beta_k(a)<\ldots<y_{n-1}(a)<\alpha_{n-1}(a)<z_n(a)< \\
&<\beta_n(a)<y_n(a) \, .
\end{aligned}$$

(g) If $g_x(x,v(x,a))$ has constant sign on $(0,\alpha_{n-1}(a)]$ then the condition $w(x,a)>0$ is necessary and sufficient for the ordering of the zeros.

We shall assume in the rest of this paper that $w(x,a)>0$.

This condition is fulfilled automatically in the autonomous case $f(x,v)\equiv f(v)$, since then $w(x,a)\equiv a>0$. It is also fulfilled in Kolodner's problem (1.2).

We shall prove now:

Lemma 4.1: The zeros $y_k(a)$, $z_k(a)$ are increasing functions of a.

Proof: We have that

$$v(y_k(a),a)\equiv 0 \quad , \; 0<a, \quad 1\leq k\leq n \; ,$$

$$v'(z_k(a),a)\equiv 0 \; , \; 0<a \quad 1\leq k\leq n \; .$$

Thus, total differentiation gives, with a selfexplanatory notation,

$$\frac{dv(y_k(a),a)}{da} = v'(y_k(a),a)\frac{dy_k(a)}{da}+\frac{\partial v(y_k(a),a)}{\partial a}\equiv 0 \; .$$

By using (c) it follows that

$$\frac{dy_k(a)}{da} = -\frac{\Delta(y_k(a),a)}{v'(y_k(a),a)} \tag{4.8}$$

Similarly, and recalling that $v''=-\lambda f(x,v)$

$$\frac{dz_k(a)}{da}=\frac{\Delta'(z_k(a),a)}{\lambda f(z_k(a),v(z_k(a),a))} \tag{4.9}$$

From (4.7), (4.8), (4.9) and the ordering (f) it follows that for $0<a$:

$$\text{sign}\frac{dy_k(a)}{da} =1 \; , \quad \text{sign}\frac{dz_k(a)}{da}=1 \; ,$$

and the Lemma is proven.

Let us denote by $a(t,y_k)$ the mapping that assign to a given number t in (0,1) the unique a such that $v(x,a)$ has t as its kth zero; also define $a(t,z_k)$ similarly.

Corollary 4.2: For each zero y_k,z_k the inverse mappings $a(t,y_k)$, $a(t,z_k)$ exist as differentiable functions, and moreover

$$\frac{da(t,y_k)}{dt} >0 \quad , \quad \frac{da(t,z_k)}{dt} >0 \; .$$

Proof: Obvious.

Furthermore,

$$\frac{da(t,y_k)}{dt} = -\frac{v'(t,a(t,y_k))}{\Delta(t,a(t,y_k))} ,$$

(4.10)

$$\frac{da(t,z_k)}{dt} = \frac{\lambda f(t,v(t,a(t,z_k)))}{\Delta'(t,a(t,z_k))} .$$

Corollary 4.2 implies then that every zero of $v(x,a)$ is an increasing function of any of them. The same is true of $v'(x,a)$.

Lemma 4.3.

$$\text{sign} \frac{dv'(1,a(t,y_k))}{dt} = \text{sign } \Delta'(1,a(t,y_k)) .$$

Proof:

$$\frac{dv'(1,a(t,y_k))}{dt} = \frac{\partial v'(1,a(t,y_k))}{\partial a} \cdot \frac{da(t,y_k)}{dt} =$$

$$= \Delta'(1,a(t,y_k)) \cdot \frac{da(t,y_k)}{dt} ,$$

and the Lemma follows since $\frac{da(t,y_k)}{dt} > 0$.

Remark: We can easily find the solution with k interior zeros in the autonomous case $f(x,v) \equiv f(v)$, just by computing a positive solution in an appropriate interval of length $\ell < 1$. In fact, because of the uniqueness and the invariance under translations we have that, by putting $\ell = 1/(2k+1)$, and calling $u_\ell(t)$ to the solution of

$$v'' + \lambda f(t,v) = 0 , \quad v(0) = 0 , \; v'(\ell) = 0 ,$$

we can obtain the solution with k interior zeros in (0,1) in the following way:

First we extend $u_\ell(t)$ to the interval $[0,2\ell]$ by reflecting it:

$$w_\ell(t) = \begin{cases} u_\ell(t) , & 0 \le t < \ell , \\ u_\ell(2l-t) , & 1 \le t < 2\ell . \end{cases}$$

Then we obtain the desired solution as:

$$u_k(t) = (-1)^s w_\ell(t-2s\ell) , \; 2s\ell \le t \le 2\,(s+1)\ell , \; s = 0,1,\ldots,k.$$

5. SHOOTING FROM THE ORIGIN ($0 \leq k \leq n$)

Shooting is a standard approach to the numerical solution of two-point boundary value problems. By shooting is meant a systematic procedure of finding the value of "a" in problem (4.3), such that the corresponding $v(x,a)$ satisfies (in this case)

$$v'(1,a)=0 \quad . \tag{5.1}$$

We assume in what follows that λ is as in (4.2).

As it is well known [5] , (5.1) is essentially a nonlinear (algebraic) equation in one unknown of which we want to find the roots. The trouble is that the mapping $a \to v'(1,a)$ includes the solution of an initial value problem which, in general, will have to be obtained by approximate means.

From now on we shall assume that the data is $C^p(0,1), p \geq 2$.

Let $0<\bar{a}$ be given and let (4.3) be solved approximately by a finite difference method of order p in h, the integration step. Let $V(x,h,\bar{a})$ be the discrete solution just obtained, defined on the grid points

$$x_i = x_{i-1} + h_i \ (i=1,\dots,s), \quad 0<h_i \leq h, \quad x_0=0, \ x_s=1 \ .$$

If a^* is the root of (5.1) being approximated then the following result holds [5]:

$$|V(x_i,h,\bar{a}) - v(x_i,a^*)| < K_1 h^p + K_2 |\bar{a}-a^*| \ , \tag{5.2}$$

K_1, K_2 nonnegative constants.

In our case we want besides, not any root of (5.1), but that one for which $v(x,a^*) \equiv v_k(x)$ has $0 \leq k \leq n$ interior zeros in (0,1). An approximation V to $v_k(x)$ can be obtained by taking a value $0<\bar{a}<\infty$ and computing an approximate solution $(V_1(x,h,\bar{a}) \ , \ V_2(x,h,\bar{a}))$ to the system

$$v_1' = v_2 \quad ,$$

$$(5.3)\qquad v_2' = -\lambda f(x,v_1)\ ,$$

$$v_1(0)=0\ ,\ v_2(0)=\bar{a}\ ,$$

(which is equivalent to (4.3)) by some stable, convergent method of order p, with maximum step size h. Care should be taken in order that x=1 be a grid point. In this way $V_2(1,h,\bar{a})$ will be an approximation to $v'(1,\bar{a})$ of order h^p .

Since f(x,v) may not be defined at the origin we must remember to take $f(0,v(0))=\lim_{x\downarrow 0} f(x,v(x))$.

In what follows we describe a process for locating an appropriate neighborhood in order to isolate the desired root and to be in a good position to start an iterative procedure to approximate it accurately. The description refers to the continuous solutions v(x,a) and it is understood that the numerical process will replace, for h sufficiently small, v(x,a) by V(x,h,a), taking care of preserving convergence throughout.

The first step consists of searching the interval $0<a<\infty$ looking for two solutions $v(x,a_1)$, $v(x,a_2)$ of (4.3) such that:

(a) $v(x,a_i)$ (i=1,2) has exactly k interior zeros in (0,1).

(b) sign $v'(1,a_1)\neq$ sign $v'(1,a_2)$.

Solutions satisfying (a) exist since there is one value a* for which v(x,a*) not only has exactly k interior zeros but even satisfies the boundary conditions (4.1b). By continuity and the monotonicity of the zeros of v(x,a) with respect to a it follows that there is a whole interval around a* such that all the solutions of (4.3) with a in that interval have exactly k zeros in (0,1).

That (b) can be attained follows from the following lemma:

Lemma 5.1. For any k=0,1,...,n let us assume that a* corresponds to the solution with exactly k zeros in (0,1).

(i) a^* <u>is a simple root</u>.

(ii) $\operatorname{sign} \frac{dv'(1,a^*)}{da} = (-1)^k$.

<u>Proof:</u>

Because of the ordering of zeros (§4(f)) we have that

(5.4) $$\beta_k < y_k < z_{k+1} = 1 < \beta_{k+1} \quad ,$$

and therefore $\Delta'(1,a^*) \neq 0$.

From the proof of Lemma 4.3 it follows that $\frac{dv'(1,a^*)}{da} \neq 0$ and a^* is a simple root.

The sign pattern (4.7) shows that

(5.5) $$\operatorname{sign} \frac{dv'(1,a^*)}{da} = \operatorname{sign} \Delta'(1,a^*) = (-1)^k \quad ,$$

and the lemma is proven.

Since (5.5) will be valid by continuity on an interval around a^*, then property (b) can be attained if a_1, a_2 belong to such an interval and $a_1 < a^* < a_2$.

In short, by direct search we can find an interval $[a_1, a_2]$ around a^* characterized by properties (a),(b). This interval will contain (and isolate) the particular root we are seeking, and from (5.5) we know precisely the shape of the graph of $v'(1,a)$ in it. Therefore we are in an optimal condition to start an iterative process like the chord method in order to obtain a good approximation to a^*.

In Section 7 we shall explain briefly the automatic searching procedure we have employed in our computer implementation.

For the approximation of the root a^* we advocate the following procedure which is a modification of the chord method, and which is guaranteed to converge, giving besides computational error bounds for $|\bar{a} - a^*|$, $\bar{a}$ being the last approximation to the zero a^*.

At the begining we have

(i_0) $v(x,a_1)$, $v(x,a_2)$ with k zeros in (0,1), but neither $v'(1,a_1)$ nor $v'(1,a_2)$ are zero (otherwise we have finished).

(ii_0) $\mathrm{sign}(v'(1,a_1)) \neq \mathrm{sign}(v'(1,a_2))$.

(iii_0) Putting $\bar{a}=a_1$ (say)

$$|\bar{a}-a^*|<(a_2-a_1)=\text{error} .$$

We take now a chord step

$$t=a_1-\frac{v'(1,a_1)(a_2-a_1)}{v'(1,a_2)-v'(1,a_1)} .$$

(i_1) Since $t\varepsilon(a_1,a_2)$ it follows that $v(x,t)$ has also k zeros in (0,1). If $v'(1,t)=0$ then we are through and $a^*=t$. If not, we choose from a_1,t,a_2 the sub-interval in whose endpoints there is a sign change, and we call it (a_{1new},a_{2new}). This is not quite the iterative version of the chord method, but preserves for us properties (ii_0) and (iii_0):

(iii_1)
$$|\bar{a}_{new}-a^*|\leq a_{2new}-a_{1new}=\text{error}_{new} .$$

If

(5.6) $\text{error}_{new}>c_1.\text{error}$,

where $0<c_1<1$ is a constant to be chosen experimentally and probably fairly close to 1, we decide that the situation is like in one of the graphs in Fig. 1 (or the similar ones changing the sign of the curvatures), i.e. the desired zero is close to one of the endpoints and therefore the process described above will not change, in general, the other endpoint, making our error bound too pesimistic. To improve our estimate in this case we perform a chord step chosing from $a_{1_{old}}$, t, $a_{2_{old}}$ the pair $a_{i_{old}}$, t for which sign $v'(1,a_{i_{old}})$ = sign $v'(1,t)$. The t_{new}

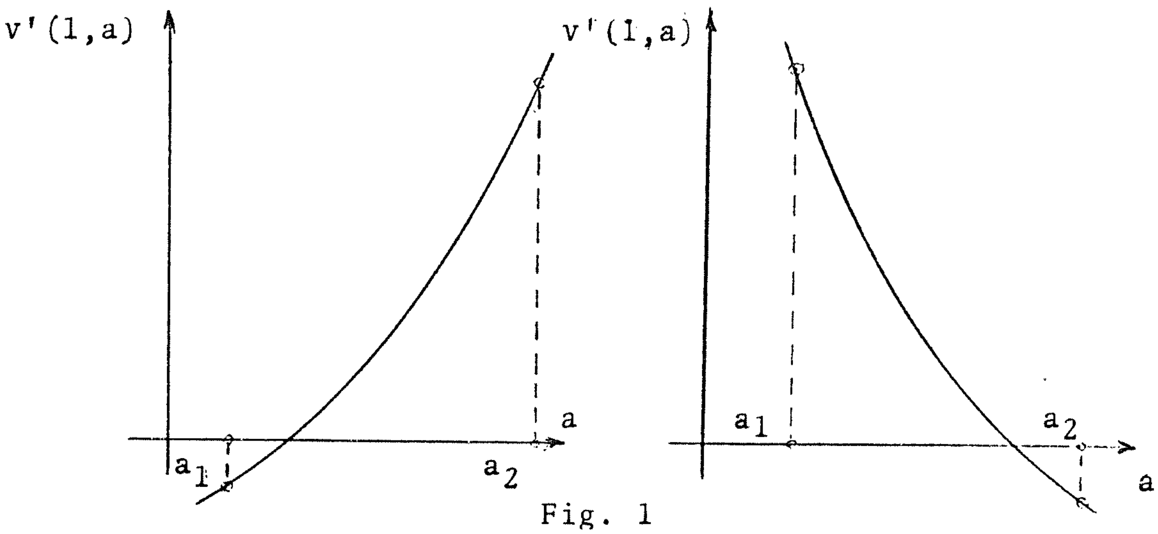

Fig. 1

thus produced is checked for consistency, i.e. we take it only if $a_{1_{new}} < t_{new} < a_{2_{new}}$. The rational here is that situation (5.6) will indicate, in general, that the standard chord method is converging monotonically from one side of the desired root, and furthermore that the latest approximations are fairly close. Thus we can presume that $v'_a(1,a)/v'_{aa}(1,a)$ has constant sign in the neighborhood in which our iterates are located and that using for a chord step two iterates <u>at the same side</u> of the root will produce a close approximation from the other side. (See [22, p.100,(18)]). This is bear out by the behaviour of the algorithms.

This strategy does not affect the theoretical convergence of the method but has an important effect in its efficiency. Recall that each evaluation of v'(1,a) is a costly affair.

The iteration ends when:

(5.7) # iterations > max. number of iterations (data) .OR.

$(|v'(1,a_1)| \leq$ EPS1 (data) .AND. error $\leq$ EPS2 (data)).

Observe that the approximate computation of t in (iii_0) yields results of order h^p when $V_2(1,h,a_i)$ is used instead of

$v'(1,a_i)$. Observe also that the choice of the constant c_1 is only important in what efficiency is concerned but that convergence does not depend upon it.

This procedure has been implemented and tested. We shall give some numerical results in §7.

The main difficulty with it is that an infinite interval must be searched in order to find the basic interval $[a_1, a_2]$ having properties (i_0, ii_0, iii_0).

If solutions with an increasing number of zeros are computed in succession then bounds can be obtained which effectively reduce the size of the interval to be initially searched.

In fact, again from the monotonicity, $v_j'(0) < v_i'(0)$ if $0 \leq i < j \leq n$. Therefore, if we compute first $v_0(x)$, the positive solution, by the direct procedures of §3, then for any j, $1 \leq j \leq n$, the corresponding a^* must belong to the interval $(0, v_0'(0))$, and the amount of search will be significantly reduced.

A different approach that combine computation of positive solutions to certain associated problems and shooting will be given in the next Section. The main features of this procedure is that no a priori information is required and that the computation of positive solutions (a very stable process in this case) provides accurate starting values for the shooting part.

6. A CHANGE OF PARAMETER.

Some of the results of §4 suggest the use of y_1, the first zero of a solution of (4.3) for a given a, as a parameter instead of a itself.

In fact from Lemma 4.1 we see that this change of variable is permissible. Furthermore, given a value $0 < y_1 < 1$, the computation of a solution with this prescribed first positive zero is equivalent to finding the unique positive solution of the two-point boundary value problem

$$\text{(6.1)} \qquad \begin{aligned} & v'' + \lambda f(x,v) = 0 \ , \\ & v(0) = v(y_1) = 0 \ , \end{aligned}$$

and then continuing this solution up to $x = 1$.

Problem (6.1) can be solved accurately by one of the algorithms described in §3. Once a discrete solution $V(x,h,y_1)$ is known in the interval $[0,y_1]$, on a mesh with step size h, we can extend it to the whole interval $[0,1]$ by using initial value techniques.

Let ε be a given positive number, and let us assume that we have obtained $V(x,h,y_1)$ for $0 \leq x \leq 1$ as explained above. Let $V'(1,h,y_1)$ be obtained by numerical differentiation as an approximation to $v'(1,y_1)$. If $V(x,h,y_1)$ has the right number of changes of sign in $(0,1)$, and $|V'(1,h,y_1)| \leq \varepsilon$ then we have finished. Otherwise, it is necessary to correct our estimate of the first zero of the desired solution and repeat the process. The corrections and, in general, the searching procedure, can be done in the same way we described in §5, since we have for the new parameter all the necessary properties.

The main difference rests in the fact that the new parameter y_1 must necessarily lie in the interval $(0,1)$ for any solution with $1 \leq k \leq n$ zeros, as compared with the infinite interval of permissible values for a. The initial search is reduced in scope and no a priori bounds or straneous computations are required.

We shall state precisely now the steps of this process which are different from the one we gave in §5.

We assume now that we are looking for that solution of (4.1) with $1 \leq k \leq n$ zeros in (0,1).

Given a value $0 < y_1 < 1$ there is a unique function $v(x,y_1)$ that satisfies

$$(6.2) \qquad \begin{aligned} &u'' + \lambda f(x,u) = 0 \ , \\ &u(0) = u(y_1) = 0 \ , \\ &u(x) \geq 0 \ , \ 0 \leq x \leq y_1 \ ; \end{aligned}$$

and

$$(6.3) \qquad \begin{aligned} &w'' + \lambda f(x,w) = 0 \ , \\ &w(y_1) = 0 \ , \ w'(y_1) = u'(y_1) \ , \\ &y_1 \leq x \leq 1 \ . \end{aligned}$$

If $y_1 = y_1^*$ is the first zero of the desired solution of (4.1), then $v(x,y_1^*)$ will be that solution and reciprocally. Thus problem (6.2), (6.3) is equivalent to (4.1) when y_1 is chosen to be the first zero of the solution of (4.1) with the prescribed number of zeros.

We discretize (6.2) by putting $b = y_1$ and proceeding as in case 1, §2.

If $0 < \bar{y} \leq y_1$ for a given $\bar{y}$, then it is clear that $h \to 0$ when $N \to \infty$, uniformly in y_1. We shall show below how to obtain $\bar{y}$. This will in fact diminish still more the interval to be searched. On the net thus obtained we solve approximately (6.2) by one of the methods described in §3 getting $V(x,h,y_1)$, a discrete solution of order $p \equiv$ order in h of $\tau(h)$:

(6.4) $$|V(x,h,y_1) - v(x,y_1)| \leq k_1 h^p, \quad x \varepsilon G_0(h) .$$

We discretize (6.3) by using a q-step method of order p for special equations of the second order (see Henrici [18, Chapt. 6]), which does not employ first derivatives. Since $x(x,y_1)$ extended to the left coincides with $v(x,y_1)$, then we can take the values $V(y_1 - s.h,h,y_1)$, $s = 0,1,\ldots,q - 1$, as the starting values, thus avoiding the need for an artificial starting procedure. Of course, if we desire to have an automatic step changing procedure then some restarting algorithm will be required. In this case we should not allow the step to be larger than h.

Since we shall have to compute $v'(1,y_1)$ numerically it is not required that $x = 1$ be a mesh point. In fact, if the data is defined and smooth, we can integrate a few steps beyond 1 in order to use a better formula to compute $V'(1\ ,h,y_1)$. Integrating up to $\bar{x} \geq 1$ we shall have now a discrete solution $V(x,h,y_1)$ which will be accurate to order h^p over the whole interval $[0,\bar{x}]$ (see for instance [5,p.23]).

It is clear that using p points we can obtain a h^{p-1} approximation to $v'(1,y_1)$. We lose one order in h because of the numerical differentiation.

Therefore, given a value $0 < y_1 < 1$, we have a way of obtaining an $O(h^p)$ approximation to $v(x,y_1)$, and an $O(h^{p-1})$ approximation to $v'(1,y_1)$.

Computation of $\bar{y}$. In order to calculate $\bar{y}$ it is necessary to compute the first positive zero of the solution $u_0(x)$ to the linear problem (4.5) with moderate accuracy.

Property (e) of §4, and the continuity of the zeros $y_s(a)$ with respect to a imply immediately that

(6.5) $$\lim_{a \downarrow 0} y_s\ (a) \downarrow \chi_s\ , \quad s = 1,\ldots,n\ ,$$

where χ_s is the s^{th} positive zero of $u_0(x)$.

In particular $\chi_1 = \lim_{a \downarrow 0} y_1(a)$, and it does not exist a solution to $v'' + \lambda f(x,v) = 0$ with $v'(0) > 0$ and first zero less than χ_1. Thus χ_1, or a one figure approximation from below can be used as $\bar{y}$, since nonnegative solutions with first zero smaller than χ_1 must vanish identically.

From now on the process is identical to the one in §5, except that the independent parameter is y_1 instead of a. Of course, the form in which $v'(1,y_1)$ is approximated has also changed.

7. SEARCHING PROCEDURE AND NUMERICAL RESULTS.

We shall briefly explain now the searching procedure used to obtain (a), (b) in Section 5. The one for the method of Section 6 is similar, though restricted to $(\bar{y},1)$.

The starting value a° is either arbitrary, in case no a priori bounds have been computed, or equal to an a priori bound (i.e. $v_s'(0)$, for $0 \leq s < k$; see p. 29). We start shooting with a° and counting the number of sign changes. The integration is interrupted if this number becomes greater than k. Since in this case there are too many zeros, we know, because of the monotonicity of the zeros with respect to a, that a° is an underestimate and we increase it in Δa, a given quantity that we have taken in our computations equal to a°. Observe that this can only happen if a° was arbitrary and not an upper bound for a.

If we integrate accross (0,1) without finding too many zeros there are two possibilities:

i) There were too few zeros, and therefore a° was an overestimate and we have to decrease it. In this case, and since we consider at the beginning that the direction of search is to the right, we diminish $\Delta a = a^\circ$ to $\Delta a/3$.
<u>We do so whenever there is a change in the direction of search</u>.

ii) We counted exactly k zeros, and a° is a candidate for an endpoint to an interval isolating a^*, the desired root. We check the sign of $v'(1,a^\circ)$, and according to Lemma 5.1, if sign $v'(1,a^\circ) = (-1)^k$ then a° is to the right of a^*, otherwise is to the left of a^*.

We keep on the search in the way described aboved until we have been able to find (a_1,a_2), the endpoints satisfying (a),(b). Once this is achieved then we are ready to start the chord iteration as explained in Section 5.

We now present some numerical results obtained with FORTRAN IV implementations of the procedures of Sections 5 and 6. They were computed on a IBM 360/40 at the Departamento de Computación of the Universidad Central de Venezuela. All the calculations were performed using 360 long words (∿16 decimal digits).

The problem we considered was Kolodner's (1.2) for different values of λ.

For instance, for $\lambda = 70$ and $k = 1,2,3,4$, we run both algorithms and the CPU times necessary to obtain all four solutions, reducing simultaneously: $a_2 - a_1 < 10^{-8}$ and $|v_k'(1,t)| < 10^{-10}$, was for SHOOT, the algorithm of Section 5: 189.5 seconds. It took an average of 10 evaluations of $v'(1,a)$ to obtain this result.

For MAXSOL, the algorithm of Section 6, the CPU time was 122.3 seconds, and the average number of evaluations was 9.

The run for SHOOT included the computation of the maximal solution ($k = 0$), used to obtain a° for the case $k = 1$.

We present also a somewhat more extreme case: $\lambda = 930$, $k = 15$.

For this case we used an arbitrary starting value of 35 and the performance of SHOOT was:

CPU Time: 65 sec.; search iterations: 5; # chord iterations: 8.
Final value of $a = 15.56199$.

For MAXSOL we had:

CPU Time: 60.6 sec.; # search iterations: 7; # chord iterations: 5; average # of iterations for positive solutions: 4 (to decrease the norm of the difference of two successive iterates below 5×10^{-7}).
Starting value for t: 0.5.
Final value for t: 0.03166132..., error in $t < 3.3 \times 10^{-12}$.

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