NUMERICAL SOLUTION OF DISCRETE NONDEGENERATE N-BODY PROBLEMS WITH AN APPLICATION TO FREE SURFACE FLUID FLOW

by

Donald Greenspan and David Schultz

Technical Report #79

January 1970
1. INTRODUCTION

The resulting dynamical behavior of n bodies under the influence of a given set of interactive forces has been of interest in pure and applied science for many years (see, e.g., references [1-3,5,7,8,10,13,15] and the additional references contained therein). Typical important n-body problems occur in the study of Brownian motion under the usual assumptions that n is relatively large and that collisions occur in accordance with an assumed probabilistic distribution, and in the study of the solar system under the usual assumptions that n is relatively small and that capture, but not collision, is admissible.

In this paper we will explore and apply a new numerical method for general, plane, nondegenerate n-body problems. The formulation will be based on a discrete approach to mechanics [4,5] in which the dynamical equations are difference equations and the solutions of the equations are discrete functions, that is, functions defined only on finite point sets. The effects of viscosity and collision will be included in the formulation and the extension of the method to three dimensions will offer no mathematical difficulty, but will require greater storage capacity for the actual computations.

*Supported by the National Science Foundation under Grant No. GJ-578
2. **DISCRETE DYNAMICAL EQUATIONS**

For $\Delta t > 0$, let $t_k^* = k \Delta t$, $k = 0, 1, \ldots$. Assume that a particle $P$ is in motion on an $x$-axis and is located at $x_k$ at time $t_k$. Then, for a given initial velocity $v_0^*$, it has been found convenient [4] to define the particle's velocity $v_k = v(t_k)$, $k = 1, 2, \ldots$, implicitly by

\[
(2.1) \quad \frac{v_k + v_{k-1}}{2} = \frac{x_k - x_{k-1}}{\Delta t}, \quad k = 1, 2, \ldots,
\]

and the particle's acceleration $a_k = a(t_k)$, $k = 1, 2, \ldots$ explicitly by

\[
(2.2) \quad a_k = \frac{v_k - v_{k-1}}{\Delta t}, \quad k = 1, 2, \ldots
\]

It follows readily [4] from (2.1) and (2.2) that

\[
(2.3) \quad v_1 = \frac{2}{\Delta t} (x_1 - x_0) - v_0
\]

\[
(2.4) \quad v_k = \frac{2}{\Delta t} \left\{ x_k + (-1)^k x_0 + 2 \sum_{j=1}^{k-1} \left[ (-1)^j x_{k-j} \right] \right\} + (-1)^k v_0, \quad k \geq 2
\]

\[
(2.5) \quad a_1 = \frac{2}{(\Delta t)^2} [x_1 - x_0 - v_0 \Delta t]
\]

\[
(2.6) \quad a_2 = \frac{2}{(\Delta t)^2} [x_2 - 3x_1 + 2x_0 + v_0 \Delta t]
\]

\[
(2.7) \quad a_k = \frac{2}{(\Delta t)^2} \left\{ x_k - 3x_{k-1} + 2(-1)^k x_0 + 4 \sum_{j=2}^{k-1} \left[ (-1)^j x_{k-j} \right] + (-1)^k v_0 \Delta t \right\}, \quad k \geq 3.
\]

It has been shown [4] that (2.3)-(2.7) and the discrete Newton's equation

\[
(2.8) \quad m \cdot a_{k+1} = F(x_k^*, t_k^*, v_k)
\]
imply the validity of the conservation of energy and momentum and that discrete solutions of initial value problems for (2.8) exist, are unique, and can be generated recursively on a digital computer whether $F$ is linear or nonlinear in its variables.

In order to study n-body problems, we will first extend (2.1) - (2.8) to systems of particles in planar motion as follows. Consider $n$ particles $P_i$, $i = 1, 2, \ldots, n$. At time $t_k = k\Delta t$, $k = 0, 1, \ldots$, let $P_i$ be at $(x_{i,k}, y_{i,k})$, have velocity $(v_{i,k,x}, v_{i,k,y})$, and have acceleration $(a_{i,k,x}, a_{i,k,y})$. Then, for each $i = 1, 2, \ldots, n$,

(2.9) $v_{i,1,x} = \frac{2}{\Delta t} [x_{i,1} - x_{i,0}] - v_{i,0,x}$

(2.10) $v_{i,1,y} = \frac{2}{\Delta t} [y_{i,1} - y_{i,0}] - v_{i,0,y}$

(2.11) $v_{i,k,x} = \frac{2}{\Delta t} [x_{i,k} + (-1)^k x_{i,0} + 2 \sum_{j=1}^{k-1} (-1)^j x_{i,k-j}] + (-1)^k v_{i,0,x}; k \geq 2$

(2.12) $v_{i,k,y} = \frac{2}{\Delta t} [y_{i,k} + (-1)^k y_{i,0} + 2 \sum_{j=1}^{k-1} (-1)^j y_{i,k-j}] + (-1)^k v_{i,0,y}; k \geq 2$

(2.13) $a_{i,1,x} = \frac{2}{(\Delta t)^2} [x_{i,1} - x_{i,0} - v_{i,0,x} \Delta t]$

(2.14) $a_{i,1,y} = \frac{2}{(\Delta t)^2} [y_{i,1} - y_{i,0} - v_{i,0,y} \Delta t]$

(2.15) $a_{i,2,x} = \frac{2}{(\Delta t)^2} [x_{i,2} - 3x_{i,1} + 2x_{i,0} + v_{i,0,x} \Delta t]$

(2.16) $a_{i,2,y} = \frac{2}{(\Delta t)^2} [y_{i,2} - 3y_{i,1} + 2y_{i,0} + v_{i,0,y} \Delta t]$
\[ a_{i,k,x} = \frac{2}{(\Delta t)^2} \left\{ x_{1,k} - 3x_{1,k-1} + 2 (-1)^k x_{1,0} + 4 \sum_{j=2}^{k-1} [(-1)^j x_{1,k-j}] \\
+ (-1)^k v_{1,0,x} \Delta t \right\}; k \geq 3 \]

\[ a_{i,k,y} = \frac{2}{(\Delta t)^2} \left\{ y_{1,k} - 3y_{1,k-1} + 2 (-1)^k y_{1,0} + 4 \sum_{j=2}^{k-1} [(-1)^j y_{1,k-j}] \\
+ (-1)^k v_{1,0,y} \Delta t \right\}; k \geq 3 . \]

In the usual manner of vector generalization, (2.8) is extended as

\[ m \ddot{a}_{k+1} = \frac{\vec{F}}{k} . \]

3. N-BODY PROBLEMS

For Newtonian n-body problems, the components of \( \vec{F} \) in (2.19) have factors of the form \( -\frac{G}{r^2} \). For our purposes, however, it will be of great advantage to replace such terms by more comprehensive expressions of the form

\[ -\frac{G}{(r + \xi)^2} + \frac{H}{(r + \xi)^p} - \alpha \sqrt{v_x^2 + v_y^2} . \]

where \( r \) is the distance between a given pair of interacting particles, \( G \) is the universal gravitational constant, \( H \) is a nonnegative constant of repulsion, \( \alpha \) is a nonnegative damping factor, \( \xi \) is a nonnegative parameter which can be used to avoid mathematical singularity when \( r \) can equal zero, and \( p \) is a constant.
Note, for example, that if $H > 0$, $p > 2$ and $\xi = 0$, then $-\frac{G}{r^2}$ dominates $\frac{H}{r^p}$ for large values of $r$, while $\frac{H}{r^p}$ dominates $-\frac{G}{r^2}$ for small values of $r$. Thus, the resulting repulsive effect for small values of $r$ can be used to simulate particle collision without the necessity of actual collision and, of course, also allows us to consider repulsion, itself, as a component of force when it does exist and does have a significant effect.

The resulting dynamical equations, which reduce to a classical $n$-body Newtonian formulation if $H = a = \xi = 0$ are derived simply by replacing terms of the form $-\frac{G}{r^2}$ in the usual formulas by terms of the form (3.1), are, for $k \geq 0$,

therefore

\begin{align*}
(3.2) \quad a_{i,k+1,x} = & \sum_{j=1 \atop j \neq i}^{n} \frac{m_j (x_{i,k} - x_{j,k})}{(r_{ij,k} + \xi)^2} \left\{ -\frac{G}{(r_{ij,k} + \xi)^2} + \frac{H}{(r_{ij,k} + \xi)^p} \right. \\
& \left. \quad - \alpha \left[ (v_{i,k,x})^2 + (v_{i,k,y})^2 \right] \right\}
\end{align*}

\begin{align*}
(3.3) \quad a_{i,k+1,y} = & \sum_{j=1 \atop j \neq i}^{n} \frac{m_j (y_{i,k} - y_{j,k})}{(r_{ij,k} + \xi)^2} \left\{ -\frac{G}{(r_{ij,k} + \xi)^2} + \frac{H}{(r_{ij,k} + \xi)^p} \right. \\
& \left. \quad - \alpha \left[ (v_{i,k,x})^2 + (v_{i,k,y})^2 \right] \right\},
\end{align*}

where $i = 1, 2, \ldots, n$ and where $r_{ij,k}$ is the distance between particles $P_i$ and $P_j$ at time $t_k$. 
4. SOLUTION OF N-BODY PROBLEMS

The following simple recursion formulas for the trajectory points

\[(x_{i,k}, y_{i,k}), i = 1, 2, \ldots, n\] then follow easily from substitution of (2.13)-(2.18) into (3.2)-(3.3):

\[
(4.1) \quad x_{i,1} = x_{i,0} + v_{i,0,x} \Delta t + \frac{(\Delta t)^2}{2} \left[ \sum_{j=1, j \neq i}^{n} \frac{m_j (x_{i,0} - x_{j,0})}{r_{ij,0} + \xi} \right] - \frac{G}{(r_{ij,0} + \xi)^2} \\
+ \frac{H}{(r_{ij,0} + \xi)^p} - \alpha \left[ (v_{i,0,x})^2 + (v_{i,0,y})^2 \right] \frac{1}{2}
\]

\[
(4.2) \quad y_{i,1} = y_{i,0} + v_{i,0,y} \Delta t + \frac{(\Delta t)^2}{2} \left[ \sum_{j=1, j \neq i}^{n} \frac{m_j (y_{i,0} - y_{j,0})}{r_{ij,0} + \xi} \right] - \frac{G}{(r_{ij,0} + \xi)^2} \\
+ \frac{H}{(r_{ij,0} + \xi)^p} - \alpha \left[ (v_{i,0,x})^2 + (v_{i,0,y})^2 \right] \frac{1}{2}
\]

\[
(4.3) \quad x_{i,2} = 3x_{i,1} - 2x_{i,0} - v_{i,0,x} \Delta t + \frac{(\Delta t)^2}{2} \left[ \sum_{j=1, j \neq i}^{n} \frac{m_j (x_{i,1} - x_{j,1})}{r_{ij,1} + \xi} \right] - \frac{G}{(r_{ij,1} + \xi)^2} \\
+ \frac{H}{(r_{ij,1} + \xi)^p} - \alpha \left( \frac{2}{\Delta t} \left[ x_{i,1} - x_{i,0} \right] - v_{i,0,x} \right)^2 \frac{1}{2}
\]

\[
+ \left( \frac{2}{\Delta t} \left[ y_{i,1} - y_{i,0} \right] - v_{i,0,y} \right)^2 \frac{1}{2} \right]
\]
(4.4) \[ y_{1,2} = 3y_{i,1} - 2y_{i,0} - v_{i,0,y} \Delta t + \frac{(\Delta t)^2}{2} \left( \sum_{j=1}^{n} \frac{m_j(y_{i,1} - y_{i,j,1})}{(r_{ij,1} + \xi)^2} \right) \left( -\frac{G}{(r_{ij,1} + \xi)^2} \right) + \left( \frac{H}{(r_{ij,1} + \xi)^p} - \alpha \left( \left[ \frac{2}{\Delta t} [x_{i,1} - x_{i,0}] - v_{i,0,x} \right]^2 \right) + \left[ \frac{2}{\Delta t} [y_{i,1} - y_{i,0}] - v_{i,0,y} \right]^2 \right) \]

and, for \( k \geq 2 \),

(4.5) \[ x_{1,k+1} = 3x_{i,k} + 2(-1)^k x_{1,0} - 4 \sum_{j=2}^{k} [(-1)^j x_{i,k-j+1} + (-1)^k v_{i,0,x} \Delta t] + \frac{(\Delta t)^2}{2} \left( \sum_{j=1}^{n} \frac{m_j(x_{i,k} - x_{i,j,k})}{(r_{ij,k} + \xi)^2} \right) \left( -\frac{G}{(r_{ij,k} + \xi)^2} + \frac{H}{(r_{ij,k} + \xi)^p} \right) - \alpha \left( \left[ \frac{2}{\Delta t} [x_{i,k} - (-1)^k x_{i,0} + 2 \sum_{j=1}^{k-1} (-1)^j x_{i,k-j} + (-1)^k v_{i,0,x}]^2 \right] + \left[ \frac{2}{\Delta t} [y_{i,k} - (-1)^k y_{i,0} + 2 \sum_{j=1}^{k-1} (-1)^j y_{i,k-j} + (-1)^k v_{i,0,y}]^2 \right] \right) }
\[ y_{i,k+1} = 3y_{i,k} + 2(-1)^{k}y_{i,0} - 4 \sum_{j=2}^{k}((-1)^{j}y_{i,k-j+1} + (-1)^{k}y_{i,0,y}^{\Delta t}} \\
+ \frac{(\Delta t)^{2}}{2} \sum_{j=1}^{n} \left\{ \frac{m_{j}(y_{i,k} - y_{j,k})}{(r_{ij,k} + \xi)^{2}} - \frac{G}{(r_{ij,k} + \xi)^{2}} + \frac{H}{(r_{ij,k} + \xi)^{2}} \right\} \\
- \alpha \left\{ \frac{2}{\Delta t} [x_{i,k}^{2} + (-1)^{k}x_{i,0} + 2 \sum_{j=1}^{k-1}((-1)^{j}x_{i,k-j}] + (-1)^{k}v_{i,0,x}^{\Delta t}} \\
+ \frac{2}{\Delta t} [y_{i,k}^{2} + (-1)^{k}y_{i,0} + 2 \sum_{j=1}^{k-1}((-1)^{j}y_{i,k-j}] + (-1)^{k}v_{i,0,y}^{2} \right\} \right\} \right\} \right\} \right\}. \]

That n-body initial value problems have unique solutions is now a consequence of the following theorem.

**Theorem 4.1** For given \( \xi > 0 \), \( \alpha \geq 0 \) and arbitrary \( x_{i,0}, y_{i,0}, v_{i,0,x} \) and \( v_{i,0,y}^{i} = 1,2,\ldots,n \), the n-body problem defined by (3.2)-(3.3) has a unique discrete solution which is given constructively by (4.1)-(4.6).

**Proof:** The proof is a direct consequence of the observation that the right-hand sides of (4.1)-(4.6) are always real, defined, and single valued.

It is important to note also that the recursive structure of (4.1)-(4.6) is what makes the solution particularly amenable to evaluation on a high speed digital computer.
5. FREE SURFACE FLUID FLOW

Free surface fluid problems have long been of wide interest in both pure and applied science (see, e.g., references [6,9,11-14,16-18] and the additional references contained therein), and liquids, in particular, have been modeled traditionally as being continuous and as consisting of an infinite number of point particles. The resulting nonlinear differential equations, even when linearized, can rarely be solved analytically.

In order to illustrate the methods and theory of Sections 1-4, we will model a particular free surface liquid flow problem as a discrete n-body problem. By Theorem 4.1, the dynamical equations of motion will be solved easily on a digital computer. Unfortunately, the present state of computer design does not as yet allow us to choose relatively large values of n. Nevertheless, it will still be of interest to note that, even though the example to be described consists of only 100 particles, still the flow does crudely approximate the actual flow constructed in our hydraulics laboratory and, indeed, a mechanism for the development of rotation is observed easily from the resulting numerical computations.

The problem to be considered, called the open gate problem, can be formulated physically as follows. The wall CD of square container ABCD, shown in Figure 5.1, is hinged at D and can be opened at will. The container is filled with liquid and then DC is opened quickly in such a fashion that C rotates on a circle with center D to the new position C', as shown in Figure 5.1. Thus, the gate has been opened. The problem is to describe the motion of the liquid as it flows out of the container.
In cgs units, consider then a liquid which is represented by 100 circular, homogeneous particles, $P_1, P_2, \ldots, P_{100}$, each of mass 8, which are arranged as shown in Figure 5.2. For simplicity, let the diameter $d$ of each particle be unity and let the $x$-axis contain the centers of the bottom row of particles while the $y$-axis contains the centers of the left column of particles. In this manner the mass centers $(x_{i,0}, y_{i,0})$ of $P_i$, $i = 1, 2, \ldots, 100$, are at nodal points of the form $(m_1 h, m_2 h)$, where $h = d = 1; m_1 = 0, 1, \ldots, 9$; and $m_2 = 0, 1, \ldots, 9$.

At time $t_k$ let the mass center of $P_i$ be $(x_{i,k}, y_{i,k})$. Then introducing the effect of gravity results in the following particular choices for (3.2) and (3.3):

\begin{equation}
(5.1) \quad a_{i,k+1, x} = 8 \sum_{j=1}^{100} \left\{ \frac{x_{i,k} - x_{j,k}}{(r_{ij,k} + .75)} \left[ \frac{H_{ij,k}}{(r_{ij,k} + .75)} - \alpha \left( v_{i,k,x}^2 + v_{i,k,y}^2 \right)^{1/2} \right] \right\}
\end{equation}

\begin{equation}
(5.2) \quad a_{i,k+1, y} = 8 \sum_{j=1}^{100} \left\{ \frac{y_{i,k} - y_{j,k}}{(r_{ij,k} + .75)} \left[ \frac{H_{ij,k}}{(r_{ij,k} + .75)} - \alpha \left( v_{i,k,x}^2 + v_{i,k,y}^2 \right)^{1/2} \right] \right\} - \frac{980 \beta_{ij}}{m}
\end{equation}
where the repulsion coefficient \( H_{ij,k} \) is defined by

\[
H_{ij,k} = \begin{cases} 
0, & \text{if } r_{ij,k} > 1 \\
1, & \text{if } r_{ij,k} \leq 1
\end{cases}
\]

and where the viscosity coefficient \( \alpha \) is defined by

\[
\alpha = \begin{cases} 
10^{-4}, & \text{if } y_{i,k} > 0 \\
10^{-3}, & \text{if } y_{i,k} = 0
\end{cases}
\]

The constant \( G \) has been set equal to zero because of the overwhelming effect of the attraction of each particle to the earth. The definition of \( \beta_{i} \) will require the following additional remarks.

At time \( t_{k} \), let the shadow of particle \( P_{i} \) be the rectangle whose vertices are \((x_{i,k} + \frac{1}{2}, 0), (x_{i,k} + \frac{1}{2}, y_{i,k}), (x_{i,k} - \frac{1}{2}, y_{i,k}), (x_{i,k} - \frac{1}{2}, 0)\), as shown in Figure 5.3. If there are no particles in the shadow of \( P_{i} \), we would wish to choose \( \beta_{i} = 1 \) and thus have the full force of gravity acting upon \( P_{i} \). On the other hand, if the shadow area of \( P_{i} \) contains portions of other particles, we would wish to choose \( \beta_{i} \) different from unity to reflect the amount of support which \( P_{i} \) has below it. The measure \( \beta_{i} \) is then defined as follows. One lists all particles \( P_{j} \) different from \( P_{i} \) whose mass centers satisfy both

\[
x_{j,k} - 1 < x_{i,k} < x_{i,k} + 1
\]

\[
y_{j,k} < y_{i,k}
\]

and defines

\[
\sigma = \sum_{j} \left[ 1 - \left| x_{i,k} - x_{j,k} \right| \right].
\]
Note that if no particle $P_j$ exists whose center satisfies (5.5) and (5.6), then $\sigma = 0$. Finally, $\beta_1$ is defined by

$$\beta_1 = \begin{cases} 
0, & \text{if } y_{i,k} = 0 \\
\left(1 - \frac{\sigma}{y_{i,k}}\right)^2, & \text{if } 0 < \frac{\sigma}{y_{i,k}} < 1 \\
(.00005) \left(1 - \frac{1+\sigma}{1+y_{i,k}}\right), & \text{if } \frac{\sigma}{y_{i,k}} \geq 1.
\end{cases} \tag{5.7}$$

FIGURE 5.3
The calculations which result after replacing (5.1) and (5.2) by recursion formulas of the types (4.1)-(4.6) are then carried out as follows. Initial velocities \( v_{1,0,x} \) and \( v_{1,0,y} \) which result from the opening of the gates are given by

\[
(5.5) \quad v_{1,0,y} = 0
\]

\[
(5.6) \quad v_{1,0,x} = (2.7) \left( 9 - y_{1,0} \right) \left( \frac{27}{40} + \frac{x_{1,0}}{40} \right)^{14-x}
\]

Formula (5.6) is only one of several which reflect the assumption that \( v_{1,0,x} \) increases with depth and with horizontal distance from the left wall when the gate is opened. If at any time \( x_{1,k} \) satisfies \( x_{1,k} \leq \frac{1}{2} \), then the associated particle is reset immediately to \( (\frac{1}{2}, y_{1,k}) \) and its velocity is reset as \( (0, v_{1,k,y}) \).

If at any time any \( y_{1,k} \) satisfies \( y_{1,k} < 0 \), then the associated particle is reset immediately to \( (x_{1,k}, 0) \) and its velocity is reset at \( (0, -0.0025v_{1,k,y}) \), which imparts some bounce to the particle.

Calculations on the UNIVAC 1108 for \( \Delta t = .01 \) and in which the left wall is kept wet, that is, no particle with \( x_{1,0} = 0 \) is allowed to move, are shown typically for time steps \( t_{25} \), \( t_{60} \), \( t_{100} \), \( t_{175} \), \( t_{250} \) and \( t_{350} \) in Figures 5.4 - 5.9, respectively. Approximations to the actual surfaces at each time step are shown in Figures 5.10-5.15, in which smooth curves have been drawn freely above the particle configurations. In Figure 5.16 is shown an actual three dimensional cavity with a string attachment for opening a left gate. In Figures 5.17 and 5.18 are shown the actual patterns as water flows to the left during and after the opening of the gate.
These figures to be inserted here.

200 copies

comes from print lab

pages 17, 19, 20 17-20.21
Note that Figures 5.12 and 5.17 compare favorably while Figures 5.15 and 5.18 both exhibit the development of a downstream bore. Also, from the very large number of other examples run, it should be noted that the computations were very sensitive to the choices of (5.4), (5.6) and magnitude of bounce allowed.

In Figures 5.19 and 5.20 are shown the positions of the particles of the 5th row at the respective time steps \( t_{50} \) and \( t_{250} \). In Figure 5.21 are shown the positions of the particles of the 4th column at time step \( t_{125} \). Note here that certain particles originally above other particles have fallen below them due to less dense shadow regions and a greater resultant force of gravity. The effect shown in Figure 5.21 is, interestingly enough, one of rotation.

Finally, because of the experimental nature of our work, we find it necessary to include our Fortran program in an Appendix, so that others can reproduce our work in its entirety.
REFERENCES


C SET UP PARAMETERS
NP=100
RR=0.75
NPP=90
CC=2.7
AM=8.0
H=1.0
A1=0.0001
A2=0.001
A3=0.0025
A4=0.0005
KL=0
LQ=0
DT=0.1
NN=50
NZP=3
LW=0
KS=1
NZ=0
Z=1
NS=1
II=0

C CALCULATE INITIAL VELOCITY
DO 33 I=1,10
DO 33 J=1,9
II=II+1
Y1(II)=I-1
VOY(II)=0
X1(II)=J
P1=14*J
R1=((27.0/40.0)+(J/40.0))*P1
33
VOX(II)=((10.0-I)*R1)*CC
DO 334 I=1,10
II=II+1
Y1(II)=I-1
334
X1(II)=0.0
30 FORMAT(9E8.1,9E8.1,9E8.1)
DO 443 K=1,NN
DO 443 I=91,100
X(I,K)=X1(I)
443
Y(I,K)=Y1(I)
DO 71 I=1,100
D(I)=0
71
ZX(I)=1
ZY(I)=1.0
Q(I)=0
F(I)=0
71
E(I)=0
C PRINT OUT LIST OF PARAMETERS
PRINT 80,AM,RR,DT,A1,A2
PRINT 81,CC,RR,A1,A2
PRINT 976,A3
976
FORMAT(1X,'A3=','E10.4')
81
FORMAT('CC=','E10.4','R=','E10.4','A1=','E10.4','A2=','E10.4')
80
FORMAT(1HL,'AM=','E10.4','H=','E10.4','DT=','E10.4','A4=','E10.4')
72
FORMAT(100('E10.4,15X,E10.4'))
PRINT 73
73
FORMAT(5X,10('X,Y'))
C PRINT OUT INITIAL POSITIONS
IZ=101
DO 120 L1=1,10
IZ=IZ-1
I=I+9*(L1-1)
IN=91-I7
IB=IN-8
120 PRINT 74,LQ,X1(IZ),Y1(IZ),(X1(I),Y1(I),I=IB,IN)
C PRINT 32
FORMAT(1X,/)  
74 FORMAT(I5,2,2,F5.2,2,1X)
C RUN EACH CASE FOR NN*NZP TIME STEPS
DO 1 IZP=1,NZP
DO 91 KK=KS,NN
K=KK
IF(KK*EQ. 1) GO TO 829
GO TO 831
829 DO 830 KQ=1,NP
X(KQ,K)=X1(KQ)
830 Y(KQ,K)=Y1(KQ)
DO 50 I=1,NP
DO 50 J=1,NP
C FIND DISTANCE BETWEEN PARTICLES
C FOR FIRST TIME STEP
50 R(I,J)=SQRT((X1(I)-X1(J))**2)+(Y1(I)-Y1(J))**2)) +RR
831 DO 91 I=1,NPP
DO 811 KP=1,2
811 KTIM(I,KP)=KTIM(I,KP)+1
ZX(I)=ZX(I)
ZY(I)=ZY(I)
K=KTIM(I,1)
IF(K-2)6,7,7
C SET X VELOCITY TO INITIAL VELOCITY
6 VX(I)=VOX(I)
7 K=KTIM(I,2)
IF(K-2)806,807,807
C SET Y VELOCITY TO INITIAL VELOCITY
806 VY(I)=VOY(I)
807 IF(KK*EQ. 1) GO TO 10
K=KK
K=K-1
C FIND DISTANCE BETWEEN PARTICLES
C FOR TIME STEPS GREATER THAN ONE
DO 54 J=1,NP
54 R(I,J)=SQRT( ((X1(I,K)-X(J,K))**2)+((Y1(I,K)-Y(J,K))**2)) +RR
K=KTIM(I,1)
IF(K=2)412,17,412
17 CONTINUE
K=KK-1
C CALCULATE X VELOCITY FOR SECOND TIME STEP
410 VX(I)=(2.*DT)*(X1(I,K)-X1(I))=VOX(I)
412 K=KTIM(I,2)
IF(K=2)53,814,53
814 K=KK-1
C CALCULATE Y VELOCITY FOR SECOND TIME STEP
510 VY(I)=(2.*DT)*(Y1(I,K)-Y1(I))=VOY(I)
53 CONTINUE
K=KTIM(I,1)
IF(K=2)816,816,815
815 K=KK-1
C CALCULATE X VELOCITY FOR TIME STEPS GREATER THAN 2
110 \[ D(I)=D(I)-X(I,K-1) \]
\[ Z=Z(I) \]
\[ VX(I)=Z+(2*\pi/DT)*X(I,K)-Z*X1(I)+Z*D(I)-Z*VOX(I) \]
816 \[ K=KTIM(I+2) \]
817 \[ IF(K=2) I=Z*817 \]
817 \[ K=KK-1 \]
C \[ \text{CALCULATE Y VELOCITY FOR TIME STEPS GREATER THAN 2} \]
112 \[ Q(I)=-Q(I)-Y(I,K-1) \]
\[ Z=ZY(I) \]
\[ VY(I)=(2*\pi/DT)*(Y(I,K)-Z*Y1(I)+Z*Q(I))-Z*VOY(I) \]
2 \[ CONTINUE \]
10 \[ CONTINUE \]
77 \[ CVX=ABS(VX(I)) \]
77 \[ CVY=ABS(VY(I)) \]
77 \[ IF(CVX<10000.)77,869,869 \]
77 \[ IF(CVY<10000.)3,869,869 \]
3 \[ A(I)=SQRT(((VX(I)**2)+((VY(I)**2))) \]
C \[ \text{DETERMINE DAMPING COEFFICIENT} \]
4019 \[ IF(KK EQ. 1) GO TO 4019 \]
4019 \[ KK=KK-1 \]
4020 \[ GO TO 4020 \]
618 \[ AA=A2 \]
618 \[ GO TO 619 \]
617 \[ AA=A1 \]
619 \[ CONTINUE \]
619 \[ D1=0 \]
619 \[ D2=0 \]
619 \[ DO 24 I1=1,NP \]
619 \[ IF(I1 EQ. 1) GO TO 24 \]
C \[ \text{DETERMINE REPULSION COEFFICIENT} \]
917 \[ IF(R(I1,K1) GE. 1.) GO TO 917 \]
917 \[ H=1. \]
917 \[ GO TO 918 \]
918 \[ H=.0 \]
918 \[ CONTINUE \]
C \[ \text{FIND ACCELERATION OF PARTICLES} \]
24 \[ D2=D2+((Y(I1,K)-Y(I1,K))/R(I1,K)+H/((R(I1,K)**2)-AA*A(I)) \]
24 \[ D1=D1+((X(I1,K)-X(I1,K))/R(I1,K)+H/((R(I1,K)**2)-AA*A(I)) \]
24 \[ CONTINUE \]
C \[ \text{DETERMINE BETA I} \]
600 \[ IF(Y(I1,K)=.0/.600,211,600 \]
600 \[ CONTINUE \]
600 \[ DO 200 I2=1,NP \]
600 \[ IF(Y(I1,K)-Y(I2,K))200,200,202 \]
202 \[ P=ABS(X(I1,K)-X(I2,K)) \]
202 \[ IF(P .GE. 1.) GO TO 200 \]
200 \[ P=P**P \]
200 \[ QQ=QQ+P \]
200 \[ CONTINUE \]
200 \[ DP=QQ/Y(I1,K) \]
202 \[ IF(DP=1.7212+.3012+.3012 \]
212 \[ BI=1.-DP \]
212 \[ BI=BI*BI \]
212 \[ GO TO 213 \]
3012 \[ BI=A4*(1.-QQ1/(Y(I1,K)+1)) \]
3012 \[ GO TO 213 \]
211 \[ BI=0. \]
C FIND X POSITION FOR THIRD TIME STEP
610 E(I)=E(I)+X(I*K-2)
   Z=ZX(I)
   X(I*K)=3*Y(I,K-1)-2*Z*X1(I)-4*E(I)-Z*DT*VOX(I)+(DT*DT/2)*AY(I)
C IF X LESS THAN .5 RESET X VALUE
   IF(X(I*K)<.5)66+66+91+91
66 X(I*K)=.5
   VOX(I)=.0
   X1(I)=.5
   E(I)=.0
   D(I)=.0
   ZX(I)=NS
   KTIM(I+1)=0
   GO TO 91
822 K=KK
   Z=ZY(I)
C FIND Y POSITION FOR THIRD TIME STEP
612 F(I)=-F(I)+Y(I,K-2)
   Y(I,K)=3*Y(I,K-1)-2*Z*Y1(I)-4*F(I)+Z*DT*VOY(I)+(DT*DT/2)*AY(I)
C IF Y LESS THAN 0 RESET Y VALUE
   IF(Y(I,K)<.0)68+68+91+91
68 Y(I,K)=.0
   Y1(I)=.0
   F(I)=.0
   Q(I)=.0
   VOY(I)=-A3*VY(I)
   KTIM(I+2)=0
   ZY(I)=NS
65 CONTINUE
91 CONTINUE
   GO TO 969
869 CONTINUE
     PRINT 870,K
870 FORMAT(20X,***I3)
969 CONTINUE
C PRINT RESULTS
   DO 288 K=K5,NN
   IZ=101
   LK=K+LW
   PRINT 32
   DO 28 L1=1,10
   IZ=IZ-1
   IT=I+9*(L1-1)
   IN=91-IT
   IB=IN-8
   28 PRINT 74,LK,X1(IZ),Y1(IZ),(X(I,K),Y(I,K),I=18,IN)
   CONTINUE
   IF(NI3P EQ. 1) GO TO 1
288 CONTINUE
   DO 29 LM=1+NPP
   X(LM+2)=X(LM,NN)
   Y(LM+2)=Y(LM,NN)
   X(LM+1)=X(LM,NN-1)
   Y(LM+1)=Y(LM,NN-1)
   KS=3
   LW=NN-2+LW
   CONTINUE
   END
CONTINUE
AX(I) = AM*D1
AY(I) = AM*D2 = (980.*BI/AM)
CONTINUE
DO 91 KQ = 1, 2
K = KT (I*KQ)
IF(K = 2) Z = 13, 14
12 IF(KQ .EQ. 2) GO TO 820
K = KK
C FIND X POSITION FOR FIRST TIME STEP
X(I*K) = X(I) + DT*VOX(I) + ((DT*DT)/2.)*AX(I)
C IF X LESS THAN .5 RESET X VALUE
26 IF(X(I*K) < .5) 26, 91, 91
X(I*K) = .5
VOX(I) = .0
X1(I) = .5
D(I) = .0
ZX(I) = NS
KT (I*1) = 0
GO TO 91
820 K = KK
C FIND Y POSITION FOR FIRST TIME STEP
Y(I*K) = Y(I) + DT*VOY(I) + ((DT*DT)/2.)*AY(I)
C IF Y LESS THAN 0 RESET Y VALUE
27 IF(Y(I*K) < 0) 75, 91, 91
75 Y(I*K) = .0
Y1(I) = .0
Q(I) = .0
VOY(I) = -A3*VY(I)
ZY(I) = NS
KT (I*2) = 0
GO TO 91
13 IF(KQ .EQ. 2) GO TO 821
K = KK
L = K = 1
C FIND X POSITION FOR SECOND TIME STEP
X(I*K) = 3.*X(I*,L) - 2.*X1(I) - DT*VOX(I) + (DT*DT/2.)*AX(I)
C IF X LESS THAN .5 RESET X VALUE
61 IF(X(I*K) < .5) 61, 91, 91
X(I*K) = .5
X1(I) = .5
D(I) = .0
VOX(I) = .0
ZX(I) = NS
KT (I*1) = 0
GO TO 91
821 K = KK
L = K = 1
C FIND Y POSITION FOR SECOND TIME STEP
Y(I*K) = 3.*Y(I*,L) - 2.*Y1(I) - DT*VOY(I) + (DT*DT/2.)*AY(I)
C IF Y LESS THAN 0 RESET Y VALUE
62 IF(Y(I*K) < 0) 63, 91, 91
63 Y(I*K) = .0
Y1(I) = .0
Q(I) = .0
VOY(I) = -A3*VY(I)
ZY(I) = NS
KT (I*2) = 0
GO TO 91
14 IF(KQ .EQ. 2) GO TO 822
K = KK