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MONOTONE SPLITTING OF MATRICES

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ABSTRACT

Given the iterative scheme $x^{i+1} = BT x^i + r$ where B, T are fixed $n \times n$ real matrices, r a fixed real n-vector and x^i a real n-vector we investigate the convergence and monotonicity of schemes of the type

$$\begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} S_{11} & -S_{12} \\ -S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} v^i \\ w^i \end{bmatrix} + \begin{bmatrix} r \\ r \end{bmatrix}$$

where S_{ij} are $n \times n$ real matrices related to T. The n-vector iterates v^i, w^i bracket in a certain sense solutions x of $x = BTx+r$. We also give necessary and sufficient conditions for the monotonicity of the original iterative scheme itself $x^{i+1} = BTx^i + r$. This leads to monotonicity results for classical iterative schemes such as the Jacobi, Gauss-Seidel and successive overrelaxation methods.

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1. INTRODUCTION

Many classical iterative schemes for finding an n-vector x that satisfies the system

1.1 $$Ax = b$$

where A and b are a given $n \times n$ real matrix and an n-vector respectively, consist of splitting the matrix A into the difference of two real $n \times n$ matrices, that is $A = M - N$, and using the iteration

1.2 $$x^{i+1} = M^{-1} N x^{i} + M^{-1} b.$$

The Jacobi, Gauss-Seidel and successive overrelaxation methods [3, Chap. 3] fall into this cateogry. If we make the identifications $B = M^{-1}$, $T = N$, $r = M^{-1} b$ we obtain the iterative scheme

1.3 $$x^{i+1} = BTx^{i} + r$$

which is the subject of this work.

With the iterative scheme 1.3 we shall associate the scheme

1.4 $$\begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} S_{11} & -S_{12} \\ -S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} v^{i} \\ w^{i} \end{bmatrix} + \begin{bmatrix} r \\ r \end{bmatrix}$$

where S_{ij} are some $n \times n$ real matrices related to T, and the n-vector iterates v^i, w^i will bracket, in a certain sense, solutions x of $x = BTx + r$. In particular we will show when the scheme 1.4 is monotonic with respect to a cone which is dual to the rows of B^{-1}, and when the iterates of 1.4 converge to a solution of $x = BTx+r$. (Theorem 2.1). We will also give some additional sufficient and necessary conditions for monotonicity of the iterates of 1.4 (Theorem 2.7). Finally we give monotonicity results for the original iteration 1.3 itself (Theorem 3.3). The conditions for the monotonicity of 1.3 in that $TB \geq 0$ and that spectral radius $\rho(BT) \leqq 1$. This immediately shows that such classical schemes as the Jacobi, Gauss-Seidel and successive overrelaxation are indeed monotonic under the standard assumptions of $T \geqq 0$, $B \geqq 0$ and $\rho(TB) < 1$ provided that we start with appropriate starting vectors. (Theorem 3.4).

The monotonicity results for 1.4 are generalizations of the results of Collatz-Schroder [1] and Tal [2] where $B = I$, $S_{11} = S_{22}$, $S_{12} = S_{21}$ and $T= S_{11} - S_{12}$.

2. MONOTONICITY OF 1.4

In this section we give various conditions related to the monotonicity of the proposed scheme 1.4.

2.1 Montonicity Theorem Let $S_{ij}, B \geqq 0$, $i,j = 1,2$, and let $S_{11} - S_{12} = S_{22} - S_{21}$.

i) If there exist v^0, w^0 such that

2.2 $B^{-1}(v^1 - v^0) \geqq 0,\ B^{-1}(-w^1 + w^0) \geqq 0,\ B^{-1}(w^0 - v^0) \geqq 0$

where v^1, w^1 are computed from 1.4 then

2.3 $B^{-1}(v^{i+1} - v^i) \geqq 0,\ B^{-1}(-w^{i+1} + w^i) \geqq 0,\ B^{-1}(w^i - v^i) \geqq 0,$

$i = 0,1,\ldots$ or equivalently

2.4 $B^{-1} v^0 \leqq B^{-1} v^1 \leqq \ldots\ldots \leqq B^{-1} v^i \leqq \ldots\ldots \leqq B^{-1} w^i \leqq$

$\ldots\ldots \leqq B^{-1} w^1 \leqq B^{-1} w^0.$

ii) If in addition to the assumption 2.2 of i)

$T = S_{11} - S_{12} = S_{22} - S_{21}$, then the system $x = BTx + r$ has a solution x such that for all i

2.5 $B^{-1} v^i \leqq B^{-1} x \leqq B^{-1} w^i$

and $x = \lim_{i \to \infty} \frac{v^i + w^i}{2}$ if

a) $S_{11} = S_{22}$ and $S_{12} = S_{21}$, or

b) 1 is not an eigenvalue of $B(S_{11} + S_{21}) = B(S_{12} + S_{22})$, or

c) There exists a real n-vector z such that

$$z(BS_{11} + BS_{21} - I) > 0$$

In cases a) and b) we have in addition that $x = \lim_{i \to \infty} v^i$

$= \lim_{i \to \infty} w^i$

Proof i) We will prove this part by induction on i. By assumption the inequalities hold for i = 0. Assume now that they hold for i and show that they hold for i + 1.

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+2} & -v^{i+1} \\ -w^{i+2} & +w^{i+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} v^{i+1} \\ -w^{i+1} \end{bmatrix} + \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} r - v^{i+1} \\ -r + w^{i+1} \end{bmatrix} \quad \text{(by 1.4)}$$

$$\geqq \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} v^{i} \\ -w^{i} \end{bmatrix} + \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} r - v^{i+1} \\ -r + w^{i+1} \end{bmatrix}$$

$$= 0 \quad \text{(by 1.4)}$$

where the inequality above follows by premultiplying the induction hypothesis inequality

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} v^{i+1} & - v^{i} \\ -w^{i+1} & + w^{i} \end{bmatrix} \geqq 0$$

by

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \geqq 0.$$

Hence

$$B^{-1} (v^{i+2} - v^{i+1}) \geqq 0 \text{ and } B^{-1} (-w^{i+2} + w^{i+1}) \geqq 0.$$

We also have that

$$B^{-1}(v^{i+1} - w^{i+1}) = (S_{11} v^i - S_{12} w^i) + (S_{21} v^i - S_{22} w^i) \qquad \text{by (1.4)}$$

$$= (S_{11} + S_{21})(v^i - w^i) \qquad (\text{since } S_{11} + S_{21} = S_{12} + S_{22})$$

$$= (S_{11}B + S_{21}B)\, B^{-1}(v^i - w^i)$$

$$\leqq 0 \qquad (\text{by induction hypothesis } B^{-1}(w^i - v^i) \geqq 0 \text{ and } S_{11}B + S_{21}B \geqq 0)$$

Hence the inequalities of 2.3 hold for $i + 1$ and the induction is complete.

ii) Because of 2.4 the bounded monotone sequences of real numbers $\{B_k^{-1} v^i\}$ and $\{B_k^{-1} w^i\}$, where $B_k^{-1} v^i$ denotes the k^{th} component of n-vector $B^{-1}v^i$, have limits a_k and b_k, $k = 1, \ldots, n$. Hence the vector sequences $\{B^{-1}v^i\}$ and $\{B^{-1}w^i\}$ have limits a and b. From the continuity of the linear operator B we also have that the vector sequences $\{v^i\}$ and $\{w^i\}$ converge to $v = Ba$ and $w = Bb$.

Hence by 1.4

$$v = BS_{11}v - BS_{12}w + r$$

$$w = -BS_{21}v + BS_{22}w + r$$

a) Define $x = \frac{v + w}{2}$. Then

$$x = \frac{v+w}{2} = B\, S_{11}\frac{v+w}{2} - BS_{12}\frac{v+w}{2} + r = BT\frac{v+w}{2} + r = BTx + r.$$

Hence $x = \frac{v+w}{2}$ is a solution of $x = BTx + r$.

b) $v - w = B(S_{11} + S_{21})v - B(S_{12} + S_{22})w = B(S_{11} + S_{21})(v - w)$.

But since 1 is not an eigenvalue of $B(S_{11} + S_{21})$, $v - w = 0$ and so $v = B(S_{11} - S_{12})v + r = BTv + r$. Hence $x = v = w$ is a solution of $x = BTx + r$.

c) If such z exists then $(v - w) = B(S_{11} + S_{21})(v - w)$ implies that $v - w = 0$, otherwise we have the contradiction

$$0 = z(BS_{11} + BS_{21} - I)(v - w) < 0$$

where the equality follows from $(BS_{11} + BS_{21} - I)(v - w) = 0$ and the inequality follows from $z(BS_{11} + BS_{21} - I) > 0$, $v - w \leqq 0$ and $v - w \neq 0$. So $v - w = 0$, $v = B(S_{11} - S_{12})v + r = BTv + r$, and $x = v = w$ is a solution of $x = BTx + r$.

In all cases a) b) and c) we have $x = \frac{v+w}{2} = \frac{1}{2} \lim_{i \to \infty} v^i + w^i$ and hence 2.5 follows from 2.4. This completes the proof.

The above theorem is a generalization of results of Collatz-Schroder [1, pp. 352-353, 361-362] where $B = I$, $S_{11} = S_{22}$ $S_{12} = S_{21}$ and $T = S_{11} - S_{12}$.

2.6 Remark Condition b) of Theorem 2.1 above is implied by the condition that 1 is not an eigenvalue of $\begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix}$.

Proof: It is sufficient to show that if λ is an eigenvalue of $B(S_{11} + S_{21})$ then it is also an eigenvalue of $\begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix}$. Let λ be an eigenvalue of $B(S_{11} + S_{21})$ and let y be the corresponding eigenvector of $(BS_{11} + BS_{21})^T$. Then

$$[y \quad y]\begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix} = [y(BS_{11} + BS_{21}) \qquad y(BS_{12} + BS_{22})] = \lambda[y \quad y]$$

and hence λ is an eigenvalue of $\begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix}$.

We observe that unlike the iteration 1.3, which under appropriate assumptions converges from any initial x^0, the iteration 1.4 must start from initial v^0, w^0 satisfying 2.2. Hence unless we can guarantee the existence of such v^0, w^0, the iteration 1.4 may be vacuous as far as producing monotonic v^i, w^i, that is v^i, w^i satisfying 2.4. The following theorem which is a generalization of a theorem of Tal [2] gives sufficient and necessary conditions for the existence of v^0, w^0 satisfying 2.2.

2.7 Theorem (Sufficient and Necessary Conditions) i) Let $S_{ij}B \geqq 0$, $i,j = 1,2$, let $T = S_{11} - S_{12} = S_{22} - S_{21}$, and let $x = BTx + r$ have a solution x. If either

a) $S_{11} = S_{22}$, $S_{12} = S_{21}$, $\rho(BS_{11} + BS_{21}) \leq 1$, or

b) $\rho \begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix} < 1$

where $\rho(A)$ denotes the spectral radius of A, then there exist v^0, w^0 satisfying 2.2, and the iterative scheme 1.4 is montonic, that is 2.4 and 2.5 hold, and $x = \lim_{i \to \infty} \frac{v^i + w^i}{2}$.

ii) If Theorem 2.1 holds and $(S_{11} + S_{21})B$ is irreducible,[1] then either $v^0 = w^0 = x$ is a solution of $x = BTx + r$, or $v^0 \neq w^0$ and

$$0 < \rho(BS_{11} + BS_{21}) = \rho(BS_{12} + BS_{22}) \leq 1$$

Remark Condition b) above, $\rho \begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix} < 1$, implies that

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1) An $n \times n$ irreducible matrix A is matrix such that there exist no $n \times n$ permutation matrix P such that $PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{11} is an $m \times m$ submatrix, A_{22} is an $(n - m) \times (n - m)$ submatrix, and $1 \leq m < n$. A permutation matrix is a square matrix which contains exactly one element which is 1 in each column and row and all other elements are zero.

$\rho(BS_{11} + BS_{21}) = \rho(BS_{12} + BS_{22}) < 1$. This follows from the fact, established under Remark 2.6, that each eigenvalue of $(BS_{11} + BS_{21})$ is also an eigenvalue of $\begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix}$.

Proof We observe first that under the assumption that x is a solution $x = BTx + r$, then the iterative scheme 1.4 is equivalent to

$$2.8 \quad \begin{bmatrix} x - v^{i+1} \\ -x + w^{i+1} \end{bmatrix} = \begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix} \begin{bmatrix} x - v^{i} \\ -x + w^{i} \end{bmatrix} \quad , \; i = 0, 1, \ldots\ldots$$

and that condition 2.2 is equivalent to

$$2.9 \quad \begin{bmatrix} B^{-1}(x-v^{1}) \\ B^{-1}(-x+w^{1}) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x-v^{0} \\ -x+w^{0} \end{bmatrix} \leqq \begin{bmatrix} B^{-1}(x-v^{0}) \\ B^{-1}(-x+w^{0}) \end{bmatrix} \text{ and } \begin{bmatrix} B^{-1}(x-v^{0}) \\ B^{-1}(-x+w^{0}) \end{bmatrix} \geqq 0$$

We also observe that the eigenvalues of $B(S_{11} + S_{21})$ and $(S_{11} + S_{21})B$ are equal because of the similarity transformation $B^{-1}B(S_{11} + S_{21})B = (S_{11} + S_{21})B$.

ia) By the Frobenius theorem [3, p. 46], [4, p. 32] there exist an eigenvector $z^0 \geqq 0$, $z^0 \neq 0$ of $S_{11}B + S_{21}B \geqq 0$ corresponding to a real nonnegative eigenvalue $\bar{\rho} = \rho(S_{11}B + S_{21}B) = \rho(BS_{11} + BS_{21}) \leqq 1$. That is

$$(S_{11}B + S_{12}B)z^0 = (S_{21}B + S_{22}B)z^0 = \bar{\rho}z^0 \leqq z^0$$

where the last inequality holds because $\bar{\rho} \leqq 1$ and $z^0 \geqq 0$. Set now

$$v^0 = x - Bz^0, \quad w^0 = x + Bz^0$$

Hence

$$\begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x + w^0) \end{bmatrix} = \begin{bmatrix} z^0 \\ z^0 \end{bmatrix} \geqq 0$$

and

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x - v^0 \\ -x + w^0 \end{bmatrix} = \begin{bmatrix} S_{11}B & S_{12}B \\ S_{21}B & S_{22}B \end{bmatrix} \begin{bmatrix} z^0 \\ z^0 \end{bmatrix} = \begin{bmatrix} \bar{\rho} z^0 \\ \bar{\rho} z^0 \end{bmatrix} \leqq \begin{bmatrix} z^0 \\ z^0 \end{bmatrix} = \begin{bmatrix} B^{-1}(x-v^0) \\ B^{-1}(-x+w^0) \end{bmatrix}.$$

Hence 2.9 holds which implies 2.2.

i b) By the Frobenius theorem [3, p. 46] there exist an eignevector $(y^1 \; y^2) \geqq 0$, $(y^1 \; y^2) \neq 0$ of $\begin{bmatrix} S_{11}B & S_{12}B \\ S_{21}B & S_{22}B \end{bmatrix} \geqq 0$ corresponding to a real nonnegative eigenvalue $\tilde{\rho}$

$$\tilde{\rho} = \rho \begin{bmatrix} S_{11}B & S_{12}B \\ S_{21}B & S_{22}B \end{bmatrix} = \rho \begin{bmatrix} BS_{11} & BS_{12} \\ BS_{21} & BS_{22} \end{bmatrix} < 1.$$

That is

$$\begin{bmatrix} S_{11}B & S_{12}B \\ S_{21}B & S_{22}B \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \tilde{\rho} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \leqq \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$$

where the last inequality holds because $\tilde{\rho} < 1$ and $(y^1 \; y^2) \geqq 0$.

Set now

$$v^0 = x - By^1, \quad w^0 = x + By^2$$

Hence

$$\begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x+w^0) \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \geqq 0$$

and

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x - v^0 \\ -x + w^0 \end{bmatrix} = \begin{bmatrix} S_{11}B & S_{12}B \\ S_{21}B & S_{22}B \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \tilde{\rho} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \leqq \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} B^{-1}(x-v^0) \\ B^{-1}(-x+w^0) \end{bmatrix}$$

Hence 2.9 holds which implies 2.2.

The rest of part i) follows by making use of Theorem 2.1 and noting that the assumptions of part ia) of this theorem imply the assumption of part iia) of Theorem 2.1, and that the assumptions of part ib) of this theorem and Remark 2.6 imply the assumptions of part iib) of theorem 2.1.

ii) If $v^0 - w^0 = 0$, then $v^{i+1} - w^{i+1} = B(S_{11} - S_{12})v^i + B(S_{21} - S_{22})v^i = 0$ for $i = 0,1,\ldots,$ and by 2.4 $v^i = w^i = v^0 = w^0$ for $i = 0,1,\ldots$ and by Theorem 2.1 $x = v^0 = w^0$ is a solution of $x = BTx + r$.

Assume now that $v^0 \neq w^0$, then $B^{-1}(w^0 - v^0) \geqq 0$ and $B^{-1}(w^0 - v^0) \neq 0$, because B^{-1} is nonsingular. By Theorem 2.1 then there exists a solution x such that $B^{-1}v^0 \leqq B^{-1}x \leqq B^{-1}w^0$. Hence

2.10 $$\begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x + w^0) \end{bmatrix} \geqq 0, \quad \begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x + w^0) \end{bmatrix} \neq 0.$$

for if equality held above then $B^{-1}(w^0 - v^0) = 0$, contrary to our assumption. We also observe that condition 2.2 is equivalent to 2.9. Hence

$$\begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x+w^0) \end{bmatrix} \geqq \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{21} \end{bmatrix} \begin{bmatrix} x - v^0 \\ -x+ w^0 \end{bmatrix} \tag{2.11}$$

Let $\bar{\rho} = \rho(S_{11}B + S_{21}B)$. Since $(S_{11}B + S_{21}B)$ is nonnegative and irreducible it follows by the Frobenius theorem [3, p. 30] that $\bar{\rho} > 0$ and that $\bar{\rho}$ is an eigenvalue of $(S_{11}B + S_{21}B)^T$ with a corresponding eigenvector $y > 0$. Hence

$$y(S_{11}B + S_{21}B) = \bar{\rho}\, y$$

and

$$[y \quad y] \begin{bmatrix} S_{11}B & S_{12}B \\ S_{21}B & S_{22}B \end{bmatrix} = [\bar{\rho} y \quad \bar{\rho} y] = \bar{\rho}[y \quad y]$$

So

$$\begin{aligned} \bar{\rho}[y \quad y] \begin{bmatrix} B^{-1}(x-v^0) \\ B^{-1}(-x+w^0) \end{bmatrix} &= [y \quad y] \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x - v^0 \\ -x + w^0 \end{bmatrix} \\ &\leqq [y \quad y] \begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x + w^0) \end{bmatrix} \qquad \text{(by 2.11)} \end{aligned} \tag{2.12}$$

From 2.10 and $y > 0$ it follows that

$$[y \quad y] \begin{bmatrix} B^{-1}(x - v^0) \\ B^{-1}(-x + w^0) \end{bmatrix} > 0$$

and hence from 2.12 we have that $\bar{\rho} \leqq 1$, and

$$0 < \bar{\rho} = \rho(BS_{11} + BS_{21}) = \rho(BS_{12} + BS_{22}) \leqq 1 .$$

This completes the proof.

3. Monotonicity of $x^{i+1} = BT x^i + r$

We can obtain monotonicity results for the scheme 1.3 itself if we observe that if we set $S_{12} = S_{21} = 0$ and $S_{11} = S_{22} = T$ in the iteration 1.4, then 1.4 uncouples into the iterative schemes

3.1 $$v^{i+1} = BT v^i + r, \quad i = 0,1,\ldots$$

3.2 $$w^{i+1} = BTw^i + r, \quad i = 0,1,\ldots$$

which are identical to 1.3 except that they are for the iterates v^i and w^i instead of x^i. We can use the methods of proof and the results of the previous section to obtain monotonicity results for 3.1 and 3.2. In addition we can sharpen these results a bit because of the uncoupling achieved above. We collect all these results in the following theorem.

3.3 Monotonicity Theorem Let $TB \geqq 0$ and let B^{-1} exist.

i) If there exists a v^0 such that

$$B^{-1}(v^1 - v^0) \geqq 0$$

where v^1 is computed from 3.1, then

$$B^{-1}(v^{i+1} - v^i) \geqq 0, \; i = 0,1,\ldots$$

ii) If there exists a w^0 such that

$$B^{-1}(w^0 - w^1) \geqq 0$$

where w^1 is computed from 3.2, then

$$B^{-1}(w^i - w^{i+1}) \geqq 0, \; i = 0,1,\ldots$$

iii) If there exists v^0, w^0 such that

$$B^{-1}(w^0 - v^0) \geqq 0$$

then

$$B^{-1}(w^i - v^i) \geqq 0, \; i = 0, 1, \ldots$$

iv) If the assumptions of i) to iii) hold then

$$B^{-1}v^0 \leqq B^{-1}v^1 \leqq \ldots \leqq B^{-1}v^i \leqq \ldots \leqq B^{-1}w^i \leqq \ldots \leqq B^{-1}w^1 \leqq B^{-1}w^0$$

and the system $x = BTx + r$ has a solution x given by

$$x = (1-\gamma) \lim_{i \to \infty} v^i + \gamma \lim_{i \to \infty} w^i$$

where γ is any real number. For $1 \geqq \gamma \geqq 0$ we also have that for all i

$$B^{-1} v^i \leqq B^{-1} x \leqq B^{-1} w^i$$

v) Let $x = BTx + r$ have a solution x. If $\rho(BT) \leqq 1$ then there exist v^0, w^0 satisfying the assumptions of i) to iii) and part iv) holds then.

vi) If iv) holds and TB is irreducible then either $v^0 = w^0 = x$ is a solution of $x = BTx + r$ or $v^0 \neq w^0$ and $0 < \rho(BT) \leqq 1$.

Proof i) The proof is by induction. $B^{-1}(v^{i+1} - v^i) \geqq 0$ holds for $i = 0$. Suppose now it holds for i. Then

$$B^{-1}(v^{i+2} - v^{i+1}) = Tv^{i+1} + B^{-1}r - B^{-1}v^{i+1} \geqq Tv^i + B^{-1}r - B^{-1}v^{i+1} = 0$$

where the inequality above follows by multiplying the induction-hypothesis inequality $B^{-1}(v^{i+1} - v^{i}) \geqq 0$ by $TB \geqq 0$, and the last inequality follows from 3.1. Hence $B^{-1}(v^{i+2} - v^{i+1}) \geqq 0$ and the induction is complete.

ii) Replace v by w in the proof of i) above and reverse all inequalities except $TB \geqq 0$.

iii) The proof again is by induction. $B^{-1}(w^{i}-v^{i}) \geqq 0$ holds for $i = 0$. Assume now that it holds for i. Then

$$B^{-1}(w^{i+1}-v^{i+1}) = T w^{i} + B^{-1}r - Tv^{i} - B^{-1}r = T(w^{i}-v^{i}) = TBB^{-1}(w^{i}-v^{i}) \geqq 0$$

where the last inequality follows from $TB \geqq 0$ and the induction-hypothesis inequality $B^{-1}(w^{i}-v^{i}) \geqq 0$. Hence $B^{-1}(w^{i+1} - v^{i+1}) \geqq 0$ and the induction is complete.

iv) By i), ii) and iii) above we have that

$$B^{-1}v^{0} \leqq B^{-1}v^{1} \leqq \ldots \leqq B^{-1}v^{i} \leqq \ldots \leqq B^{-1}w^{i} \leqq \ldots \leqq B^{-1}w^{1} \leqq B^{-1}w^{0}$$

Again as in proof of Theorem 2.1 ii) we have that the monontonicity of the v^{i} and w^{i} insure the existence of these limits v and w that is $v = \lim_{i \to \infty} v^{i}$ and $w = \lim_{i \to \infty} w^{i}$. From 3.1 and 3.2 we get

$$v = BT\,v + r \qquad\qquad w = BT\,w + r$$

Hence for any real number γ

$$(1 - \gamma)\,v + \gamma w = BT\,((1 - \gamma)v + \gamma w) + r.$$

and $x = (1 - \gamma)v + \gamma w$ is a solution of $x = BT\,x + r$.

We also have that for all i

$$B^{-1}v^i \leqq B^{-1}v \leqq B^{-1}w^i$$

$$B^{-1}v^i \leqq B^{-1}w \leqq B^{-1}w^i$$

Hence for $1 \geqq \gamma \geqq 0$ we have that

$$B^{-1}v^i \leqq B^{-1}((1-\gamma)\ v + \gamma w) = B^{-1}x \leqq B^{-1}w^i.$$

v) From Theorem 2.7ia) we have, by setting $T = S_{11} = S_{22}$ and $S_{12} = S_{21} = 0$, that there exist v^0, w^0 satisfying 2.2. Hence the assumptions of i) to iii) above hold and part iv) of this theorem holds.

vi) From Theorem 2.7 ii) we have, by setting $T = S_{11} = S_{22}$ and $S_{12} = S_{21} = 0$, that $0 < \rho(BT) \leqq 1$.

This completes the proof of the theorem.

As an interesting application of Theorem 3.3 we show that under the standard assumptions for the regular splitting of matrices [3, pp. 87-90], such classical methods as the Jacobi, Gauss-Seidel and successive overrelaxation methods <u>are all monotonic schemes provided that we start with appropriate initial vectors</u> v^0, w^0, <u>the existence of which is guaranteed by the standard assumptions</u>. We state this result as the following theorem.

3.4 Theorem <u>(Monotonicity of Classical Schemes)</u> Let $A = M-N$ be a regular splitting of the $n \times n$ matrix A, that is M^{-1} exists, $M^{-1} \geqq 0$ and $N \geqq 0$. Consider the iterative scheme

$$x^{i+1} = M^{-1}N x^i + M^{-1}b$$

for solving $Ax = b$

i) For any v^0, w^0 such that

3.5 $M(v^1 - v^0) \geqq 0, \quad M(w^0 - w^1) \geqq 0, \quad M(w^0 - v^0) \geqq 0$

where v^1, w^1 are computed from

$$v^{i+1} = M^{-1}N v^i + M^{-1}b \qquad i = 0, 1, \ldots$$

$$w^{i+1} = M^{-1}N w^i + M^{-1}b \qquad i = 0, 1, \ldots$$

we have

$$Mv^0 \leqq Mv^1 \leqq \ldots \leqq Mv^i \leqq \ldots \leqq Mw^i \leqq \ldots \leqq Mw^1 \leqq Mw^0$$

and the system $Ax = b$ has a solution given by

$$x = (1 - \gamma) \lim_{i \to \infty} v^i + \gamma \lim_{i \to \infty} w^i$$

where γ is any real number. For $1 \geqq \gamma \geqq 0$ we also have that for all i

$$B^{-1}v^i \leqq B^{-1}x \leqq B^{-1}w^i$$

ii) If $A^{-1} \geqq 0$, then $\rho(M^{-1}N) < 1$ and the existence of v^0, w^0 satisfying 3.5 is assured.

Proof i) This follows from Theorem 3.3 iv) by making the identifications $B^{-1} = M$, $T = N$, $r = M^{-1}b$

ii) By Theorem 3.13 [3 , p. 89] we have that $\rho(M^{-1}N) < 1$. Hence the system $x = M^{-1}Nx + M^{-1}b$ has a solution x, and by Theorem 3.3v) there exist v^0, w^0 satisfying 3.5. This completes the proof.

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